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LINES ON HOLOMORPHIC CONTACT MANIFOLDS AND A GENERALIZATION OF (2,3,5)-DISTRIBUTIONS TO HIGHER DIMENSIONS

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Abstract. Since the celebrated work by Cartan, distributions with small growth vector $(2,3,5)$ have been studied extensively. In the holomorphic setting, there is a natural correspondence between holomorphic $(2,3,5)$-distributions and nondegenerate lines on holomorphic contact manifolds of dimension 5. We generalize this correspondence to higher dimensions by studying nondegenerate lines on holomorphic contact manifolds and the corresponding class of distributions of small growth vector $(2m,3m,3m+2)$ for any positive integer m.

§1. Introduction

We work in the holomorphic setting. All manifolds and maps are holomorphic, unless stated otherwise. Open subsets refer to Euclidean topology. A Zariski-open subset of a complex manifold is the complement of a closed analytic subset. We use the following terminology on distributions.

DEFINITION 1.1. A *distribution* on a complex manifold M is a vector subbundle $D \subset TM$ of the tangent bundle of M.

(i) Lie brackets of local sections of D give a homomorphism

$$\mathrm{Levi}^D : \wedge^2 D \to TM/D,$$

called the *Levi tensor* of D.

(ii) There is a sequence of vector bundles, called the *weak derived system* of D,

$$D = \partial^{(0)} D \subset \partial D = \partial^{(1)} D \subset \partial^{(2)} D \subset \cdots \subset \partial^{(d)} D = \partial^{(d+1)} D$$

defined on a Zariski-open subset of M such that their associated sheaves satisfy

$$\mathcal{O}(\partial^{(i+1)} D) = [\mathcal{O}(\partial^{(i)} D), \mathcal{O}(D)] + \mathcal{O}(\partial^{(i)} D)$$

for each $0 \leq i \leq d+1$. We say that D is *regular* at $x \in M$, if the weak derived system of D is a sequence of vector bundles in a neighborhood of x.

(iii) The sequence of integers

$$(\mathrm{rank}(D), \mathrm{rank}(\partial D), \mathrm{rank}(\partial^{(2)} D), \ldots, \mathrm{rank}(\partial^{(d)} D))$$

is called the *small growth vector* of D.

(iv) When D is regular at $x \in M$, the graded vector space

$$\mathrm{symb}_x(D) := \oplus_{i=1}^{d} (\partial^{(i)} D)_x / (\partial^{(i-1)} D)_x$$

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has a natural structure of a nilpotent graded Lie algebra induced by Lie brackets of local sections. It is called the *symbol algebra* of D at x.

In the celebrated paper [C], Cartan studied distributions on five-dimensional manifolds with the small growth vector $(2,3,5)$, commonly called $(2,3,5)$-distributions. Cartan investigated the local equivalence problem for $(2,3,5)$-distributions and many mathematicians have developed this study further (see the references in [IKTY]). A remarkable development in 1990s was the theory of abnormal extremals of $(2,3,5)$-distributions, originated from geometric control theory (see [BH] and [Z] and the references therein). It associates a certain contact manifold of dimension 5 to each $(2,3,5)$-distribution. By reinterpreting this result from the viewpoint of complex geometry, the following one-to-one correspondence has been discovered in Theorems 5.10 and 5.12 of [HL], where *small growth vector* is abbreviated to s.g.v.:

$$\left\{\begin{array}{c}\text{distributions of}\\ \text{s.g.v.}(2,3,5)\end{array}\right\} \overset{(\mathsf{Corr.}1)}{\Longleftrightarrow} \left\{\begin{array}{c}\text{nondegenerate lines on contact}\\ \text{manifolds of dimension } 5\end{array}\right\}.$$

See Definition 3.8 for the precise meaning of the right-hand side. Note that nondegenerate lines are called "contact unbendable rational curves of Cartan type" in Definition 5.8 of [HL]. A local version of this correspondence, where the right-hand side is replaced by Lagrangian cone structures on contact manifolds of dimension 5 satisfying certain conditions, is given in Theorem 3.1 of [IKTY]. From the viewpoint of [HL], the Lagrangian cone structure is a local description of the VMRT of the nondegenerate lines (see Lemma 3.3).

In the current paper, we generalize this correspondence to higher dimensions as the following one-to-one correspondence:

$$\left\{\begin{array}{c}\text{some distributions of}\\ \text{s.g.v.}(2m,3m,3m+2)\end{array}\right\} \overset{(\mathsf{Corr.}m)}{\Longleftrightarrow} \left\{\begin{array}{c}\text{nondegenerate lines on contact}\\ \text{manifolds of dimension } 2m+3\end{array}\right\}.$$

The major difference from the case $m=1$ is that the distributions on the left-hand side are not determined by the small growth vector alone: their symbol algebras must be of the form $\mathfrak{g}_+(F)$ described in Definition 3.14, where F is a nondegenerate cubic form on a vector space of dimension m. For simplicity, we call distributions on the left-hand side $\mathfrak{g}_+$-distributions. Our main results are Theorems 3.15 and 4.4, which give a precise statement of the correspondence ($\mathsf{Corr.}$ m). When $m=1$, the Lie algebra $\mathfrak{g}_+(F)$ is determined by the small growth vector because all nondegenerate cubic forms on a vector space of dimension 1 are isomorphic. Thus, $\mathfrak{g}_+$-distributions in dimension 5 are just $(2,3,5)$-distributions. There is more than one isomorphism type of nondegenerate cubic forms when $m \geq 2$. So when $m \geq 2$, the small growth vector alone cannot determine the type of our distributions. As a matter of fact, when $m \geq 3$, there are nontrivial moduli of nondegenerate cubic forms and the isomorphism types of symbol algebras of a $\mathfrak{g}_+$-distribution may vary from point to point. A classical example of ($\mathsf{Corr.}$ m) is the following.

EXAMPLE 1.2. Let $\mathfrak{g}$ be a simple Lie algebra, and let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Let $X^{\mathfrak{g}}$ be the *adjoint variety* of $\mathfrak{g}$, namely, the highest weight orbit of the coadjoint representation on $\mathbb{P}\mathfrak{g}^\vee$, and let $2m+3$ be the dimension of $X^{\mathfrak{g}}$. The variety $X^{\mathfrak{g}}$ has a natural G-invariant contact structure and is covered by nondegenerate lines. The space of lines on $X^{\mathfrak{g}} \subset \mathbb{P}\mathfrak{g}^\vee$ is a rational homogeneous space $Y^{\mathfrak{g}} = G/P$ for some parabolic subgroup $P \subset G$. If $\mathfrak{g}$ is not of type A or C, there exists a G-invariant distribution $D^{\mathfrak{g}} \subset TY^{\mathfrak{g}}$

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of rank $2m$ whose symbol algebras are isomorphic to $\mathfrak{g}_+(F)$ for a homaloidal EKP cubic F (using the terminology in Theorem 3 of [D]), which is the cubic form whose associated cubic hypersurface is either

(i) the union of a hyperplane and a quadratic cone with one isolated singular point outside the hyperplane, or
(ii) the secant variety of one of the four Severi varieties.

The correspondence $(\mathsf{Corr}.m)$ in this case is

$$\begin{array}{c}\text{distribution } D^{\mathfrak{g}} \text{ on } Y^{\mathfrak{g}} \text{ of} \\ \text{s.g.v.}(2m, 3m, 3m+2)\end{array} \iff \begin{array}{c}\text{lines on the contact manifold} \\ X^{\mathfrak{g}} \text{ of dimension } 2m+3.\end{array}$$

One interesting consequence of $(\mathsf{Corr}.m)$ is that there are many examples of holomorphic contact manifolds covered by nondegenerate lines, which is already nontrivial when $m=1$. Another interesting aspect of $(\mathsf{Corr}.m)$ is its potential application in Riemannian geometry. A well-known conjecture in complex geometry, which is equivalent to the LeBrun–Salamon conjecture (page 110 of [LS]) on quaternionic-Kähler manifolds, is that a Fano contact manifold whose automorphism group is reductive is biholomorphic to an adjoint variety in Example 1.2. There is an approach to this conjecture using lines on Fano contact manifolds of Picard number 1 (see, e.g., [BKK], [K1], [K2]). The lines in this case are expected to be nondegenerate (see Remark 3.9) and we have the associated $\mathfrak{g}_+$-distributions via $(\mathsf{Corr}.m)$. We believe that it is important to understand the geometry of these distributions for this approach to the LeBrun–Salamon conjecture. Note that, if the cubic forms are homaloidal EKP cubics, it is already proved that the Fano contact manifold is an adjoint variety (by Main Theorem in §2 of [M]).

