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Positive Solutions for the Generalized Nonlinear Logistic Equations

Leszek Gasiński and Nikolaos S. Papageorgiou

Abstract. We consider a nonlinear parametric elliptic equation driven by a nonhomogeneous differential operator with a logistic reaction of the superdiffusive type. Using variational methods coupled with suitable truncation and comparison techniques, we prove a bifurcation type result describing the set of positive solutions as the parameter varies.

1 Introduction

Let $\Omega \in \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Dirichlet problem:

$$(P_\lambda) \qquad \begin{cases} -\operatorname{div} a\big(\nabla u(z)\big) = \lambda g(z,u(z)) - f\big(z,u(z)\big) & \text{in } \Omega,\\ u|_{\partial\Omega} = 0, \; u > 0, \lambda > 0. \end{cases}$$

In this problem, $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone map that satisfies some other growth regularity conditions. The precise assumptions on $a(\cdot)$ are listed in hypotheses $H(a)$ below and are general enough to include many differential operators of interest, such as the p-Laplacian. We stress that in problem (P_λ) the differential operator is nonhomogeneous, and this is a source of difficulties in dealing with problem (P_λ). The two functions g and f involved in the reaction are both Carathéodory functions; that is, for all $\zeta \in \mathbb{R}$, the functions $z \mapsto f(z,\zeta)$ and $z \mapsto g(z,\zeta)$ are measurable, and for almost all $z \in \Omega$, the functions $\zeta \mapsto f(z,\zeta)$ and $\zeta \mapsto g(z,\zeta)$ are continuous. They satisfy certain asymptotic conditions as $\zeta \to +\infty$ and as $\zeta \searrow 0$ that incorporate in our framework the so-called superdiffusive reaction of the nonlinear logistic equation, which has the form

$$\lambda \zeta^{q-1} - \zeta^{r-1} \quad \text{for } \zeta \geqslant 0, \tag{1.1}$$

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with $1 < p < q < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geqslant N. \end{cases}$$

Our goal is to study the dependence on the parameter $\lambda > 0$ of the set of positive solutions. This question was investigated by Takeuchi [25, 26] and by Dong [6] for Dirichlet p-Laplacian equations with the classical superdiffusive reaction (1.1). They proved bifurcation type results describing the dependence of positive solutions on the parameter $\lambda > 0$. Their work was extended to more general reactions of the form $\lambda\zeta^{q-1} - f(z,\zeta)$ for all $\zeta \geqslant 0$ by Filippakis–O'Regan–Papageorgiou [8] (Dirichlet problems) and Cardinali–Papageorgiou–Rubbioni [3] (Neumann problems). We should also mention the recent works of Gasiński–Papageorgiou [13,15] and Gasiński–O'Regan–Papageorgiou [9]. In [13], Dirichlet p-Laplacian equations are studied with a reaction of the form $\lambda f(z,\zeta)$ and positive solutions are produced. In [15], $\lambda g(z,\zeta) = \lambda\zeta^{q-1}$ with $q < p$, and the hypotheses on a and f are more restrictive. In [9], nonlinear Dirichlet logistic equations are studied driven by a nonhomogeneous differential operator (as in (P_λ)) and a reaction that is either subdiffusive or equidiffusive. The emphasis is on the existence of nodal (sign changing) solutions. For some other problems containing the nonhomogeneous differential operator we refer to Gasiński–Papageorgiou [12, 14, 16].

Here, we extend all the aforementioned works on superdiffusive logistic equations. First, our differential operator is in general nonhomogeneous and includes as a special case the p-Laplacian. Second, our relation is considerably more general than all the reactions used in the previous superdiffusive works. Furthermore, we produce additional information concerning the positive solutions, since we generate minimal positive solutions $\overline{u}_\lambda$ and show that the map $\lambda \mapsto \overline{u}_\lambda$ is strictly increasing and left-continuous. For this, we use a strong comparison principle. In addition, in the next section, for easy reference, we recall the main mathematical tools that we will use in the sequel, state our hypotheses on the map $a(\cdot)$, and we present some useful consequences of these hypotheses. In Section 3, we prove the bifurcation type result describing the set of positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies.

2 Mathematical Background – Hypotheses

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\phi \in C^1(X)$, we say that ϕ satisfies the Palais–Smale condition if the following is true:

> Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\phi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and with $\phi'(x_n) \to 0$ in X^* admits a strongly convergent subsequence.

This compactness type condition on the functional ϕ leads to a deformation theorem from which one can derive the minimax theory of the critical values of ϕ. Prominent in this theory is the so-called mountain pass theorem.

Theorem 2.1 *Suppose $\phi \in C^1(X)$ satisfies the Palais–Smale condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > \rho > 0$,*

$$\max\{\phi(x_0), \phi(x_1)\} < \inf\left\{\phi(x) : \|x - x_0\| = \rho\right\} = m_\rho,$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \phi(\gamma(t))$, where

$$\Gamma = \left\{\gamma \in C\big([0,1];X\big) : \gamma(0) = x_0,\ \gamma(1) = x_1\right\}.$$

Then $c \geqslant m_\rho$ and c is a critical value of ϕ.

In the analysis of problem (P_λ), we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}.$$

By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \qquad \text{for } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}.$$

Here, $n(\cdot)$ denotes the outward unit normal to $\partial\Omega$.

