Canad. Math. Bull.Vol. 34 (3), 1991 pp. 405–411

THE CONVERSE OF THE DOMINATED ERGODIC THEOREM IN HUREWICZ SETTING

LÁSZLÓ I. SZABÓ

ABSTRACT. The converse of the dominated ergodic theorem in infinite measure spaces is extended to non-singular transformations, i.e. transformations that only preserve the measure of null sets. An inverse weak maximal inequality is given and then applied to obtain related results in Orlicz spaces.

Introduction. According to Birkhoff's ergodic theorem, the ergodic averages $A_n(f) = (1/n)\sum_{i=0}^{n-1} T^i f$ converge a.e. if T is a measure preserving point transformation and $f \in L_1$. Wiener's [10] dominated ergodic theorem asserts that if $f \in L_p, p > 1$, then $\sup_n A_n(f) \in L_p$, and, in finite measure spaces, if $f \in L \log L$, then $\sup_n A_n(f) \in L_1$. The converse of this last result was obtained by D. S. Ornstein [8], after the particular case of translations in $\mathbf{R}^d$ was proved by Stein [9]. In infinite measure spaces, of interest are the ratios $R_n(f,g) = (\sum_{i=0}^{n-1} T^i f)/(\sum_{i=0}^{n-1} T^i g)$, and the right form of the converse of the dominated theorem, still for measure preserving transformations T, was found by Derriennic [2]. Using his method we extend this to the Hurewicz setting: the point transformation τ only preserves the measure of null sets, and T appearing in $R_n(f,g)$ is the Markov operator induced by τ. At the same time, we generalize the result so that the correspondence is established not only between L_p and $L_p (p > 1)$, or between $L \log L$ and L_1, but, more generally, between appropriate Orlicz spaces. Here our approach is similar to that of Edgar-Sucheston [3], and we prove below the needed variant of their Orlicz norm *inverse* inequality.

1. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. That is, X is an abstract set, $\mathcal{F}$ is a σ-algebra of *measurable* subsets of X, μ is a σ-finite measure on $\mathcal{F}$. All considered sets and functions will be assumed measurable. A mapping $\tau: X \longrightarrow X$ is called measurable if $\tau^{-1}\mathcal{F} \subset \mathcal{F}$. A transformation is a bijection of X such that both τ and τ^{-1} are measurable. Thus we consider here only invertible transformations. A transformation τ is called *non-singular* if $\mu(A) = 0$ implies $\mu(\tau^{-1}A) = \mu(\tau A) = 0$. A set W is called *wandering* if the sets $\tau^{-n}W$ $(n = 0, 1, 2, \ldots)$ are disjoint. A non-singular transformation τ is defined to be *conservative* if there are no wandering sets of positive measure; *incompressible* if $A \subset \tau^{-1}A$ implies $\mu(\tau^{-1}A \setminus A) = 0$. It is easy to see that τ is conservative if and only

This paper is part of the Ph.D. dissertation of the author written under the guidance of Prof. Louis Sucheston at the Ohio State University, Columbus, Ohio.

Research was supported in part by NSF Grant 88-02126.

Received by the editors April 1, 1989; revised: June 18, 1990.

AMS subject classification: 28D99, 47A35.

https://doi.org/10.4153/CMB-1991-065-0 Published online by Cambridge University Press

if it is incompressible (see [7], p. 16). From now on we assume that τ is a non-singular transformation. Denote by ω_n the a.e. uniquely defined function satisfying for all A

$$\mu(A) = \int_{\tau^{-n}A} \omega_n \, d\mu.$$

Write ω for ω_1. Define an operator T by

$$Tf(x) = \omega(x) f\big(\tau(x)\big)$$

for $f \in L_1$. Then T is a *Markovian* operator: a positive linear operator on L_1 that preserves the integral: $\int Tf \, d\mu = \int f d\mu$ for all $f \in L_1$. Such an operator is necessarily a *contraction* on L_1: $\|Tf\|_1 \leq \|f\|_1$.