Let us discuss briefly the content of the paper and the methods employed. The key ingredients in establishing $(\mathsf{Corr}.m)$ come from deformation theory of rational curves, in particular, the theory of VMRT, some standard results of which are recalled in §2. Section 3 discusses how to go from the right-hand side to the left-hand side of $(\mathsf{Corr}.m)$. The proof of the main result, Theorem 3.15, is much more involved than the proof in the case of $m=1$, because the structure of the symbol algebras of the distributions is more intricate in higher dimensions. Its proof uses some special features of deformation theory of rational curves on contact manifolds. Section 4 discusses how to go from the left-hand side to the right-hand side of $(\mathsf{Corr}.m)$. The proof of the main result, Theorem 4.4, is a generalization of that of Theorem 5.10 in [HL] to higher dimensions and the idea has originated from [Z].

§2. Unbendable rational curves and VMRT

Definition 2.1. Let X be a complex manifold of dimension n, and let Douady(X) be its Douady space parameterizing all compact analytic subspaces of X.

(i) A smooth rational curve $\mathbb{P}^1 \cong C \subset X$ is *unbendable* if its normal bundle N_C is isomorphic to $\mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus(n-1-p)}$ for some nonnegative integer $p \leq n-1$. Consequently,

$$TX|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus(n-1-p)}$$

with $TC \subset TX|_C$ corresponding to $\mathcal{O}(2)$. In this case, we denote by N_C^+ (resp. $TX|_C^+$) the subbundle of N_C (resp. $TX|_C$) corresponding to $\mathcal{O}(1)^{\oplus p}$ (resp. $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p}$).

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(ii) Let $\mathrm{URC}(X)$ be the subset of $\mathrm{Douady}(X)$ parameterizing all unbendable rational curves on X. It is an open subset of $\mathrm{Douady}(X)$ because the vector bundle $N_C \cong \mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus(n-1-p)}$ on $C \cong \mathbb{P}^1$ has no nontrivial deformation. Using the basic deformation theory of rational curves (e.g., Main Theorem of [K3]) and $H^1(C, N_C) = 0$, we see that $\mathrm{URC}(X)$ is an open subset in the smooth locus of $\mathrm{Douady}(X)$.

It is convenient to use the following notion.

Definition 2.2. Let C be a complex manifold, and let $\mathcal{E}$ be a vector bundle on C. For a point $x \in C$, denote by $H^0(C, \mathcal{E} \otimes \mathbf{m}_x)$ the vector space of sections of $\mathcal{E}$ vanishing at x. The homomorphism

$$\mathrm{jet}_x^{\mathcal{E}} : H^0(C, \mathcal{E} \otimes \mathbf{m}_x) \to T_x^\vee C \otimes \mathcal{E}_x$$

is defined by taking the derivative at x of sections vanishing at x.

The following is well known (see, e.g., page 58 of [HM] and Lemma 3.3 of [HL]).

Proposition 2.3. *In Definition 2.1, let Y be a connected open subset of* $\mathrm{URC}(X)$. *Let* $Y \stackrel{\rho}{\leftarrow} Z \stackrel{\mu}{\rightarrow} X$ *be the associated universal family morphisms. Define the following distributions on Z:*

$$\begin{aligned} \mathcal{V} &:= \mathrm{Ker}(\mathrm{d}\mu), \\ \mathcal{F} &:= \mathrm{Ker}(\mathrm{d}\rho), \\ \mathcal{T}^0 &= \mathcal{V} \oplus \mathcal{F}. \end{aligned}$$

For the point $y = [C] \in Y$ corresponding to an unbendable rational curve $C \subset X$ and a point $z \in \rho^{-1}(y) \subset Z$, set $\mu(z) = x \in C$.

(i) *We have the following natural identifications:*

$$\begin{aligned} T_y Y &= H^0(C, N_C), \\ T_z Z &= H^0(C, TX|_C)/H^0(C, TC \otimes \mathbf{m}_x), \\ \mathcal{V}_z &= H^0(C, N_C^+ \otimes \mathbf{m}_x), \\ \mathcal{F}_z &= H^0(C, TC)/H^0(C, TC \otimes \mathbf{m}_x) = T_x C, \end{aligned}$$

where the third (resp. fourth) identification is induced by the differential $\mathrm{d}_z\rho : T_z Z \to T_y Y$ (resp. $\mathrm{d}_z\mu : T_z Z \to T_x X$).

(ii) *Define $\mathcal{T}^1 := \partial \mathcal{T}^0$. It is a vector subbundle of TZ and the Lie brackets of sections induce a natural isomorphism of vector bundles $\psi : \mathcal{F} \otimes \mathcal{V} \to \mathcal{T}^1/\mathcal{T}^0$.*

(iii) *The differential $\mathrm{d}_z\mu$ sends $\mathcal{T}_z^1/\mathcal{T}_z^0$ to $N_{C,x}^+ \subset T_x X/T_x C$ such that in combination with (i) and (ii), we have the commutative diagram*

$$\begin{array}{ccc} \mathcal{F}_z \otimes \mathcal{V}_z & = & T_x C \otimes H^0(C, N_C^+ \otimes \mathbf{m}_x) \\ \psi \downarrow & & \downarrow \mathbf{j}_x \\ \mathcal{T}_z^1/\mathcal{T}_z^0 & \xrightarrow{\mathrm{d}_z\mu} & N_{C,x}^+, \end{array}$$

where $\mathbf{j}_x$ is the contraction of vectors in $T_x C$ with the image of

$$\mathrm{jet}_x^{N_C^+} : H^0(C, N_C^+ \otimes \mathbf{m}_x) \to T_x^\vee C \otimes N_{C,x}^+.$$

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(iv) *We have isomorphisms* $\mathcal{F}|_{\rho^{-1}(y)} = TC \cong \mathcal{O}(2)$ *and*

$$\mathcal{V}|_{\rho^{-1}(y)} = \mathrm{Hom}(TC, N_C^+) \cong \mathcal{O}(-1)^{\oplus m}.$$

Definition 2.4. In Proposition 2.3, the tangent map $\tau : Z \to \mathbb{P}TX$ sending $z \in Z$ to $[\mathrm{d}\mu(\mathcal{F}_z)] \in \mathbb{P}T_xX$ is an immersion whose image $\mathcal{C} \subset \mathbb{P}TX$ is called the *variety of minimal rational tangents* (to be abbreviated as VMRT) of the family Y. The fiber $\mathcal{C}_x \subset \mathbb{P}T_xX$ at $x \in \mu(Z)$ is called the *VMRT* at x. Often, we replace Y by a suitable connected open subset to assume that $\mathcal{C} \subset \mathbb{P}TX$ and $\mathcal{C}_x$ are submanifolds in $\mathbb{P}TX$.

Notation 2.5. Let W be a vector space, and let $S \subset \mathbb{P}W$ be a (not necessarily closed) submanifold. For a point $s \in S$, we denote by $\widehat{s} \subset \widehat{S}$ the corresponding affine cones in W. Let N_S be the normal bundle of S in $\mathbb{P}W$. The second fundamental form $\mathrm{II}_S : \mathrm{Sym}^2 TS \to N_S$ is a homomorphism of vector bundles. Let $\mathrm{Dom}(\mathrm{III}_S) \subset S$ be the Zariski-open subset where the image of II_S is a vector subbundle $N_S^{(2)}$ of N_S. Then we have the third fundamental form

$$\mathrm{III}_{S,s} : \mathrm{Sym}^3 T_sS \to N_{S,s}/N_{S,s}^{(2)}$$

for each $s \in \mathrm{Dom}(\mathrm{III}_S)$. Denoting by $\widehat{T}_sS \subset W$ the affine tangent space of S at s, we write $\widehat{N}_{S,s}$ for $W/\widehat{T}_sS$ and $\widehat{N}_{S,s}^{(2)}$ for the vector subspace of $\widehat{N}_{S,s}$ such that we have natural identifications

$$N_{S,s} = \widehat{s}^\vee \otimes \widehat{N}_{S,s} \text{ and } N_{S,s}^{(2)} = \widehat{s}^\vee \otimes \widehat{N}_{S,s}^{(2)}.$$

The following is well known (see, e.g., the proof of Proposition 1.4 in [H1] and Corollary 3.14 and Proposition 3.16 in [HL])

Proposition 2.6. *In Definitions 2.1 and 2.4, replace* Y *by a neighborhood of* y *to assume that* $\mathcal{C}$ *and* $\mathcal{C}_x$ *are submanifolds of* $\mathbb{P}TX$ *and the immersion* τ *in Definition 2.4 is an embedding. Let us identify* Z *with* $\mathcal{C} \subset \mathbb{P}TX$ *by the embedding* τ.

(i) *The affine tangent space* $\widehat{T}_z\mathcal{C}_x \subset T_xX$ *satisfies*

$$\widehat{T}_z\mathcal{C}_x = \mathrm{d}_z\mu(\mathcal{T}_z^1),$$
$$\widehat{T}_z\mathcal{C}_x/T_xC = N_{C,x}^+.$$

(ii) *The natural isomorphism*

$$\mathrm{d}_z(\tau|_{\mu^{-1}(x)}) : \mathcal{V}_z = H^0(C, N_C^+ \otimes \mathbf{m}_x) \ \to \ T_z\mathcal{C}_x = T_x^\vee C \otimes N_{C,x}^+$$

coincides with $\mathrm{jet}_x^{N_C^+}$.