Let $\vartheta \in C^1(0,+\infty)$ be a function that satisfies

$$0 < \widehat{c} \leqslant \frac{t\vartheta'(t)}{\vartheta(t)} \leqslant c_0 \qquad \text{for } t > 0, \tag{2.1}$$

$$c_1 t^{p-1} \leqslant \vartheta(t) \leqslant c_2(1 + t^{p-1}) \qquad \text{for } t > 0, \tag{2.2}$$

for some $\widehat{c}, c_0, c_1, c_2 > 0$.

The hypotheses on the map $a(\cdot)$ are the following:

$H(a)$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

- (i) $a_0 \in C^1\big(0,+\infty\big)$, $\lim_{t\to 0^+} t a_0(t) = 0$, $\lim_{t\to 0^+} \frac{t a_0'(t)}{a_0(t)} > -1$ and the function $(0,+\infty) \ni t \mapsto t a_0(t)$ is strictly increasing;
- (ii) $|\nabla a(y)| \leqslant c_3 \frac{\vartheta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$ and some $c_3 > 0$;
- (iii) $\frac{\vartheta(|y|)}{|y|}|\xi|^2 \leqslant \big(\nabla a(y)\xi, \xi\big)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$.

Remark 2.2 These conditions on the map $a(\cdot)$ were motivated by the global nonlinear regularity theory of Lieberman [20, p. 320] and the nonlinear maximum principle of Pucci–Serrin [24, pp. 111, 120]. They are weaker than the ones used by Gasiński–O'Regan–Papageorgiou [9] to describe their nonlinear nonhomogeneous differential operator. If

$$G_0(t) = \int_0^t s\, a_0(s)\, ds,$$

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then G_0 is strictly increasing and strictly convex. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Evidently, the map $y \mapsto G(y)$ is convex and differentiable, with

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for } y \in \mathbb{R}^N \setminus \{0\}, \quad \nabla G(0) = 0.$$

So G is the primitive of a.

By virtue of the convexity of G and since $G(0) = 0$, we have

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N} \quad \text{for } y \in \mathbb{R}^N. \tag{2.3}$$

The next lemma summarizes the main properties of $a(\cdot)$ and is a straightforward consequence of hypotheses $H(a)$.

Lemma 2.3 *If hypotheses $H(a)$ hold, then*

(i) *the map $y \mapsto a(y)$ is continuous, strictly monotone, hence also maximal monotone;*
(ii) *there exists $c_4 > 0$, such that $|a(y)| \leqslant c_4(1 + |y|^{p-1})$ for all $y \in \mathbb{R}^N$;*
(iii) *$(a(y), y)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.*

This lemma together with (2.1), (2.2), and (2.3) leads to the following growth conditions on the primitive G.

Corollary 2.4 *If hypotheses $H(a)$ hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5\big(1 + |y|^p\big) \quad \textit{for } y \in \mathbb{R}^N,$$

for some $c_5 > 0$.

Example 2.5 The following maps satisfy hypotheses $H(a)$.

(i) $a(y) = |y|^{p-2}y$, with $1 < p < +\infty$. This map corresponds to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \quad \text{for } u \in W_0^{1,p}(\Omega).$$

(ii) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < +\infty$. This map corresponds to the (p,q)-Laplace differential operator, defined by

$$\Delta_p u + \Delta_q u \quad \text{for } u \in W_0^{1,p}(\Omega).$$

These operators arise in many physical applications (see Cherfils–Il′yasov [4]), and equations driven by such operators were studied by Gasiński–Papageorgiou [15], Mugnai–Papageorgiou [21], Papageorgiou–Rădulescu [22], and Papageorgiou–Winkert [23].

(iii) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}} y$ with $1 < p < +\infty$. This map corresponds to the generalized p-mean curvature differential operator, defined by

$$\operatorname{div}\Big(\big(1 + |\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u\Big) \quad \text{for } u \in W_0^{1,p}(\Omega).$$

(iv) $a(y) = |y|^{p-2}y\big(1 + \frac{1}{1+|y|^p}\big)$ with $1 < p < +\infty$.

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(v) $a(y) = \begin{cases} 2|y|^{\tau-2}y & \text{if } |y| \leqslant 1, \\ |y|^{p-2}y + |y|^{q-2}y & \text{if } 1 < |y|, \end{cases}$ with $1 < q < p$, $\tau = \frac{p+q}{2}$.

We consider the nonlinear map $A: W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$ (where $\frac{1}{p} + \frac{1}{p'} = 1$), defined by

$$\langle A(u), h \rangle = \int_\Omega \big(a(\nabla u), \nabla h \big)_{\mathbb{R}^N} \, dz \quad \text{for } u, h \in W_0^{1,p}(\Omega).$$

For every $\zeta \in \mathbb{R}$, we set $\zeta^\pm = \max\{\pm\zeta, 0\}$. Then for every $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

We denote the Lebesgue measure on $\mathbb{R}^N$ by $|\cdot|_N$. Finally, if $h: \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example a Carathéodory function), then

$$N_h(u)(\cdot) = h\big(\cdot, u(\cdot)\big) \quad \text{for } u \in W_0^{1,p}(\Omega)$$

(the Nemytskii map corresponding to h).