The following properties are easy to prove. For every $f \in L_1$, $A \in \mathcal{F}$ and $n = 0, 1, 2, \ldots$, we have

$$\text{(1)} \qquad \int_A f \, d\mu = \int_{\tau^{-n}A} (f \circ \tau^n) \cdot \omega_n \, d\mu,$$

$$\text{(2)} \qquad T^n f = (f \circ \tau^n) \cdot \omega_n,$$

and the adjoint operator $T^*: L_\infty \longrightarrow L_\infty$ is given by

$$T^* h = h \circ \tau^{-1}.$$

A positive contraction T is defined to be *conservative* if $h \geq 0$, $T^*h \leq h$ implies $T^*h = h$. A transformation τ is called *ergodic* if $\mu(A \triangle \tau^{-1}A) = 0$ implies $\mu(A) = 0$ or $\mu(A^c) = 0$. An operator T is called *ergodic* if $T^* 1_A = 1_A$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$. We note that the operator T defined above is conservative (resp. ergodic) if and only if τ is conservative (resp. ergodic).

Let I_A denote the operator of multiplication by 1_A. One can prove by induction that if $f \in L_1$ and $A, E \in \mathcal{F}$, then

$$\text{(3)} \qquad \int_E (TI_A)^n f \, d\mu = \int_{E_n} T^n f \, d\mu,$$

for $n = 0, 1, 2, \ldots$, where $E_n = E \cap \bigcap_{i=1}^n \tau^{-i} A$.

From now on let us assume that two functions f and g are given with $f \in L_1, f \geq 0$, $g > 0$. Define

$$s(f, g) = \sup_n \frac{\sum_{i=0}^n T^i f}{\sum_{i=0}^n T^i g}.$$

The next lemma and Theorem 1, for measure preserving transformations, were proved by Derriennic [2]. Here our method is similar to his.

LEMMA 1. *Assume that τ is conservative and ergodic. Let λ be a positive number and define $A = \{ s(f, g) > \lambda \}$. If $\mu(A^c) > 0$ then*

$$\int_A f \, d\mu \leq \lambda \int_{A \cup \tau^{-1}A} g \, d\mu.$$

PROOF. Since T is conservative and ergodic,

$$1_A = \sum_{n=1}^{\infty} (I_A T^*)^n 1_{A^c}. \tag{4}$$

This is known, but for completeness we give a proof:

$$\sum_{n=1}^{N} (I_A T^*)^n 1 = \sum_{n=1}^{N} (I_A T^*)^n 1_A + \sum_{n=1}^{N} (I_A T^*)^n 1_{A^c}.$$

On the other hand, $T^* 1 = 1$, hence

$$\sum_{n=1}^{N} (I_A T^*)^n 1 = 1_A + \sum_{n=1}^{N-1} (I_A T^*)^n 1_A.$$

Therefore

$$1_A = (I_A T^*)^N 1_A + \sum_{n=1}^{N} (I_A T^*)^n 1_{A^c}.$$

Since

$$(I_A T^*)^N 1_A = 1_{A \cap \tau A \cap \cdots \cap \tau^N A},$$

$(I_A T^*)^N 1_A$ converges a.e. as $N \longrightarrow \infty$ to 1_D where $D = \bigcap_{n=1}^{\infty} \tau^n A$. Now $D \subset \tau(D)$ implies that $D = \tau(D)$ a.e. because τ is conservative. But τ is also ergodic, so $\mu(D) = 0$, because $D \subset A$ and $\mu(A^c) > 0$. This proves (4).