(iii) *For a local section* $\vec{f}$ *of* $\mathcal{F}$ *and local sections* $\vec{v}_1, \vec{v}_2$ *of* $\mathcal{V}$ *near* $z \in \mathcal{C}$ *with values* $v_1, v_2 \in \mathcal{V}_z$ *at* z,

$$([\vec{v}_2, [\vec{v}_1, \vec{f}]]_z \mod \mathcal{T}_z^1) \ = \ \vec{f}_z \otimes \mathrm{II}_{\mathcal{C}_x,z}(v_1, v_2),$$

where the left-hand side is regarded as an element of $\widehat{N}_{\mathcal{C}_x,z} = N_{C,x}/N_{C,x}^+$ *by (i) and Proposition 2.3(iii), and* $\vec{f}_z \otimes$ *stands for the contraction of* $\vec{f}_z \in \mathcal{F}_z = T_xC$ *with*

$$\mathrm{II}_{\mathcal{C}_x,z}(v_1, v_2) \in T_x^\vee C \otimes \widehat{N}_{\mathcal{C}_x,z}^{(2)} \subset T_x^\vee C \otimes \widehat{N}_{\mathcal{C}_x,z}.$$

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(iv) *Assume that* $z \in \mathcal{C}_x$ *is in* $\mathrm{Dom}(\mathrm{III}_{\mathcal{C}_x})$*. Then* $\mathcal{T}^2 = \partial\mathcal{T}^1 = \partial^{(2)}\mathcal{T}^0$ *is a vector subbundle of* TZ *in a neighborhood of* $\rho^{-1}(y)$ *in* Z *and*

$$\widehat{N}^{(2)}_{\mathcal{C}_x,z} = (\mathrm{d}_z\mu(\mathcal{T}^2_z) \mod \widehat{T}_z\mathcal{C}_x)$$

as subspaces of $\widehat{N}_{\mathcal{C}_x,z} = N_{C,x}/N^+_{C,x}$.

(v) *In (iv), for a local section* $\vec{f}$ *of* $\mathcal{F}$ *and local sections* $\vec{v}_1, \vec{v}_2, \vec{v}_3$ *of* $\mathcal{V}$ *near* $z \in \mathcal{C}$ *with values* $v_1, v_2, v_3 \in \mathcal{V}_z$ *at* z*,*

$$([\vec{v}_3, [\vec{v}_2, [\vec{v}_1, \vec{f}]]]_z \mod \mathcal{T}^2_z) = \vec{f}_z \otimes \mathrm{III}_{\mathcal{C}_x,z}(v_1, v_2, v_3),$$

where the left hand-side is regarded as an element of $\widehat{N}_{\mathcal{C}_x,z}/\widehat{N}^{(2)}_{\mathcal{C}_x,z}$ *by (iv), and* $\vec{f}_z\otimes$ *stands for the contraction of* $\vec{f}_z \in \mathcal{F}_z = T_xC$ *with*

$$\mathrm{III}_{\mathcal{C}_x,z}(v_1, v_2, v_3) \in T^\vee_x C \otimes \widehat{N}_{\mathcal{C}_x,z}/\widehat{N}^{(2)}_{\mathcal{C}_x,z}.$$

DEFINITION 2.7. In Proposition 2.3, define $\mathcal{D}_y := H^0(C, N^+_C) \subset T_yY$. This determines a distribution $\mathcal{D} \subset TY$ of rank $2m$. For each $x \in C$, define

$$U_x := H^0(C, \mathcal{O}(x)),$$

the two-dimensional vector space of rational functions on C with at most one pole at x and define

$$V_x := H^0(C, N^+_C \otimes \mathbf{m}_x) \subset \mathcal{D}_y,$$

such that we have a canonical tensor decomposition $\mathcal{D}_y = U_x \otimes V_x$.

PROPOSITION 2.8. *In Definition 2.7, let* $\mathbf{1}_x \in U_x$ *be the constant function on* C *with value 1. The differential* $\mathrm{d}_z\rho : T_zZ \to T_yY$ *induces an identification* $\mathcal{T}^1_z/\mathcal{F}_z = \mathcal{D}_y$ *such that*

$$\mathcal{V}_z = \mathbf{1}_x \otimes V_x \ \subset \ U_x \otimes V_x = \mathcal{D}_y.$$

Moreover, when $z \in \mathrm{Dom}(\mathrm{III}_{\mathcal{C}_x})$*, it induces an identification* $\mathcal{T}^2_z/\mathcal{F}_z = (\partial\mathcal{D})_y$.

Proof. The identification $\mathcal{T}^1_z/\mathcal{F}_z = \mathcal{D}_y$ follows from Propositions 1 and 8 of [HM] (see Proposition 3.7 of [HL]). Then $\mathcal{V}_z = \mathbf{1}_x \otimes V_x$ follows from Proposition 2.3(i) and the identification $\mathcal{T}^2_z/\mathcal{F}_z = (\partial\mathcal{D})_y$ comes from $\mathcal{T}^2 = \partial\mathcal{T}^1$ in Proposition 2.6(iv). ∎

We skip the proof of the following two elementary lemmata.

LEMMA 2.9. *In Proposition 2.8, let* $f \in U_x$ *be a nonconstant rational function on* C*.*

(i) *The homomorphism*

$$\mathbf{res}_x : \wedge^2 H^0(C, \mathcal{O}(x)) \to T_xC$$

that sends $\mathbf{1}_x \wedge f$ *to*

$$\mathrm{Res}_x(f\ \mathrm{d}t)\frac{\partial}{\partial t},$$

where t *is a local holomorphic coordinate on* C *centered at* x *and* $\mathrm{Res}_x(f\ \mathrm{d}t)$ *is the residue of the logarithmic form* $f\ \mathrm{d}t$ *at* x*, is independent of the choice of the coordinate* t *and gives a canonical identification of* $\wedge^2 U_x$ *with* T_xC*.*

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(ii) *Write* $\vec{f}_x := \mathbf{res}_x(\mathbf{1}_x \wedge f) \in T_x C$. *For any vector bundle* $\mathcal{E}$ *on* C *and a section* $v \in H^0(C, \mathcal{E} \otimes \mathbf{m}_x)$, *let* $\vec{f}_x(v) \in \mathcal{E}_x$ *be the contraction of* $\vec{f}_x$ *with* $\mathrm{jet}_x^{\mathcal{E}}(v) \in T_x^\vee C \otimes \mathcal{E}_x$. *Then*

$$\vec{f}_x(v) = (f \otimes v)(x) \ \in \mathcal{E}_x,$$

where $f \otimes v$ *stands for the section of* $H^0(C, \mathcal{E})$ *under the natural homomorphism*

$$U_x \otimes H^0(C, \mathcal{E} \otimes \mathbf{m}_x) \to H^0(C, \mathcal{E}).$$

LEMMA 2.10. *In Lemma 2.9, for any point* $x' \in C$ *different from* x, *define*

$$A_x^{x'} := \{f \in U_x \mid f(x') = 0\} \subset U_x.$$

Then we have canonical isomorphisms

$$A_x^{x'} \otimes U_{x'} = U_x \text{ and } A_x^{x'} \otimes V_x = V_{x'}$$

coming from multiplication by rational functions. Moreover, for any $f \in A_x^{x'}$ *and* $v \in V_x$, *we have* $f \otimes v = \mathbf{1}_{x'} \otimes fv$ *under the two tensor decompositions* $\mathcal{D}_y = U_x \otimes V_x = U_{x'} \otimes V_{x'}$.