Let $f_0: \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $|f_0(z,\zeta)| \leqslant a_0(z)(1 + |\zeta|^{r-1})$ for almost all $z \in \Omega$ all $\zeta \in \mathbb{R}$, with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$. We set

$$F_0(z,\zeta) = \int_0^\zeta f_0(z,s)\, ds$$

and consider the C^1-functional $\phi_0: W_0^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$\phi_0(u) = \int_\Omega G\big(\nabla u(z)\big)\, dz - \int_\Omega F_0\big(z, u(z)\big)\, dz \quad \text{for } u \in W_0^{1,p}(\Omega).$$

From Gasiński–Papageorgiou [12], we have the following result.

Proposition 2.6 *If hypotheses $H(a)$ hold and $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of ϕ_0,* i.e., *there exists $\rho_0 > 0$ such that $\phi_0(u_0) \leqslant \phi_0(u_0 + h)$ for all $h \in C_0^1(\overline{\Omega})$, $\|h\|_{C_0^1(\overline{\Omega})} \leqslant \rho_0$, then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of ϕ_0,* i.e., *there exists $\rho_1 > 0$, such that $\phi_0(u_0) \leqslant \phi_0(u_0 + h)$ for all $h \in W_0^{1,p}(\Omega)$, $\|h\| \leqslant \rho_1$.*

Let $\widehat{h}, h \in L^\infty(\Omega)$. We write $\widehat{h} \prec h$ if for every compact set $K \subseteq \Omega$, there exists $\varepsilon = \varepsilon(K) > 0$ such that $\widehat{h}(z) + \varepsilon \leqslant h(z)$ for almost all $z \in K$. Clearly, if $\widehat{h}, h \in C(\Omega)$ and $\widehat{h}(z) < h(z)$ for all $z \in \Omega$, then $\widehat{h} \prec h$. Using this notion, we can have a strong comparison principle that extends Arcoya–Ruiz [2, Proposition 2.6] and Cuesta-Takač [5, Theorem 2.1]. For the proof we refer to Gasiński–Papageorgiou [15, Lemma 2.9, p. 195].

Proposition 2.7 *Suppose hypotheses $H(a)$ hold, $\xi \geqslant 0$, $\widehat{h}, h \in L^\infty(\Omega)$, $\widehat{h} \prec h$, and $u, v \in W_0^{1,p}(\Omega)$ are solutions of the problems*

$$\begin{cases} -\operatorname{div} a\big(\nabla u\big) + \xi|u|^{p-2}u = \widehat{h} & \text{in } \Omega,\ u|_{\partial\Omega} = 0, \\ -\operatorname{div} a\big(\nabla v\big) + \xi|v|^{p-2}v = h & \text{in } \Omega,\ v|_{\partial\Omega} = 0, \end{cases}$$

and $v \in \operatorname{int} C_+$. *Then* $v - u \in \operatorname{int} C_+$.

3 Positive Solutions

The hypotheses on the functions g and f are the following.

$H(g)$: $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that $g(z,0) = 0$ for almost all $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)_+$ and $r \in (p, p^*)$, such that

$$g(z,\zeta) \leqslant a(z)(1+\zeta^{r-1}) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0;$$

(ii) there exists $q > p$ such that

$$0 < \widetilde{\beta} \leqslant \liminf_{\zeta \to +\infty} \frac{g(z,\zeta)}{\zeta^{q-1}} \leqslant \limsup_{\zeta \to +\infty} \frac{g(z,\zeta)}{\zeta^{q-1}} \leqslant \widehat{\beta},$$

uniformly for almost all $z \in \Omega$;

(iii) we have

$$\lim_{\zeta \to 0^+} \frac{g(z,\zeta)}{\zeta^{p-1}} = 0 \quad \text{uniformly for almost } z \in \Omega;$$

(iv) for every $\rho > 0$, we can find $\eta_\rho > 0$, such that $g(z,\zeta) \geqslant \eta_\rho$ for almost all $z \in \Omega$ and all $\zeta \geqslant \rho$.

$H(f)$: $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$, $f(z,\zeta) \geqslant 0$ for all $\zeta \geqslant 0$ and almost all $z \in \Omega$, and

(i) there exist $a_0 \in L^\infty(\Omega)_+$ and $r_0 \in (p, p^*)$, such that

$$f(z,\zeta) \leqslant a_0(z)(1+\zeta^{r_0-1}) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0;$$

(ii) if $q > p$ is as in hypothesis $H(g)$(ii), then

$$\lim_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta^{q-1}} = +\infty \quad \text{uniformly for almost } z \in \Omega;$$

(iii) we have

$$0 \leqslant \liminf_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta^{p-1}} \leqslant \limsup_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta^{p-1}} \leqslant \beta_0,$$

uniformly for almost all $z \in \Omega$.

H_0: for every $\lambda > 0$ and every $\rho > 0$, there exists $\xi_\rho^\lambda > 0$ such that for almost all $z \in \Omega$ the function $\zeta \mapsto \lambda g(z,\zeta) - f(z,\zeta) + \xi_\rho^\lambda \zeta^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 3.1 Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis, we may assume without any loss of generality that $g(z,\zeta) = f(z,\zeta) = 0$ for almost all $z \in \Omega$, all $\zeta \leqslant 0$.