By (3), we have

$$\begin{aligned} \int_A f \, d\mu &= \int f \sum_{n=1}^{\infty} (I_A T^*)^n 1_{A^c} \, d\mu \\ &= \int_{A^c} \sum_{n=1}^{\infty} (T I_A)^n f \, d\mu \\ &= \sum_{n=1}^{\infty} \int_{B_n} T^n f \, d\mu, \end{aligned}$$

where $B_n = A^c \cap \bigcap_{i=1}^{n} \tau^{-i} A$. The set $E_k = B_k \cap \tau^{-(k+1)} A^c$ contains exactly the points of A^c which first return to A^c at time $k+1$. The sets E_k for $k = n, n+1, \ldots$ form a partition of B_n, hence

$$\int_A f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} \Big(\sum_{i=1}^{n} T^i f \Big) \, d\mu. \tag{5}$$

On A^c we have

$$\sum_{i=1}^{n} T^i f \leq \lambda \sum_{i=0}^{n} T^i g$$

for every n. Therefore, applying (5) with g instead of f, one has

$$\begin{aligned} \int_A f \, d\mu &\leq \lambda \sum_{n=1}^{\infty} \int_{E_n} \Big(\sum_{i=0}^{n} T^i g \Big) \, d\mu \\ &= \lambda \sum_{n=1}^{\infty} \int_{E_n} g \, d\mu + \lambda \sum_{n=1}^{\infty} \int_{E_n} \Big(\sum_{i=1}^{n} T^i g \Big) \, d\mu \\ &= \lambda \Big(\int_{A^c \cup \tau^{-1} A} g \, d\mu + \int_A g \, d\mu \Big) \\ &= \lambda \int_{A \cup \tau^{-1} A} g \, d\mu. \end{aligned}$$

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https://doi.org/10.4153/CMB-1991-065-0 Published online by Cambridge University Press

THEOREM 1. *There exists* $\lambda_0 \geq 0$ *such that for all* $\lambda > \lambda_0$,

$$\int_{\{f>\lambda g\}} f\,d\mu \leq \lambda \int_{\{s(f,g)+s(Tf,Tg)>\lambda\}} g\,d\mu. \tag{6}$$

PROOF. Let $\lambda_0 = \inf\{\lambda > 0 : \mu\big(s(f,g) \leq \lambda\big) > 0\}$. Note that $f > \lambda g$ implies $s(f,g) > \lambda$, and also $\tau^{-1}\{s(f,g) > \lambda\} = \{s(Tf,Tg) > \lambda\}$, because

$$\begin{aligned} s(f,g)(\tau x) &= \sup_n \frac{\sum_{i=0}^{n-1} T^i f(\tau x)}{\sum_{i=0}^{n-1} T^i g(\tau x)} \\ &= \sup_n \frac{\sum_{i=0}^{n-1} T^i f(\tau x)\omega(x)}{\sum_{i=0}^{n-1} T^i g(\tau x)\omega(x)} \\ &= \sup_n \frac{\sum_{i=1}^{n} T^i f}{\sum_{i=1}^{n} T^i g} = s(Tf,Tg). \end{aligned}$$

Therefore (6) follows from Lemma 1 for $\lambda > \lambda_0$. ∎

2. An Orlicz function is defined as an increasing, convex function $\Phi\colon [0,\infty) \longrightarrow [0,\infty)$, satisfying $\Phi(0) = 0$, nontrivial in the sense that $\Phi(u) > 0$ for some $u > 0$. Let φ be the left-continuous derivative of Φ. If φ is unbounded, or equivalently $\Phi(u)/u \longrightarrow \infty$ as $u \longrightarrow \infty$, then the generalized inverse ψ of φ exists, defined as follows: $\psi(y) = \inf\{x \in (0,\infty) : \varphi(x) \geq y\}$. Let $\Psi(u) = \int_0^u \psi(t)\,dt$. Then Ψ is also an Orlicz function, called the conjugate of Φ. Recall Young's inequality: for all $u, v > 0$,

$$uv \leq \Phi(u) + \Psi(v). \tag{7}$$

Equality holds if and only if $v = \varphi(u)$ or $u = \psi(v)$. Let

$$\xi(u) = u\varphi(u) - \Phi(u) = \Psi\big(\varphi(u)\big).$$

The *modular* for ξ of a real-valued function h on X is defined as $M_\xi(h) = \int \xi(|h|)\,d\gamma$. Assume that ξ is an Orlicz function, then the *Luxemburg norm* of h is $\|h\|_\xi = \inf\{a > 0, M_\xi(h/a) \leq 1\}$. Let $L_\xi = \{h : \|h\|_\xi < \infty\} = \{h : M_\xi(h/a) < \infty$ for some $a > 0\}$. L_ξ is a Banach lattice, called the Orlicz space for ξ. Define $H_\xi = \{h : M_\xi(h/a) < \infty$ for all $a > 0\}$. H_ξ is called the heart of the Orlicz space L_ξ. It is a Banach lattice, closure in L_ξ of integrable simple functions. The spaces H_ξ are often the right setting for ergodic theorems (see [1] and [3]).