PROPOSITION 2.11. *In Definition 2.7, for a local section* $\vec{f}$ *of* $\mathcal{F}$ *and* $\vec{v}$ *of* $\mathcal{V}$ *near* $z \in \mathcal{C}$, *denote by* $f \in U_x$ *a rational function satisfying* $\mathrm{d}_z\mu(\vec{f}_z) = \mathbf{res}_x(\mathbf{1}_x \wedge f)$ *in Lemma 2.9 and by* $v \in \mathcal{V}_z = V_x$ *the value of* $\vec{v}$ *at* z. *Then*

$$\mathrm{d}_z\rho([\vec{v}, \vec{f}]_z) \ \equiv \ f \otimes v \mod \mathbf{1}_x \otimes V_x (\subset U_x \otimes V_x = \mathcal{D}_y). \tag{2.1}$$

Proof. The vector $[\vec{v}, \vec{f}]_z \in T_z Z$ is represented by a section $w \in H^0(C, TX)$ by Proposition 2.3(i). Regarding v as an element of $H^0(C, N_C^+ \otimes \mathbf{m}_x)$, Proposition 2.3(iii) says that

$$(\mathrm{d}_z\mu([\vec{v}, \vec{f}]_z) \mod T_x C) = \vec{f}_z \otimes \mathrm{jet}_x^{N_C^+}(v) = \vec{f}_z(v) \in N_{C,x}^+.$$

This implies that the value w_x at x of the section w of $TX|_C$ satisfies

$$(w_x \mod T_x C) = \vec{f}_z(v).$$

Since $\mathrm{d}_z\rho([\vec{v}, \vec{f}]_z)$ in $T_y Y = H^0(C, N_C)$ is represented by w modulo $H^0(C, TC)$, its value modulo $\mathbf{1}_x \otimes V_x = H^0(C, N_C \otimes \mathbf{m}_x)$ is just

$$(w_x \mod T_x C) \in H^0(C, N_C)/H^0(C, N_C \otimes \mathbf{m}_x) = N_{C,x}.$$

So the left-hand side of (2.1) is just $\vec{f}_z(v)$ in Lemma 2.9(ii). Then Lemma 2.9(ii) says that this is equal to $(f \otimes v)(x)$, the right-hand side of (2.1). ∎

PROPOSITION 2.12. *In Proposition 2.11, for* $u_1, u_2 \in U_x$ *and* $v_1, v_2 \in V_x$, *let* $u_1 \otimes v_1, u_2 \otimes v_2$ *be elements of* $\mathcal{D}_y = U_x \otimes V_x$. *Then we have*

$$\mathrm{Levi}_y^{\mathcal{D}}(u_1 \otimes v_1, u_2 \otimes v_2) = (u_1 \wedge u_2) \otimes \mathrm{II}_{\mathcal{C}_x, z}(v_1, v_2), \tag{2.2}$$

where the second fundamental form $\mathrm{II}_{\mathcal{C}_x, z} : \mathrm{Sym}^2 V_x \to T_x^\vee C \otimes (N_{C,x}/N_{C,x}^+)$ *is interpreted as a homomorphism*

$$\mathrm{Sym}^2 V_x \to \wedge^2 U_x^\vee \otimes (H^0(C, N_C)/H^0(C, N_C^+)) = \wedge^2 U_x^\vee \otimes (T_y Y/\mathcal{D}_y)$$

via the isomorphism $\mathbf{res}_x : T_x C \cong \wedge^2 U_x$ *in Lemma 2.9(i) and the natural isomorphism* $H^0(C, N_C/N_C^+) = N_{C,x}/N_{C,x}^+$ *coming from* $N_C/N_C^+ \cong \mathcal{O}^{\oplus(n-m-1)}$.

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Proof. We may check (2.2) assuming $u_1 = f$ and $u_2 = \mathbf{1}_x$. It is a direct consequence of

$$([\vec{v}_2,[\vec{v}_1,\vec{f}]]_z \mod \mathcal{V}_z) \;=\; \vec{f}_z \otimes \mathrm{II}_{\mathcal{C}_x,z}(v_1,v_2) \tag{2.3}$$

from Proposition 2.6(iii). In fact, the vector $v_2 \in \mathcal{V}_z$ is sent by $\mathrm{d}_z\rho$ to the corresponding element $v_2 \in V_x = H^0(C, N_C^+ \otimes \mathbf{m}_x)$, which is just $\mathbf{1}_x \otimes v_2$ in $U_x \otimes V_x = \mathcal{D}_y$. The vector $[\vec{v}_1,\vec{f}]_z$ is sent to $f \otimes v_1$ modulo $\mathbf{1}_x \otimes V_x$ by Proposition 2.11. Then the left-hand side of (2.3) is equal to $\mathrm{Levi}^{\mathcal{D}}(u_2 \otimes v_2, u_1 \otimes v_1)$. By Lemma 2.9, the isomorphism $\mathbf{res}_x$ identifies $\vec{f}_z$ with $-\mathbf{1}_x \wedge f = -u_1 \wedge u_2$. Thus, the right-hand side of (2.3) is

$$-(u_1 \wedge u_2) \otimes \mathrm{II}_{\mathcal{C}_x,z}(v_1,v_2).$$

This proves (2.2). ∎

PROPOSITION 2.13. *In Proposition 2.11, assume that $\mathcal{D}$ is regular at $y \in Y$. Then for any $f \in U_x, v_1, v_2, v_3 \in V_x$, the Lie bracket in* $\mathrm{symb}_y(\mathcal{D})$ *satisfies*

$$[\mathbf{1}_x \otimes v_3, [\mathbf{1}_x \otimes v_2, f \otimes v_1]] = (\mathbf{1}_x \wedge f) \otimes \mathrm{III}_{\mathcal{C}_x,z}(v_1,v_2,v_3),$$

where $\mathrm{III}_{\mathcal{C}_x,z} : \mathrm{Sym}^3 V_x \to T_x^\vee C \otimes (\widehat{N}_{\mathcal{C}_x,z}/\widehat{N}^{(2)}_{\mathcal{C}_x,z})$ *is interpreted as a homomorphism*

$$\mathrm{Sym}^3 V_x \to \wedge^2 U_x^\vee \otimes (T_yY/(\partial\mathcal{D})_y),$$

via the isomorphism $\mathbf{res}_x : T_xC \cong \wedge^2 U_x$ *and the isomorphisms*

$$\widehat{N}_{\mathcal{C}_x,z}/\widehat{N}^{(2)}_{\mathcal{C}_x,z} \overset{\mathrm{d}_z\mu}{=} T_z z/\mathcal{T}_z^2 \overset{\mathrm{d}_z\rho}{=} T_yY/(\partial\mathcal{D})_y.$$

Proof. The proof follows the same argument as the proof of Proposition 2.12, using Proposition 2.6(v) in place of Proposition 2.6(iii). ∎

§3. From nondegenerate lines on contact manifolds to distributions with symbols $\mathfrak{g}_+(F)$

DEFINITION 3.1. Let X be a complex manifold of dimension $2m+3, m \geq 1$, and let $\mathcal{H} \subset TX$ be a contact distribution, namely, a distribution of rank $2m+2$ such that the Levi tensor $\mathrm{Levi}^{\mathcal{H}} : \wedge^2\mathcal{H} \to TX/\mathcal{H}$ gives a symplectic form, that is, a nondegenerate anti-symmetric form, on each fiber of $\mathcal{H}_x$. The pair $(X,\mathcal{H})$ is called a *contact manifold*. The quotient line bundle $\mathcal{L} := TX/\mathcal{H}$ is called the *contact line bundle* on X. An unbendable smooth rational curve $C \subset X$ satisfying $\mathcal{L}|_C \cong \mathcal{O}(1)$ is called a *line*. Denote by $\mathrm{Lines}(X,\mathcal{H})$, the open subset of $\mathrm{URC}(X) \subset \mathrm{Douady}(X)$ parameterizing lines.

DEFINITION 3.2. Fix a one-dimensional vector space $\mathbf{L}$, and let $\omega : \wedge^2 W \to \mathbf{L}$ be a symplectic form on a vector space W of dimension $2m+2$.

(i) For a subspace $B \subset W$, define $B^\perp := \{w \in W \mid \omega(w,B) = 0\}$.
(ii) A subspace $B \subset W$ is *Lagrangian* if $B^\perp = B$. In this case, the dimension of B is $m+1$.
(iii) A submanifold $S \subset \mathbb{P}W$ is *Legendrian* if its affine tangent space $\widehat{T}_sS \subset W$ at every point $s \in S$ is a Lagrangian subspace of W.

LEMMA 3.3. *In Definition 3.1, choose a connected open subset $Y \subset \mathrm{Lines}(X,\mathcal{H})$ such that $\mathcal{C} \subset \mathbb{P}TX$ in Definition 2.4 is a submanifold.*

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(i) *For each line $C \subset X$, the normal bundle N_C is isomorphic to $\mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus(m+2)}$ and $\mathcal{H}|_C$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus m} \oplus \mathcal{O}(-1)$. In other words, the integer p in Definition 2.1 is exactly $m = \frac{1}{2}(\dim X - 3)$.*
(ii) *For each $x \in C$, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_xX$ is contained in $\mathbb{P}\mathcal{H}_x$ and is Legendrian with respect to the symplectic form $\mathrm{Levi}_x^{\mathcal{H}}$ on $\mathcal{H}_x$.*

Proof. From the well-known relation $\det TX = \mathcal{L}^{\otimes(m+2)}$ (e.g., from (2.2) of [LB]), we have the isomorphism $N_C \cong \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus(m+2)}$. Consequently,

$$TX|_C \cong TC \oplus N_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus(m+2)}.$$