Example 3.2 The following functions satisfy the above hypotheses:

(i) $g(\zeta) = \zeta^{q-1}$ and $f(\zeta) = \zeta^{r-1}$ for all $\zeta \geqslant 0$ with $p < q < r < p^*$. This pair corresponds to the classical superdiffusive reaction (see (1.1))).

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(ii) $g(\zeta) = \zeta^{q-1}$ for all $\zeta \geqslant 0$ and $f(\zeta) = \begin{cases} \zeta^{q-1} & \text{if } \zeta \in [0,1], \\ \zeta^{q-1}\ln\zeta & \text{if } 1 < \zeta, \end{cases}$ with $p < q < p^*$.

(iii) $g(\zeta) = \begin{cases} \zeta^{\tau-1} - \zeta^{\eta-1} & \text{if } \zeta \in [0,1], \\ \zeta^{q-1} - \zeta^{p-1} & \text{if } 1 < \zeta, \end{cases}$ with $p < \tau < \eta$ and $p < q < p^*$ and $f(\zeta) = \begin{cases} \zeta^{p-1} & \text{if } \zeta \in [0,1], \\ \zeta^{r-1} & \text{if } 1 < \zeta, \end{cases}$ with $p < r < p^*$.

We introduce

$$\mathcal{L} = \left\{ \lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution} \right\}$$

and let $S(\lambda)$ be the set of positive solutions of (P_λ). Also, we set $\lambda_* = \inf \mathcal{L}$ (as always, if $\mathcal{L} = \varnothing$, then $\inf \mathcal{L} = +\infty$).

Proposition 3.3 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold, then for all $\lambda > 0$, $S(\lambda) \subseteq \operatorname{int} C_+$ and $\lambda_* > 0$.*

Proof Clearly, we can assume that $\lambda \in \mathcal{L}$. Let $u_\lambda \in S(\lambda)$. Then

$$\begin{cases} -\operatorname{div} a\big(\nabla u_\lambda(z)\big) = \lambda g(z, u_\lambda(z)) - f\big(z, u_\lambda(z)\big) & \text{for a.a. } z \in \Omega, \\ u_\lambda|_{\partial\Omega} = 0, \end{cases}$$

(see [12]). From Ladyzhenskaya–Uraltseva [18, p. 288], we have $u_\lambda \in L^\infty(\Omega)$. So, the regularity result of Lieberman [20, p. 320], implies $u_\lambda \in C_+ \setminus \{0\}$. Let $\rho = \|u_\lambda\|_\infty$ and let $\xi_\rho^\lambda > 0$ be as postulated by hypothesis H_0. We have

$$-\operatorname{div} a(\nabla u_\lambda(z)) + \xi_\rho^\lambda u_\lambda(z)^{p-1} = \lambda g(z, u_\lambda(z)) - f\big(z, u_\lambda(z)\big) + \xi_\rho^\lambda u_\lambda(z)^{p-1} \geqslant 0$$

for almost all $z \in \Omega$, so

$$\operatorname{div} a\big(\nabla u_\lambda(z)\big) \leqslant \xi_\rho^\lambda u_\lambda(z)^{p-1} \quad \text{for almost all } z \in \Omega.$$

Let $\gamma(t) = t a_0(t)$ for all $t > 0$. Hypothesis $H(a)$(ii) and (2.1)–(2.2) entail that

$$t\gamma'(t) = t^2 a_0'(t) + t a_0(t) \geqslant c_1 t^{p-1} \quad \text{for all } t \geqslant 0.$$

Integrating by parts, we obtain

$$\begin{aligned} \int_0^t s\gamma'(s)\,ds &= t\gamma(t) - \int_0^t \gamma(t)\,dt \\ &= t^2 a_0(t) - G_0(t) \geqslant \frac{c_1}{p} t^p \quad \text{for all } t \geqslant 0. \end{aligned} \tag{3.1}$$

Let

$$H(t) = t^2 a_0(t) - G_0(t) \quad \text{and} \quad H_0(t) = \frac{c_1}{p} t^p \quad \text{for all } t \geqslant 0.$$

Both are strictly increasing functions. Let $\tau > 0$. We introduce the sets

$$C_1(\tau) = \left\{ t \in (0,1) : H(t) \geqslant \tau \right\} \quad \text{and} \quad C_2(\tau) = \left\{ t \in (0,1) : H_0(t) \geqslant \tau \right\}.$$

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Evidently, $C_2(\tau) \subseteq C_1(\tau)$ and so $\inf C_1(\tau) \leqslant \inf C_2(\tau)$. Then from Leoni [19, p. 6], we have $H^{-1}(\tau) \leqslant H_0^{-1}(\tau)$. It follows that for $\delta \in (0,1)$ we have

$$\int_0^\delta \frac{1}{H^{-1}(\frac{\xi_p^\lambda}{p}\tau^p)}d\tau \geqslant \int_0^\delta \frac{1}{H_0^{-1}(\frac{\xi_p^\lambda}{p}\tau^p)}d\tau = c_6\int_0^\delta \frac{1}{t}dt = +\infty \tag{3.2}$$

for some $c_6 > 0$. Because of (3.1) and (3.2), we can apply the strong maximum principle of Pucci-Serrin [24, p. 111] and infer that $u_\lambda(z) > 0$ for all $z \in \Omega$. Then, applying the boundary point theorem of Pucci-Serrin [24, p. 120], we conclude that $u_\lambda \in \operatorname{int} C_+$. So, we have that $S(\lambda) \subseteq \operatorname{int} C_+$ for all $\lambda > 0$.