THEOREM 2. *Let* Φ *be an Orlicz function with* $\Phi(u)/u \longrightarrow \infty$ *as* $u \longrightarrow \infty$, *and let r, s be positive functions on X. Suppose there is* $c > 0$ *and* $\lambda_0 \geq 0$ *such that*

$$(c/\lambda)\int_{\{r>\lambda\}} r\,d\gamma \leq \gamma(s > \lambda) \tag{8}$$

for all $\lambda \geq \lambda_0$. *Then for each* $\lambda > 0$ *there exists* $y_0 \geq 0$ *such that*

$$M_\xi(s/\lambda) \geq cM_\Phi(r/\lambda) + cM_\xi(r/\lambda) - I, \tag{9}$$

https://doi.org/10.4153/CMB-1991-065-0 Published online by Cambridge University Press

where

$$I = (c/\lambda)\int_0^{y_0}\int_{\{r>\lambda\psi(y)\}} r\,d\gamma\,dy.$$

PROOF. Let C_φ denote the set of points of continuity of φ, and let C_ψ be the set of points of continuity of ψ. Since φ and ψ are nondecreasing, both C_φ and C_ψ are countable. Let $A = \lambda C_\varphi \cup \{0\}$, and suppose that r and s have countably many values, all in the set A. Note that for each $x \in X$, $r(x)/\lambda \in C_\varphi$. For $y \in C_\psi$,

$$y \leq \varphi\big(r(x)/\lambda\big) \text{ iff } \psi(y) \leq r(x)/\lambda.$$

Hence, for almost all $y \in (0,\infty)$,

$$\{x \in X : y \leq \varphi\big(r(x)/\lambda\big)\} = \{x \in X : \psi(y) \leq r(x)/\lambda\}.$$

Now Fubini's theorem implies that the sets

$$\{(x,y) \in X \times (0,\infty) : y \leq \varphi\big(r(x)/\lambda\big)\}$$

and

$$\{(x,y) \in X \times (0,\infty) : \psi(y) \leq r(x)/\lambda\}$$

agree except for a set of measure zero. The same holds for the sets

$$\{(x,y) \in X \times (0,\infty) : y \leq \varphi\big(s(x)/\lambda\big)\},$$
$$\{(x,y) \in X \times (0,\infty) : \psi(y) \leq s(x)/\lambda\}.$$

Fix $\lambda > 0$ and let $y_0 = \inf\{y \geq 0 : \lambda\psi(y) \geq \lambda_0\}$. Now we have

$$\begin{aligned}
\lambda M_\xi(s/\lambda) &= \lambda \int \xi(s/\lambda)\,d\gamma \\
&= \lambda \int \Psi(\varphi(s/\lambda))\,d\gamma \\
&= \lambda \int_X \int_0^{\varphi(s/\lambda)} \psi(y)\,dy\,d\gamma \\
&= \lambda \int_0^\infty \int_{\{\lambda\psi(y)\leq s\}} d\gamma\,\psi(y)\,dy \quad \text{(by Fubini's theorem)} \\
&\geq \lambda \int_{y_0}^\infty \int_{\{\lambda\psi(y)\leq s\}} d\gamma\,\psi(y)\,dy \\
&= \int_{y_0}^\infty \gamma\big(\lambda\,\psi(y) \leq s\big)\lambda\psi(y)\,dy \\
&\geq c\int_{y_0}^\infty \int_{\{r>\lambda\psi(y)\}} r\,d\gamma\,dy \quad \text{(by the assumption (8))} \\
&= c\int_0^\infty \int_{\{r>\lambda\psi(y)\}} r\,d\gamma\,dy - c\int_0^{y_0} \int_{\{r>\lambda\psi(y)\}} r\,d\gamma\,dy.
\end{aligned}$$