Since $TC \cong \mathcal{O}(2)$, it must belong to the kernel of the projection $TX|_C \to (\mathcal{L} = TX/\mathcal{H})|_C \cong \mathcal{O}(1)$. Since this holds for all lines represented by elements in the parameter space Y, we have $\mathcal{C} \subset \mathbb{P}\mathcal{H}$. By Proposition 2.6(i), $\widehat{T}_z\mathcal{C}_x/T_xC = N^+_{C,x}$, where $x \in C$ and $z = \mathbb{P}T_xC \in \mathcal{C}_x$. It follows that $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m}$ is contained in $\mathcal{H}|_C$. This gives $\mathcal{H}|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus m} \oplus \mathcal{O}(-1)$, completing the proof of (i). By (i), any line is tangent to $\mathcal{H}$, and $\mathcal{C}_x \subset \mathbb{P}\mathcal{H}_x$. Moreover, the Levi tensor gives a family of nondegenerate antisymmetric forms on the bundle $\mathcal{H}$, that is, a homomorphism

$$\wedge^2\mathcal{H}_C \cong \wedge^2(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus m} \oplus \mathcal{O}(-1)) \to \mathcal{L}|_C \cong \mathcal{O}(1).$$

The subspace $(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m})_x$ is a Lagrangian subspace of $\mathcal{H}_x$ with respect to this antisymmetric form. As this subspace corresponds to the affine tangent space of $\mathcal{C}_x$ at $\mathbb{P}T_xC$ by Proposition 2.6(i), the VMRT must be Legendrian, proving (ii). ∎

Definition 3.4. In Definition 3.1, let $Y \subset \mathrm{Lines}(X, \mathcal{H})$ be a connected open subset, and let $Y \xleftarrow{\rho} Z \xrightarrow{\mu} X$ be the universal family. Given $z \in Z$, write $x = \mu(z), y = \rho(z)$ and $C = \mu(\rho^{-1}(y)) \subset X$. Using the symplectic form $\mathrm{Levi}_x^{\mathcal{H}}$ on $\mathcal{H}_x$, define

$$\mathcal{R}_z := (T_xC)^{\perp} \subset \mathcal{H}_x \text{ and } \mathcal{R}_z^+ := \widehat{T}_z\mathcal{C}_x \subset \mathcal{R}_z \subset \mathcal{H}_x.$$

Then $\mathcal{R}^+ \subset \mathcal{R}$ are vector subbundles of the vector bundle $\mathcal{H}$ on Z, of rank $m+1$ and $2m+1$, respectively. Denote by $\mathcal{R}_C^+ \subset \mathcal{R}_C \subset \mathcal{H}|_C$ the corresponding vector subbundles on C.

Lemma 3.5. *In Definition 3.4, we have the following.*

(i) *The vector bundle $\mathcal{R}_C$ (resp. $\mathcal{R}_C^+$) is isomorphic to the subbundle*

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus m} \ (\ \textit{resp.}\ \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m})$$

of $\mathcal{H}|_C$ from Lemma 3.3(i).
(ii) *If $s \in H^0(C, TX|_C)$ satisfies $s_x \in \mathcal{R}_{C,x}$ for a point $x \in C$, then $s \in H^0(C, \mathcal{H}|_C)$.*
(iii) $\mathrm{Levi}^{\mathcal{H}}$ *induces a perfect pairing of vector bundles on C*

$$N_C^+ \otimes \mathcal{R}_C/(TX|_C)^+ \to \mathcal{L}|_C.$$

Proof. Note that $\mathrm{Levi}^{\mathcal{H}}$ gives a perfect paring

$$TC \otimes (\mathcal{H}|_C/\mathcal{R}_C) \cong \mathcal{L}|_C \cong \mathcal{O}(1).$$

This implies (i). From (i), the quotient bundle $TX|_C/\mathcal{R}_C$ is isomorphic to $\mathcal{O}^{\oplus 2}$. The assumption $s_x \in \mathcal{R}_{C,x}$ implies that s modulo $\mathcal{R}_C$ defines a section of $\mathcal{O}^{\oplus 2}$ vanishing at x. Hence, it vanishes identically. This shows that $s \in H^0(C, \mathcal{H}) = H^0(C, \mathcal{R})$, proving (ii). (iii) is immediate from (i). ∎

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DEFINITION 3.6. A cubic form $F \in \mathrm{Sym}^3 V^\vee$ on a vector space V is *nondegenerate* if

$$\{v \in V \mid F(v, v_1, v_2) = 0 \text{ for all } v_1, v_2 \in V\} = 0.$$

DEFINITION 3.7. In Definition 3.2, let $S \subset \mathbb{P}W$ be a Legendrian submanifold. We say that S has *nondegenerate fundamental forms* at $s \in S$ if the following are satisfied.

(i) The point s is contained in $\mathrm{Dom}(\mathrm{III}_S)$.
(ii) The image of the second fundamental form $\widehat{N}^{(2)}_{S,s}$ is equal to $\widehat{s}^\perp/\widehat{T}_sS$.
(iii) The third fundamental form $\mathrm{III}_{S,s} : \mathrm{Sym}^3 T_sS \to \widehat{s}^\vee \otimes W/\widehat{s}^\perp$ is a nondegenerate cubic form on T_sS in the sense of Definition 3.6.
(iv) For any $v_1, v_2 \in T_sS$, the element

$$\mathrm{III}_{S,s}(v_1, v_2, \cdot) \in T_s^\vee S \otimes \widehat{s}^\vee \otimes W/\widehat{s}^\perp$$

coincides with

$$\mathrm{II}_{S,s}(v_1, v_2) \in \widehat{s}^\vee \otimes \widehat{N}^{(2)}_{S,s} = \widehat{s}^\vee \otimes \widehat{s}^\perp/\widehat{T}_sS$$

via the natural isomorphisms

$$\widehat{s}^\perp/\widehat{T}_sS = (\widehat{T}_sS/\widehat{s})^\vee \otimes \mathbf{L} \text{ and } W/\widehat{s}^\perp = \widehat{s}^\vee \otimes \mathbf{L}$$

induced by ω.

DEFINITION 3.8. A line C on a contact manifold $(X, \mathcal{H})$ is a *nondegenerate line*, if for some point $x \in C$, the VMRT $\mathcal{C}_x$ at x has nondegenerate fundamental forms at the point $\mathbb{P}T_xC \in \mathcal{C}_x$. Let $\mathrm{NDL}(X, \mathcal{H})$ be the subset of $\mathrm{Lines}(X, \mathcal{H})$ parameterizing nondegenerate lines. It is a Zariski-open subset in $\mathrm{Lines}(X, \mathcal{H})$.

REMARK 3.9. The conditions in Definition 3.7 may look technical, but actually they hold at general points of a general Legendrian submanifold. For example, §3 of [LM] shows that if the Legendrian submanifold $S \subset \mathbb{P}W$ is a smooth projective variety different from a linear subspace, then S has nondegenerate fundamental forms at a general point $s \in S$. When $(X, \mathcal{H})$ is a Fano contact manifold of Picard number 1, Theorem 1.1 of [K2] implies that the VMRT at a general point is a smooth projective variety. Thus a general line is nondegenerate in the sense of Definition 3.8, unless the VMRT's are linear. It is expected that the latter situation does not occur and [BKK] proposes an approach to exclude this possibility.

PROPOSITION 3.10. *In Definitions 3.8, assume that $Y \subset \mathrm{NDL}(X, \mathcal{H})$. Then the second fundamental forms of the VMRT determine a surjective homomorphism $\mathrm{II} : \mathrm{Sym}^2 \mathcal{V} \to \mathcal{F}^\vee \otimes \mathcal{R}/\mathcal{R}^+$ and the third fundamental forms of the VMRT determine a surjective homomorphism $\mathrm{III} : \mathrm{Sym}^3 \mathcal{V} \to \mathcal{F}^\vee \otimes (\mathcal{H}/\mathcal{R})$. In particular, we have*

$$\mathcal{T}^1_z = (\mathrm{d}_z\mu)^{-1}(\mathcal{R}^+_z) \subset T_zZ \text{ and } \mathcal{T}^2_z = (\mathrm{d}_z\mu)^{-1}(\mathcal{R}_z) \subset T_zZ$$

for any $z \in Z$.

Proof. It is immediate from the definition of a nondegenerate line that the second and the third fundamental forms of VMRT determine homomorphisms II and III, which are surjective at general points of Z. By Proposition 2.3(iv) and $(\mathcal{H}|_C/\mathcal{R}_C) \cong \mathcal{O}(-1)$ from Lemma 3.5(i), the two homomorphisms are surjective at every point of Z. Then $\mathcal{T}^2_z = (\mathrm{d}_z\mu)^{-1}(\mathcal{R}_z)$ follows from Proposition 2.6(iv). ∎

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PROPOSITION 3.11. *Under the hypotheses of Proposition 3.10, the two subspaces* $(\partial\mathcal{D})_y$ *and* $H^0(C,\mathcal{H}|_C/TC)$ *of* $T_yY = H^0(C,N_C)$ *coincide. Consequently, the two quotient spaces* $T_yY/(\partial\mathcal{D})_y$ *and*

$$H^0(C,\mathcal{L}) = H^0(C,N_C)/H^0(C,\mathcal{H}|_C/TC)$$

can be identified in a natural way.