Hypotheses $H(g)$(i)–(iii) and $H(f)$(i)–(iii) imply that we can find $\lambda_0 > 0$ such that

$$\lambda_0 g(z,\zeta) - f(z,\zeta) \leqslant \frac{c_1}{p-1}\widehat{\lambda}_1(p)\zeta^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \geqslant 0, \tag{3.3}$$

with $\widehat{\lambda}_1(p) > 0$ being the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Let $\lambda \in (0,\lambda_0)$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u \in S(\lambda) \subseteq \operatorname{int} C_+$ such that

$$A(u) = \lambda N_g(u) - N_f(u). \tag{3.4}$$

We act on (3.4) with $u \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\frac{c_1}{p-1}\|\nabla u\|_p^p &\leqslant \int_\Omega (\lambda g(z,u) - g(z,u))u\,dz < \int_\Omega \big(\lambda_0 g(z,u) - f(z,u)\big)u\,dz \\ &\leqslant \frac{c_1}{p-1}\widehat{\lambda}_1(p)\|u\|_p^p\end{aligned}$$

(see Lemma 2.3, hypothesis $H(g)$(iv), (3.3) and recall that $\lambda < \lambda_0$), so

$$\|\nabla u\|_p^p < \widehat{\lambda}_1(p)\|u\|_p^p,$$

which contradicts the variational characterization of the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ (see *e.g.*, Gasiński-Papageorgiou [11, p. 732]). Therefore, $\lambda \notin \mathcal{L}$ and so we conclude that $0 < \lambda_0 \leqslant \lambda_* = \inf \mathcal{L}$. ■

Proposition 3.4 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold and $\lambda \in \mathcal{L}$, then $[\lambda, +\infty) \subseteq \mathcal{L}$.*

Proof Let $\mu > \lambda$. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 3.3). We have

$$\begin{aligned}-\operatorname{div} a\big(\nabla u_\lambda(z)\big) &= \lambda g\big(z,u_\lambda(z)\big) - f(z,u_\lambda(z)) \\ &\leqslant \mu g\big(z,u_\lambda(z)\big) - f\big(z,u_\lambda(z)\big) \quad \text{for almost all } z \in \Omega.\end{aligned} \tag{3.5}$$

We consider the following truncation of the reaction of problem (P_μ):

$$k_\mu(z,\zeta) = \begin{cases} \mu g(z,u_\lambda(z)) - f(z,u_\lambda(z)) & \text{if } \zeta \leqslant u_\lambda(z), \\ \mu g(z,\zeta) - f(z,\zeta) & \text{if } u_\lambda(z) < \zeta. \end{cases} \tag{3.6}$$

This is a Carathéodory function. We set

$$K_\mu(z,\zeta) = \int_0^\zeta k_\mu(z,s)\,ds$$

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and consider the C^1-functional $\psi_\mu: W_0^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$\psi_\mu(u) = \int_\Omega G\big(\nabla u(z)\big)\, dz - \int_\Omega K_\mu\big(z, u(z)\big)\, dz, \quad u \in W_0^{1,p}(\Omega).$$

From (3.6) and hypotheses $H(g)$(ii), $H(f)$(ii), we see that ψ_λ is coercive. Also, using the Sobolev embedding theorem, we see that ψ_μ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \inf_{u \in W_0^{1,p}(\Omega)} \psi_\mu(u),$$

so $\psi'_\mu(u_\mu) = 0$, and thus

$$A(u_\mu) = N_{k_\mu}(u_\mu). \tag{3.7}$$

On (3.7) we act with $(u_\lambda - u_\mu)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \big\langle A(u_\mu), (u_\lambda - u_\mu)^+ \big\rangle &= \int_\Omega k_\mu(z, u_\mu)(u_\lambda - u_\mu)^+\, dz \\ &= \int_\Omega \big(\mu g(z, u_\lambda) - f(z, u_\lambda)\big)(u_\lambda - u_\mu)^+\, dz \\ &\geqslant \big\langle A(u_\lambda), (u_\lambda - u_\mu)^+ \big\rangle \end{aligned}$$

(see (3.6), (3.5)), so

$$\int_{\{u_\lambda > u_\mu\}} \big(a(\nabla u_\lambda) - a(\nabla u_\mu), \nabla u_\lambda - \nabla u_\mu\big)_{\mathbb{R}^N} \leqslant 0.$$

Thus $|\{u_\lambda > u_\mu\}|_N = 0$ (see Lemma 2.3), and hence $u_\lambda \leqslant u_\mu$.

Using (3.6), equation (3.7) becomes

$$A(u_\mu) = \mu N_g(u_\mu) - N_f(u_\mu),$$

so $u_\mu \in S(\mu) \subseteq \operatorname{int} C_+$, hence $\mu \in \mathcal{L}$, and thus $[\lambda, +\infty) \subseteq \mathcal{L}$. ∎

As a consequence of Proposition 3.4, we have $(\lambda_*, +\infty) \subseteq \mathcal{L}$.