By Fubini's theorem, the first term is equal to

$$
\begin{aligned}
c\int_X\int_0^{\varphi(r/\lambda)} dy\, r\, d\gamma &= \lambda c\int \varphi(r/\lambda) r/\lambda\, d\gamma \\
&\qquad \text{(applying the case of equality in Young's inequality)} \\
&= \lambda c\int \Phi(r/\lambda)\, d\gamma + \lambda c\int \Psi\big(\varphi(r/\lambda)\big)\, d\gamma \\
&= \lambda c M_\Phi(r/\lambda) + \lambda c M_\xi(r/\lambda).
\end{aligned}
$$

The case of general r and s follows by approximating r and s with functions of the above special form, as in [3]. ∎

REMARK. If (8) holds for all $\lambda > 0$, then one can take $y_0 = 0$ and $I = 0$. This case was proved in [3].

Let γ be the measure $g \cdot \mu$, i.e. for $A \in \mathcal{F}$, let $\gamma(A) = \int_A g\, d\mu$.

We are now ready to state our main result.

THEOREM 3. *Let Φ be an Orlicz function with $\Phi(u)/u \longrightarrow \infty$ as $u \longrightarrow \infty$, and assume that ξ is also an Orlicz function. Let τ be a non-singular, conservative and ergodic transformation. Assume that $f \in L_1, f \geq 0$, $g > 0$. Then:*

(*a*) $s(f,g) + s(Tf,Tg) \in L_\xi(\gamma)$ *implies* $f/g \in L_\Phi(\gamma)$;

(*b*) $s(f,g) + s(Tf,Tg) \in H_\xi(\gamma)$ *implies* $f/g \in H_\Phi(\gamma)$.

PROOF. Theorem 1 implies that assumption (8) holds for $r = f/g$ and $s = s(f,g) + s(Tf,Tg)$ with $c = 1$. Note that for $\lambda > 0$,

$$
\begin{aligned}
I &\leq (c/\lambda)\int_0^{y_0}\int r\, d\gamma\, dy \\
&= (y_0 c/\lambda)\int r\, d\gamma = (y_0 c/\lambda)\int f\, d\mu < \infty.
\end{aligned}
$$

Hence, by (9), if there is $\lambda > 0$ with $M_\xi(s/\lambda) < \infty$ (i.e. $s \in L_\xi$), then $M_\Phi(r/\lambda) < \infty$, therefore $r \in L_\Phi$. Similarly, if $M_\xi(s/\lambda) < \infty$ for all $\lambda > 0$ (i.e. $s \in H_\xi$), then $M_\Phi(r/\lambda) < \infty$ for all $\lambda > 0$, hence $r \in H_\Phi$. ∎

Several statements follow from Theorem 3 by taking particular cases of Orlicz functions. Let $1 < p < \infty$; by considering $\Phi(u) = (1/p)u^p$, we get

COROLLARY 1. *If $s(f,g) + s(Tf,Tg) \in L_p(\gamma)$ then $f/g \in L_p(\gamma)$.*

COROLLARY 2. *If $s(f,g)+s(Tf,Tg) \in L\log^{k-1}L(\gamma)$ then $f/g \in L\log^k L(\gamma)$. If $s(f,g)+s(Tf,Tg) \in H_{\Phi_{k-1}}$ then $f/g \in H_{\Phi_k}$.*

PROOF. Let $k \geq 1$. Recall that the Orlicz space $L\log^k L(\gamma)$ is defined by the Orlicz function $\Phi_k(u) = u(\log^+ u)^k$. Then $\xi(u) = k\Phi_{k-1}(u)$. ∎

The spaces H_{Φ_k} are Fava's spaces R_k (Fava [4]); see also Frangos and Sucheston [5].

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The Ohio State University
Columbus, Ohio
USA