Proof. By Lemma 3.5(ii), the subspace $\mathcal{T}_z^2 = (\mathrm{d}_z\mu)^{-1}\mathcal{R}_z \subset T_zZ$ in Proposition 3.10 can be identified with

$$H^0(C,\mathcal{H})/H^0(C,TC\otimes\mathbf{m}_x) \ \subset\ H^0(C,TX)/H^0(C,TC\otimes\mathbf{m}_x)$$

under the identification in Proposition 2.3(i). Since $\mathrm{d}_z\rho : T_zZ \to T_yY$ is the quotient map

$$H^0(C,TX)/H^0(C,TC\otimes\mathbf{m}_x) \to H^0(C,N_C),$$

the subspace $(\mathrm{d}_z\mu)^{-1}\mathcal{R}$ is sent to $H^0(C,\mathcal{H}|_C/TC) \subset T_yY$. This agrees with $(\partial\mathcal{D})_y$ because $\mathrm{d}_z\rho(\mathcal{T}_z^2) = \partial\mathcal{D}$ by Proposition 2.8. ∎

LEMMA 3.12. *In Proposition 3.11, define*

$$I_x := T_x^\vee C \otimes H^0(C,\mathcal{L}\otimes\mathbf{m}_x) = (T_x^\vee C)^{\otimes 2}\otimes\mathcal{L}_x$$

and write $\mathbf{F}_x : \mathrm{Sym}^3 V_x \to I_x$ *for* $\mathrm{III}_{\mathcal{C}_x,z}$. *For each pair* $x \neq x' \in C$, *recall from Lemma 2.10 the one-dimensional vector space* $A_x^{x'}$ *of rational functions on* C *with pole at* x *and zero at* x'.

(i) *There is a canonical isomorphism between* $(A_x^{x'})^{\otimes 3}\otimes I_x$ *and* $I_{x'}$. *For* $f \in A_x^{x'}$ *and* $j \in I_x$, *denote by* $f^3\cdot j$ *the element of* $I_{x'}$ *corresponding to* $f^3\otimes j \in (A_x^{x'})^{\otimes 3}\otimes I_x$.
(ii) *For any* $v_1,v_2,v_3 \in V_x$ *and* $f \in A_x^{x'}$,

$$f^3\cdot\mathbf{F}_x(v_1,v_2,v_3) = \mathbf{F}_{x'}(f\cdot v_1, f\cdot v_2, f\cdot v_3).$$

Proof. Fix a base point $x_0 \in C$ and define a line bundle $\mathcal{A}$ on C whose fiber at x is $A_x^{x_0}$. Then $\mathcal{A}$ is isomorphic to $\mathcal{O}(1)$. Let $\mathcal{I}$ be the line bundle on C whose fiber at $x \in C$ is I_x. Then $\mathcal{I}$ is isomorphic to $\mathcal{O}(-3)$. Thus we have a canonical trivialization of the line bundle $\mathcal{A}^{\otimes 3}\otimes\mathcal{I} = I_{x_0}\times C$, which shows (i).

Note that $\mathcal{V}|_{\rho^{-1}(y)} \cong \mathcal{O}(-1)^{\oplus m}$ from Proposition 2.3(iv). Thus the collection of the homomorphisms $\mathbf{F}_x$ for all $x \in C$ gives rise to a surjective homomorphism between trivial vector bundles $\mathrm{Sym}^3(\mathcal{V}\otimes\mathcal{A}) \to \mathcal{A}^{\otimes 3}\otimes\mathcal{I}$, which corresponds to $\mathbf{F}_{x_0} : \mathrm{Sym}^3 V_{x_0} \to I_{x_0}$ under the canonical trivialization $\mathcal{A}^{\otimes 3}\otimes\mathcal{I}$. This shows (ii). ∎

PROPOSITION 3.13. *In Proposition 3.11, there are natural identifications:*

(i) $\mathcal{D}_y = U_x\otimes V_x$,
(ii) $(\partial\mathcal{D})_y/\mathcal{D}_y = \wedge^2 U_x\otimes I_x\otimes V_x^\vee$, *and*
(iii) $T_yY/(\partial\mathcal{D})_y = \wedge^2 U_x\otimes I_x\otimes U_x$.

In particular, for any pair $x \neq x' \in C$, *there is a natural identification of* $\wedge^2 U_x\otimes I_x\otimes U_x$ *and* $\wedge^2 U_{x'}\otimes I_{x'}\otimes U_{x'}$ *compatible with the identification* $(\mathcal{A}_x^{x'})^{\otimes 3}\otimes I_x = I_{x'}$ *in Lemma 3.12 and the identification* $\mathcal{A}_{x'}^{x}\otimes U_{x'} = U_x$ *in Lemma 2.10.*

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Proof. We have already seen (i) in Definition 2.7. (ii) is from $(\partial\mathcal{D})_y/\mathcal{D}_y = \mathcal{T}_z^2/\mathcal{T}_z^1$ in Proposition 2.8, the identification $\wedge^2 U_x = T_x C$ from Lemma 2.9, and

$$\mathcal{T}_z^2/\mathcal{T}_z^1 = (N_{C,x}^+)^\vee \otimes \mathcal{L}_x = V_x^\vee \otimes T_x^\vee C \otimes \mathcal{L}_x$$

from Lemma 3.5(iii). (iii) follows from

$$T_y Y/(\partial\mathcal{D})_y = H^0(C,\mathcal{L}) = H^0(C,\mathcal{L}\otimes \mathbf{m}_x)\otimes U_x,$$

where the first equality is by Proposition 3.11. ∎

DEFINITION 3.14. Let $\mathbb{U}$ be a vector space of dimension 2, and let $\mathbb{V}$ be a vector space of dimension $m \geq 1$. Fix a one-dimensional vector space $\mathbb{I}$. Let $\mathfrak{g}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ be the vector space defined by

$$\mathfrak{g}_1 := \mathbb{U}\otimes\mathbb{V},\ \mathfrak{g}_2 := (\wedge^2\mathbb{U})\otimes\mathbb{I}\otimes\mathbb{V}^\vee,\ \mathfrak{g}_3 := (\wedge^2\mathbb{U})\otimes\mathbb{I}\otimes\mathbb{U}.$$

Fix a cubic form $F : \mathrm{Sym}^3\,\mathbb{V} \to \mathbb{I}$ on $\mathbb{V}$ which is nondegenerate in the sense of Definition 3.6. For $v_1, v_2 \in \mathbb{V}$, define $F_{v_1v_2} \in \mathbb{I}\otimes\mathbb{V}^\vee$ by

$$F_{v_1v_2}(v) := F(v_1,v_2,v) \in \mathbb{I}.$$

We define a graded Lie algebra structure on $\mathfrak{g}_+ := \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ by

$$[u_1\otimes v_1, u_2\otimes v_2] = (u_1\wedge u_2)\otimes F_{v_1,v_2} \text{ and} \tag{3.1}$$

$$[[u_1\otimes v_1, u_2\otimes v_2], u_3\otimes v_3] = (u_1\wedge u_2)\otimes F(v_1,v_2,v_3)\otimes u_3. \tag{3.2}$$

By the nondegeneracy of F, (3.1) implies $[\mathfrak{g}_1,\mathfrak{g}_1] = \mathfrak{g}_2$. Thus (3.2) is sufficient to determine the Lie bracket $[\mathfrak{g}_1,\mathfrak{g}_2]$. It is easy to check that this gives a graded Lie algebra structure on $\mathfrak{g}_+$. Note that the Lie bracket $[\mathfrak{g}_1,\mathfrak{g}_2]\to\mathfrak{g}_3$ is independent of F and satisfies

$$[(u_1\wedge u_2)\otimes v^*, u_3\otimes v_3] = (u_1\wedge u_2)\otimes u_3\otimes v^*(v_3) \tag{3.3}$$

for any $u_1,u_2,u_3 \in \mathbb{U}, v_3 \in \mathbb{V}$ and $v^* \in \mathbb{I}\otimes V^\vee$. Sometimes, we write $\mathfrak{g}_+ = \mathfrak{g}_+(F)$ to indicate that the Lie algebra structure depends on the cubic form F.

The following is the precise formulation of going from the right-hand side to the left-hand side of $(\mathsf{Corr}.m)$ in §1.

THEOREM 3.15. *Let $(X,\mathcal{H})$ be a contact manifold of dimension $2m+3$. Let $C \subset (X,\mathcal{H})$ be a nondegenerate line, and let $y \in \mathrm{NDL}(X,\mathcal{H})$ be the corresponding point. Then the symbol algebra $\mathrm{symb}_y(\mathcal{D})$ of the natural distribution $\mathcal{D}$ on $\mathrm{NDL}(X,\mathcal{H})$ is isomorphic to $\mathfrak{g}_+(F_y)$ in Definition 3.14 for some nondegenerate cubic form F_y on an m-dimensional vector space $\mathbb{V}$.*

Proof. Let $\mathbf{F}_x : \mathrm{Sym}^3\, V_x \to I_x$ be the cubic form given by

$$\mathrm{III}_{\mathcal{C}_x,z} : \mathrm{Sym}^3\,\mathcal{V}_x \to (\mathcal{F}_z^\vee)^{\otimes 2}\otimes\mathcal{L}_x$$

via the natural isomorphism $V_x \cong \mathcal{V}_x$ for a point $x \in C$. Using Proposition 3.13, let us identify $\mathrm{symb}_y(\mathcal{D})$ with $\mathfrak{g}_+$ as graded vector spaces by setting $\mathbb{U} = U_x, \mathbb{V} = V_x$ and $\mathbb{I} = I_x$. It suffices to show that the Lie algebra structure of $\mathrm{symb}_y(\mathcal{D})$ agrees with that of $\mathfrak{g}_+(\mathbf{F}_x)$, namely, it satisfies (3.1) and (3.2).