Proposition 3.5 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold and $\mu > \lambda_*$, then problem (P_μ) has at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+$.*

Proof Let $\lambda \in (\lambda_*, \mu)$ and let $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 3.3 and recall that $(\lambda_*, +\infty) \subseteq \mathcal{L}$). From the proof of Proposition 3.4, we know that we can find $u_0 \in S(\mu) \subseteq \operatorname{int} C_+$, such that $u_\lambda \leqslant u_0$. Moreover, we know that u_0 is a minimizer of the functional ψ_μ (see the proof of Proposition 3.4). Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho^\lambda > 0$ be as postulated by hypothesis H_0. We have

$$\begin{aligned} & -\operatorname{div} a\big(\nabla u_\lambda(z)\big) + \xi_\rho^\lambda u_\lambda(z)^{p-1} \\ &= \lambda g(z, u_\lambda(z)) - f\big(z, u_\lambda(z)\big) + \xi_\rho^\lambda u_\lambda(z)^{p-1} \\ &\leqslant \lambda g\big(z, u_0(z)\big) - f\big(z, u_0(z)\big) + \xi_\rho^\lambda u_0(z)^{p-1} \\ &= \mu g(z, u_0(z)) - f\big(z, u_0(z)\big) + \xi_\rho^\lambda u_0(z)^{p-1} - (\mu - \lambda) g\big(z, u_0(z)\big) \\ &\leqslant \mu g(z, u_0(z)) - f\big(z, u_0(z)\big) + \xi_\rho^\lambda u_0(z)^{p-1} \\ &= -\operatorname{div} a\big(\nabla u_0(z)\big) + \xi_\rho^\lambda u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega \end{aligned} \tag{3.8}$$

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(see hypothesis H_0, $H(g)$(iv) and recall that $u_\lambda \leqslant u_0$, $\lambda < \mu$ and $u_0 \in S(\mu)$). Recall that $u_0 \in \operatorname{int} C_+$. So, for $K \subseteq \Omega$ compact, we have $u_0(z) \geqslant m_K > 0$ for all $z \in K$, so

$$g(z, u_0(z)) \geqslant \eta_K > 0 \quad \text{for almost all } z \in K$$

(see hypothesis $H(g)$(iv)). This fact and (3.8) permit the use of Proposition 2.7, from which we infer that

$$u_0 - u_\lambda \in \operatorname{int} C_+. \tag{3.9}$$

Let

$$\widehat{G}(z,\zeta) = \int_0^\zeta g(z,s)\,ds, \quad F(z,\zeta) = \int_0^\zeta f(z,s)\,ds$$

and consider $\phi_\mu \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ the energy functional for problem (P_μ) defined by

$$\phi_\mu(u) = \int_\Omega G(\nabla u(z))\,dz - \mu \int_\Omega \widehat{G}(z,u(z))\,dz + \int_\Omega F(z,u(z))\,dz, \quad u \in W_0^{1,p}(\Omega).$$

We have $\phi_\mu \in C^1(W_0^{1,p}(\Omega))$. We define

$$[u_\lambda) = \left\{ u \in W_0^{1,p}(\Omega) : u_\lambda(z) \leqslant u(z) \text{ for almost all } z \in \Omega \right\}.$$

From (3.6) we see that

$$\phi_\mu\big|_{[u_\lambda)} = \psi_\mu\big|_{[u_\lambda)} + \xi_\lambda^*, \tag{3.10}$$

with $\xi_\lambda^* \in \mathbb{R}$. Recall that $u_0 \in \operatorname{int} C_+$ is a minimizer of ψ_μ. Then from (3.9) and (3.10), it follows that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of ϕ_μ and from Proposition 2.6, we get

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \phi_\mu. \tag{3.11}$$

Hypotheses $H(g)$(iii) and $H(f)$(iii) imply that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\widehat{G}(z,\zeta) \leqslant \frac{\varepsilon}{p}\zeta^p \quad \text{and} \quad F(z,\zeta) \geqslant -\frac{\varepsilon}{p}\zeta^p \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in [0,\delta]. \tag{3.12}$$

Let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$. Then using (3.12), Corollary 2.4, and the variational characterization of $\widehat{\lambda}_1(p) > 0$, we have

$$\phi_\mu(u) \geqslant \frac{c_1}{p(p-1)}\|\nabla u\|_p^p - \frac{(\mu+1)\varepsilon}{p}\|u\|_p^p \geqslant \frac{1}{p}\Big(\frac{c_1}{p-1} - \frac{(\mu+1)\varepsilon}{\widehat{\lambda}_1(p)}\Big)\|\nabla u\|_p^p. \tag{3.13}$$

If we choose $\varepsilon \in (0, \frac{c_1\widehat{\lambda}_1(p)}{(p-1)(\mu+1)})$, then from (3.13) it follows that $u = 0$ is a local $C_0^1(\overline{\Omega})$-minimizer of ϕ_μ, so

$$u = 0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \phi_\mu.$$

Without any loss of generality, we may assume that $0 = \phi_\mu(0) \leqslant \phi_\mu(u_0)$ (the reasoning is similar if the opposite inequality holds). Also, we assume that the set K_{ϕ_μ} of critical points of ϕ_μ is finite (otherwise, we already have infinitely many positive solutions). Then because of (3.11), we can find $\rho \in (0,1)$ small, such that

$$0 = \phi_\lambda(0) \leqslant \phi_\lambda(u_0) < \inf\{\phi_\lambda(u) : \|u - u_0\| = \rho\} = m_\rho, \quad \|u_0\| \geqslant \rho \tag{3.14}$$

(see the proof of Aizicovici–Papageorgiou–Staicu [1, Proposition 29] or proof of Gasiński–Papageorgiou [10, Theorem 3.4]). From hypotheses $H(g)$(ii) and $(H(f)$(ii)),

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it follows that ϕ_μ is coercive. So, it satisfies the Palais–Smale condition. This fact and (3.14) permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\phi_\mu} \quad \text{and} \quad m_\rho \leqslant \phi_\mu(\widehat{u}). \tag{3.15}$$

From (3.14) and (3.15) we see that $\widehat{u} \notin \{0, u_0\}$ and $\widehat{u} \in S(\mu) \subseteq \operatorname{int} C_+$. ∎

In fact we can show that for every $\lambda > \lambda_*$, problem (P_λ) has a smallest positive solution.

Proposition 3.6 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold and $\lambda > \lambda_*$, then problem (P_λ) admits a smallest positive solution $\overline{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$, and the map $(\lambda_*, +\infty) \ni \lambda \mapsto \overline{u}_\lambda \in C_0^1(\overline{\Omega})$ is strictly increasing (that is, if $\lambda < \mu$, then $\overline{u}_\mu - \overline{u}_\lambda \in \operatorname{int} C_+$).*

Proof As in Filippakis–Kristaly–Papageorgiou [7], exploiting the monotonicity of the map A, we have that the solution set $S(\lambda)$ is downward directed (*i.e.*, if $u_1, u_2 \in S(\lambda)$, then we can find $u \in S(\lambda)$ such that $u \leqslant u_1$, $u \leqslant u_2$). Then since we want to produce the minimal element of $S(\lambda)$, without any loss of generality, we can assume that

$$\|u\|_\infty \leqslant c_7 \quad \text{for } u \in S(\lambda) \tag{3.16}$$

for some $c_7 > 0$. From Hu–Papageorgiou [17, p. 178], we know that we can find a sequence $\{u_n\}_{n \geqslant 1} \subseteq S(\lambda)$ such that

$$\inf S(\lambda) = \inf_{n \geqslant 1} u_n.$$

We have

$$A(u_n) = \lambda N_g(u_n) - N_f(u_n) \quad \text{for } n \geqslant 1. \tag{3.17}$$

From (3.16), (3.17), and Lieberman [20, p. 320], we know that we can find $\alpha \in (0,1)$ and $c_8 > 0$ such that $u_n \in C_0^{1,\alpha}(\overline{\Omega})$ and $\|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_8$ for all $n \geqslant 1$. Exploiting the compactness of the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$ and by passing to a suitable subsequence if necessary, we can say that

$$u_n \longrightarrow \overline{u}_\lambda \quad \text{in } C_0^1(\overline{\Omega}). \tag{3.18}$$

Suppose that $\overline{u}_\lambda = 0$. Using hypotheses $H(g)$(iii) and $H(f)$(iii), we see that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$g(z,\zeta) \leqslant \varepsilon \zeta^{p-1} \quad \text{and} \quad f(z,\zeta) \geqslant -\varepsilon \zeta^{p-1} \tag{3.19}$$

for almost all $z \in \Omega$, all $\zeta \in [0,\delta]$. From (3.18) and since we have assumed that $\overline{u}_\lambda = 0$, we can find $n_0 \in \mathbb{N}$ such that

$$u_n(z) \in [0,\delta] \quad \text{for } n \geqslant n_0, z \in \overline{\Omega}. \tag{3.20}$$

On (3.17) we act with u_n and using Lemma 2.3 and (3.19), (3.20), we obtain

$$\frac{c_1}{p-1}\|\nabla u\|_p^p \leqslant (\lambda+1)\varepsilon \|u_n\|_p^p \quad \text{for all } n \geqslant n_0.$$

Choosing $\varepsilon \in (0, \frac{c_1}{p-1}\frac{\widehat{\lambda}_1(p)}{\lambda+1})$, we obtain

$$\|\nabla u_n\|_p^p < \widehat{\lambda}_1(p)\|u_n\|_p^p \quad \text{for all } n \geqslant n_0,$$

which contradicts the variational characterization of $\widehat{\lambda}_1(p)$ (see Gasiński–Papageorgiou [11, p. 732]). This proves that $\overline{u}_\lambda \neq 0$.