For any $x \in C$ and $v_1,v_2 \in V_x$, let $F_{v_1v_2} \in I_x\otimes V_x^\vee$ be the contraction of $F = \mathbf{F}_x$ with v_1,v_2. Fix a nonconstant rational function $f \in U_x$. For $v_1,v_2 \in V_x$, Proposition 2.12, combined with

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Definition 3.7(iv), says that in $\mathrm{symb}_y(\mathcal{D})$,

$$[\mathbf{1}_x \otimes v_2, f \otimes v_1] = (\mathbf{1}_x \wedge f) \otimes F_{v_1 v_2} \ \in (\wedge^2 U_x) \otimes (I_x \otimes V_x^\vee).$$

Thus the Lie algebra $\mathrm{symb}_y(\mathcal{D})$ satisfies (3.1) with $F = \mathbf{F}_x$.

For $v_1, v_2, v_3 \in V_x$, Proposition 2.13 says

$$[\mathbf{1}_x \otimes v_3, [\mathbf{1}_x \otimes v_2, f \otimes v_1]] = (\mathbf{1}_x \wedge f) \otimes \mathbf{F}_x(v_1, v_2, v_3) \otimes \mathbf{1}_x \tag{3.4}$$

as elements in $(\wedge^2 U_x) \otimes I_x \otimes U_x$. Thus, the Lie algebra $\mathrm{symb}_y(\mathcal{D})$ satisfies (3.2) with $F = \mathbf{F}_x$ when $u_3 = \mathbf{1}_x$. It remains to check (3.2) when u_3 is a nonconstant function in U_x.

For a given nonconstant function $f \in U_x$, let $x' \in C$ be the zero of f such that $f \in \mathcal{A}_x^{x'}$ in the notation of Lemma 3.12. For $v_1, v_2, v_3 \in V_x$, we have $w_1, w_2, w_3 \in V_{x'}$ such that $w_i = f \cdot v_i$ for $i = 1, 2, 3$. Let $h \in \mathcal{A}_{x'}^x$ be the rational function $\frac{1}{f}$. Then

$$\mathbf{1}_x \otimes v_i = h \otimes w_i \text{ and } \mathbf{1}_{x'} \otimes w_i = f \otimes v_i \tag{3.5}$$

for $i = 1, 2, 3$, under the tensor decomposition $H^0(C, N_C^+) = U_x \otimes V_x = U_{x'} \otimes V_{x'}$. From (3.4) applied to the point $x' \in C$,

$$[\mathbf{1}_{x'} \otimes w_3, [\mathbf{1}_{x'} \otimes w_2, h \otimes w_1]] = (\mathbf{1}_{x'} \wedge h) \otimes \mathbf{F}_{x'}(w_1, w_2, w_3) \otimes \mathbf{1}_{x'} \tag{3.6}$$

as elements of $T_y Y/(\partial \mathcal{D})_y = (\wedge^2 U_{x'}) \otimes I_{x'} \otimes U_{x'}$. By (3.5), the left-hand side of (3.6) is equal to $[f \otimes v_3, [f \otimes v_2, \mathbf{1}_x \otimes v_1]]$. On the other hand, the right-hand side of (3.6) is

$$(\mathbf{1}_{x'} \wedge h) \otimes f^3 \cdot \mathbf{F}_x(v_1, v_2, v_3) \otimes \mathbf{1}_{x'}$$

by Lemma 3.12. Using the compatibility of the tensor multiplication by $\mathcal{A}_x^{x'}$ in the identification (iii) of Proposition 3.13, this is equal to

$$\begin{aligned} & f^2 \cdot (\mathbf{1}_{x'} \wedge h) \otimes \mathbf{F}_x(v_1, v_2, v_3) \otimes (f \cdot \mathbf{1}_{x'}) \\ & = (f \wedge \mathbf{1}_x) \otimes \mathbf{F}_x(v_1, v_2, v_3) \otimes f. \end{aligned}$$

Thus, (3.6) gives

$$[f \otimes v_3, [f \otimes v_2, \mathbf{1}_x \otimes v_1]] = -(\mathbf{1}_{x_0} \wedge f) \otimes \mathbf{F}_x(v_1, v_2, v_3) \otimes f.$$

This proves (3.2) when u_3 is a nonconstant function $f \in U_x$. ∎

§4. From distributions with symbols $\mathfrak{g}_+(F)$ to nondegenerate lines on contact manifolds

Definition 4.1. Let $D \subset TM$ be a distribution on a complex manifold M, and let $D^\perp \subset T^\vee M$ be its annihilator.

(i) Consider the null space of the Levi tensor at $y \in M$,

$$\mathrm{Null}(\mathrm{Levi}_y^D) := \{v \in D_y \mid \mathrm{Levi}_y^D(v, u) = 0 \text{ for all } u \in D_y\}.$$

As y varies over M, the null spaces define a distribution $\mathrm{Ch}(D)$ on a Zariski-open subset of M, called the *Cauchy characteristic* of D. It is easy to see that this is an integrable distribution, defining a holomorphic foliation on the Zariski-open subset of M.

(ii) For each $y \in M$ and $\alpha \in D_y^\perp \subset T_y^\vee M$, define the anti-symmetric form on D_y,

$$\alpha \circ \mathrm{Levi}_y^D : \wedge^2 D_y \to \mathbb{C}$$

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and its null-space by

$$\mathrm{Null}(\alpha\circ \mathrm{Levi}^D_y) = \{v \in D_y \mid \alpha\circ \mathrm{Levi}^D_y(v,u) = 0 \text{ for all } u \in D_y\}.$$

NOTATION 4.2. The projectivization $\mathbb{P}T^\vee M$ of the cotangent bundle of a complex manifold M has a natural $\mathcal{O}_{\mathbb{P}T^\vee M}(1)$-valued 1-form ϑ^M such that $\mathrm{Ker}(\vartheta^M)$ is a contact structure on $\mathbb{P}T^\vee M$ (see Notation 2.4 of [H2] for more details).

The following is Proposition 3.6 of [H2] (note that $\mathrm{Levi}^{\widetilde{H}}_\alpha$ is written as $\check{\mathrm{d}}_\alpha(\vartheta^M|_{\mathbb{P}D^\perp})$ in [H2]).

PROPOSITION 4.3. *In the setting of Definition 4.1, the restriction $\vartheta^M|_{\mathbb{P}D^\perp}$ defines a distribution $\widetilde{\mathcal{H}}$ of corank 1 on the manifold $\mathbb{P}D^\perp$. For any $y \in M$ and any nonzero $a \in D^\perp_y$ with the corresponding point $\alpha \in \mathbb{P}D^\perp_y$, we have an isomorphism*

$$\mathrm{Null}(\mathrm{Levi}^{\widetilde{\mathcal{H}}}_\alpha) \cong \mathrm{Null}(\alpha\circ \mathrm{Levi}^D_y) \subset D_y$$

induced by the natural projection $\mathbb{P}D^\perp \to M$.

The following is the precise formulation of going from the left-hand side to the right-hand side of ($\mathsf{Corr}.m$) in §1.

THEOREM 4.4. *Let $\mathcal{D} \subset TY$ be a distribution regular at every point of Y such that the symbol algebra $\mathrm{symb}_y(\mathcal{D})$ is isomorphic to $\mathfrak{g}_+(F_y)$ in Definition 3.14 for a nondegenerate cubic form F_y for each $y \in Y$. Let $\mathcal{W} \subset T^\vee Y$ be the annihilator of $\partial\mathcal{D}$. Denote by $\varrho : \mathbb{P}\mathcal{W} \to Y$ the $\mathbb{P}^1$-bundle obtained by the projectivization of $\mathcal{W}$. Then, for each $y \in Y$, there exists an open neighborhood $O^y \subset Y$ of y and a submersion $\mu_y : \varrho^{-1}(O^y) \to X^y$ of relative dimension m onto a complex manifold X^y equipped with a contact structure $\mathcal{H}^y \subset TX^y$ with the following properties.*

- (i) *The distribution of corank 1 on $\varrho^{-1}(O^y)$ given by the restriction of ϑ^Y to the submanifold $\mathbb{P}\mathcal{W} \subset \mathbb{P}T^\vee$ has the fiber at $\alpha \in \varrho^{-1}(O^y) \subset \mathbb{P}\mathcal{W}$ equal to $(\mathrm{d}_\alpha\mu_y)^{-1}(\mathcal{H}^y_{\mu_y(\alpha)})$.*
- (ii) *For two distinct points $\alpha_1 \neq \alpha_2 \in \mathbb{P}\mathcal{W}_y$, the two vector spaces $\mathrm{Ker}(\mathrm{d}_{\alpha_1}\mu_y) \subset T_{\alpha_1}\mathbb{P}\mathcal{W}$ and $\mathrm{Ker}(\mathrm{d}_{\alpha_2}\mu_y) \subset T_{\alpha_2}\mathbb{P}\mathcal{W}$ satisfy*

$$\mathrm{d}_{\alpha_1}\varrho(\mathrm{Ker}(\mathrm{d}_{\alpha_1}\mu_y)) \ \cap\ \mathrm{d}_{\alpha_2}\varrho(\mathrm{Ker}(\mathrm{d}_{\alpha_2}\mu_y)) \ = \ 0.$$

- (iii) *The submersion μ_y sends the fibers of ϱ to lines on $(X^y, \mathcal{H}^y)$.*
- (iv) *The lines on $(X^y, \mathcal{H}^y)$ in (iii) form a $(3m+2)$-dimensional family, that is, all of their small deformations on X^y come from the μ_y-images of the fibers of ϱ.*
- (v) *The lines in (iii) are nondegenerate.*

We need the following two lemmata. We skip the proof of the first one, which is straightforward. The second is well known (e.g., Lemma 3.5 of [K1]), but we reproduce the proof for the reader's convenience.

LEMMA 4.5. *In Definition 3.14, denote by $\varsigma : \wedge^2(\mathfrak{g}_1 + \mathfrak{g}_2) \to \mathfrak{g}_3$ be the homomorphism defined by the Lie bracket of $\mathfrak{g}$ modulo $\mathfrak{g}_1 + \mathfrak{g}_2$. It is independent of F and for any $a \in \mathfrak{g}_3^\vee$ the anti-symmetric form $a \circ \varsigma$ on $\mathfrak{g}_1 + \mathfrak{g}_2$ satisfies*

$$\mathrm{Null}(a\circ\varsigma) = a^\perp \otimes V \ \subset\ \mathfrak{g}_1,$$

where $a^\perp = \{u \in \mathbb{U} \mid a(u) = 0\}$.

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LEMMA 4.6. *For a contact manifold $(X,\mathcal{H})$ of dimension $2m+3$ with the contact line bundle $\mathcal{L}=TX/\mathcal{H}$, let $C\subset X$ be a smooth rational curve such that $\mathcal{L}|_C\cong\mathcal{O}(1)$ and $TX|_C$ is semipositive, that is,*

$$TX|_C\cong\mathcal{O}(l_1)\oplus\cdots\oplus\mathcal{O}(l_{2m+3})$$

for some nonnegative integers $l_1\geq\cdots\geq l_{2m+3}\geq 0$. Then C is a line on $(X,\mathcal{H})$.

Proof. It suffices to show that C is unbendable. Let $\lambda: TX\to\mathcal{L}$ be the quotient homomorphism. Since $TC\cong\mathcal{O}(2)$ and $\mathcal{L}|_C\cong\mathcal{O}(1)$, we have $\lambda(TC)=0$, which implies that C is tangent to $\mathcal{H}=\mathrm{Ker}(\lambda)$. Since $\det TX=\mathcal{L}^{\otimes(m+2)}$, we have

$$l_1+\cdots+l_{2m+3}=m+2. \tag{4.1}$$

Let

$$f^*\mathcal{H}\cong\mathcal{O}(k_1)\oplus\cdots\oplus\mathcal{O}(k_{2m+2})$$

for $k_1\geq\cdots\geq k_{2m+2}$. Since $\mathrm{Levi}^{\mathcal{H}}$ gives symplectic forms on fibers of $\mathcal{H}$ with values in $\mathcal{L}$, we have

$$k_1+k_{2m+2}=k_2+k_{2m+1}=\cdots=k_m+k_{m+3}=k_{m+1}+k_{m+2}=1.$$

Thus $k_1\geq\cdots\geq k_{m+1}\geq 1$ and $0\geq k_{m+2}\geq\cdots\geq k_{2m+2}$. Combining $2\leq k_1\leq l_1$ and $k_i\leq l_i$ for $1\leq i\leq 2m+2$ with (4.1), we conclude

$$l_1=k_1=2,\ l_2=\cdots=l_{m+1}=k_2=\cdots=k_{m+1}=1,$$
$$l_{m+2}=\cdots=l_{2m+3}=k_{m+1}=\cdots k_{2m+1}=0,\ k_{2m+2}=-1.$$

Thus, C is unbendable. ∎

Proof of Theorem 4.4. By Lemma 4.5, for any $y\in Y$, any $\alpha_1\neq\alpha_2\in\mathbb{P}\mathcal{W}_y$ and corresponding nonzero vectors $a_1\neq a_2\in\mathcal{W}_y$, we have

$$\dim\mathrm{Null}(a_i\circ\mathrm{Levi}_y^{\partial\mathcal{D}})=m \tag{4.2}$$

and

$$\mathrm{Null}(a_1\circ\mathrm{Levi}_y^{\partial\mathcal{D}})\cap\mathrm{Null}(a_2\circ\mathrm{Levi}_y^{\partial\mathcal{D}})=0. \tag{4.3}$$

Applying Proposition 4.3 with $M=Y, D=\partial\mathcal{D}, D^{\perp}=\mathcal{W}$, (4.2) shows that the Cauchy characteristic of $\widetilde{\mathcal{H}}$ is a distribution defined on the whole $\mathbb{P}(\partial\mathcal{D})^{\perp}$. For each $y\in Y$, we can choose a neighborhood O^y such that the space of leaves of the foliation $\mathrm{Ch}(\widetilde{\mathcal{H}})$ in $\varrho^{-1}(O^y)$ becomes a complex manifold X^y (e.g., by Lemma 5.6 of [HL]) and the quotient map becomes a submersion $\mu_y:\varrho^{-1}(O^y)\to X^y$. Then $\widetilde{\mathcal{H}}$ descends to a contact structure $\mathcal{H}^y\subset TX^y$ on X^y, satisfying (i). (ii) follows from (4.3).

To check (iii), let $C\subset X^y$ be the image of $\varrho^{-1}(y)$ under μ_y. Then $\mathcal{L}^y|_C\cong\mathcal{O}(1)$ for the contact line bundle $\mathcal{L}^y=TX^y/\mathcal{H}^y$ by (i) together with the fact that $\vartheta^M|_{\mathbb{P}W}$ is $\mathcal{O}_{\mathbb{P}\mathcal{W}}(1)$-valued 1-form. Since μ_y is a submersion, the vector bundle $TX^y|_C$ is semipositive. This shows that C is a line by Lemma 4.6.

To prove (iv), we need to check that there exists no arc $\Delta\subset Y$ with $y\in\Delta$ such that the surface $\varrho^{-1}(\Delta)$ is sent to a single curve in X^y by μ_y. But if such an arc Δ exists, for two

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distinct point $\alpha_1 \neq \alpha_2 \in \mathbb{P}\mathcal{W}_y$, we have

$$T_y\Delta \subset \mathrm{d}_{\alpha_1}\varrho(\mathrm{Ker}(\mathrm{d}_{\alpha_1}\mu_y)) \cap \mathrm{d}_{\alpha_2}\varrho(\mathrm{Ker}(\mathrm{d}_{\alpha_2}\mu_y)),$$

a contradiction to (ii).

By (iv), after shrinking O^y if necessary, we may regard O^y as an open subset of $\mathrm{Lines}(X^y, \mathcal{H}^y)$ and identify $\varrho^{-1}(O^y)$ with the open subset $Z := \rho^{-1}(O^y)$ in the universal family. By Proposition 4.3 and Lemma 4.5, the distribution $\mathcal{D}$ is spanned by the images of $\mathrm{d}\varrho(\mathrm{Ker}(\mathrm{d}_\alpha\mu_y))$ for $\alpha \in \varrho^{-1}(y)$. Thus $\mathcal{D}|_{O^y}$ agrees with the distribution $\mathcal{D}$ of Definition 2.7 under the identification of O^y as an open subset of $\mathrm{URC}(X^y)$. Let $C := \mu_y(\varrho^{-1}(y))$ be the line in $(X^y, \mathcal{H}^y)$ corresponding to y. By Proposition 2.12, we see that $\mathrm{II}_{\mathcal{C}_x,z}$ for a point $z \in \varrho^{-1}(y)$ satisfies (i) and (ii) of Definition 3.7. By Proposition 2.13, we see that $\mathrm{III}_{\mathcal{C}_x,z}$ satisfies (iii) and (iv) of Definition 3.7. It follows that C is a nondegenerate line. □

The arguments in the proof of Theorem 4.4 show that the constructions in Theorems 3.15 and 4.4 are the inverse of each other.

Competing Interests. The authors declare none.

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