If in (3.17) we pass to the limit as $n \to +\infty$ and use (3.18), then

$$A(\overline{u}_\lambda) = \lambda N_g(\overline{u}_\lambda) - N_f(\overline{u}_\lambda),$$

so $\overline{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ and $\overline{u}_\lambda = \inf S(\lambda)$. Moreover, if $\lambda < \mu$, then as in the proof of Proposition 3.5, using hypothesis H_0 and Proposition 2.7 (the strong comparison principle), we obtain

$$\overline{u}_\mu - \overline{u}_\lambda \in \operatorname{int} C_+,$$

so the map $\lambda \mapsto \overline{u}_\lambda$ is strictly increasing. ∎

Proposition 3.7 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold, then $\lambda_* \in \mathcal{L}$ and so $\mathcal{L} = [\lambda_*, +\infty)$.*

Proof Let $\lambda_n \in (\lambda_*, +\infty)$ be such that $\lambda_n \searrow \lambda_*$. Let $\overline{u}_n = \overline{u}_{\lambda_n} \in \operatorname{int} C_+$ for $n \geqslant 1$ be the corresponding minimal positive solution of problem (P_{λ_n}) for $n \geqslant 1$. From Proposition 3.6 we know that the sequence $\{\overline{u}_n\} \subseteq \operatorname{int} C_+$ is strictly decreasing. So, we have

$$\|\overline{u}_n\|_\infty \leqslant \|\overline{u}_1\|_\infty \quad \text{for } n \geqslant 1. \tag{3.21}$$

We have

$$A(\overline{u}_n) = \lambda_n N_g(\overline{u}_n) - N_f(\overline{u}_n) \quad \text{for } n \geqslant 1. \tag{3.22}$$

From (3.21) and the regularity result of Lieberman [20, p. 320], we know that there exists $\alpha \in (0,1)$ and $c_0 > 0$ such that

$$\overline{u}_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|\overline{u}_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_9 \quad \text{for } n \geqslant 1.$$

So, we can assume that

$$\overline{u}_n \longrightarrow u_* \quad \text{in } C_0^1(\overline{\Omega}). \tag{3.23}$$

As in the proof of Proposition 3.6, using (3.23) and hypothesis $H(g)$(iii) and $H(f)$(iii), we can show that $u_* \neq 0$. Therefore, by passing to the limit as $n \to +\infty$ in (3.22) and using (3.23), we obtain that $u_* \in S(\lambda_*) \subseteq \operatorname{int} C_+$, hence $\lambda_* \in \mathcal{L}$. ∎

Reasoning as in the proof of Proposition 3.6, we can produce a minimal solution for problem (P_{λ_*}).

Proposition 3.8 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold, then problem (P_{λ_*}) admits a smallest positive solution $\overline{u}_{\lambda_*} \in S(\lambda_*) \subseteq \operatorname{int} C_+$. The map $[\lambda_*, +\infty) \ni \lambda \mapsto \overline{u}_\lambda \in C_0^1(\overline{\Omega})$ is strictly increasing.*

Proposition 3.9 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold, then the map $(\lambda_*, +\infty) \ni \lambda \mapsto \overline{u}_\lambda \in C_0^1(\overline{\Omega})$ is left continuous.*

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Proof Let $\{\lambda_n\}_{n\geqslant 1} \subseteq (\lambda_*, +\infty)$ be a sequence such that $\lambda_n \nearrow \lambda$ and $n \to +\infty$. Let $\overline{u}_n = \overline{u}_{\lambda_n} \in S(\lambda_n) \subseteq \operatorname{int} C_+$ for all $n \geqslant 1$ (see Proposition 3.6). We have

$$\|\overline{u}_n\|_\infty \leqslant \|\overline{u}_\lambda\|_\infty \quad \text{for } n \geqslant 1.$$

So, as before (see *e.g.*, the proof of Proposition 3.7), we have

$$\overline{u}_n \nearrow \widetilde{u} \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{and} \quad \widetilde{u} \in S(\lambda) \subseteq \operatorname{int} C_+. \tag{3.24}$$

Suppose that $\widetilde{u} \neq \overline{u}_\lambda$. Then we can find $z_0 \in \Omega$ such that $\overline{u}_\lambda(z_0) < \widetilde{u}(z_0)$, so for some $n_0 \geqslant 1$, we have $\overline{u}_\lambda(z_0) < \overline{u}_n(z_0)$ for $n \geqslant n_0$ (see (3.24)), which contradicts the strict monotonicity of $\lambda \mapsto \overline{u}_\lambda$ (see Proposition 3.6). Therefore, $\widetilde{u} = \overline{u}_\lambda$ and so we have proved the left-continuity of the map $\lambda \mapsto \overline{u}_\lambda$. ∎

So, we can state the following bifurcation type result, summarizing the situation for the positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies.

Theorem 3.10 *If hypotheses $H(a)$, $H(g)$, $H(f)$, and H_0 hold, then there exists $\lambda_* > 0$, such that*

(i) *for all $\lambda > \lambda_*$ problem (P_λ) has at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+$;*
(ii) *for $\lambda = \lambda_*$ problem (P_λ) has at least one positive solution $u_* \in \operatorname{int} C_+$;*
(iii) *for $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solution.*

Moreover, for every $\lambda \geqslant \lambda_*$ problem (P_λ) has a smallest positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$; if $\mu > \lambda \geqslant \lambda_*$, then $\overline{u}_\mu - \overline{u}_\lambda \in \operatorname{int} C_+$ and the map $(\lambda_*, +\infty) \ni \lambda \mapsto \overline{u}_\lambda \in C_0^1(\overline{\Omega})$ is left continuous.

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Jagiellonian University, Faculty of Mathematics and Computer Science, Kraków, Poland
e-mail: leszek.gasinski@ii.uj.edu.pl

National Technical University,, Department of Mathematics, Zografou Campus, Athens 15780, Greece
e-mail: npapg@math.ntua.gr