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p-adic Families of Cohomological Modular Forms for Indefinite Quaternion Algebras and the Jacquet–Langlands Correspondence

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Abstract. We use the method of Ash and Stevens to prove the existence of small slope p-adic families of cohomological modular forms for an indefinite quaternion algebra B. We prove that the Jacquet–Langlands correspondence relating modular forms on $\mathbf{GL}_2/\mathbb{Q}$ and cohomomological modular forms for B is compatible with the formation of p-adic families. This result is an analogue of a theorem of Chenevier concerning definite quaternion algebras.

1 Introduction

This paper deals with a basic instance of the compatibility between two of the major themes in the study of automorphic forms:

- Langlands' *principle of functoriality*: this (mostly conjectural) principle describes the precise relationships between automorphic representations of different groups.
- *p-adic variation*: systems of Hecke eigenvalues associated with automorphic forms often vary in p-adic analytic families.

The *Jacquet–Langlands correspondence*, perhaps the simplest nontrivial instance of the principle of functoriality, gives precise conditions under which a classical cuspidal eigenform f can be lifted to an eigenform f^B on a Shimura curve attached to a quaternion algebra B. Both classical cuspidal eigenforms and eigenforms on Shimura curves are known to display p-adic variation. Our main result is that the correspondence $f \rightsquigarrow f^B$ is compatible with moving f and f^B in p-adic analytic families when B is indefinite.[1] Sections 2–6 deal with the local theory, and Sections 7–12 with the global thing.

Remark 1.1 We began studying these issues with applications to p-adic L-functions, p-adic Abel–Jacobi maps and Stark–Heegner points/Darmon cycles in mind (see, for

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[1]The corresponding result for definite B was proved by Chenevier [6]. His work was an inspiration for ours, and we adapt many of his techniques.

example, [8,12,15,19] for the definition of these points/cycles). For these applications, we refer the interested reader to [9,13,20].

1.1 p-adic Families

Let p be a prime and let M_0 be a positive integer with $p \nmid M_0$. Let $\mathcal{X}$ be the p-adic weight space and let $\Omega \subset \mathcal{X}$ be an affinoid subset defined over $\mathbb{Q}_p$, and let E be a p-adic field. A *p-adic analytic family of overconvergent eigenforms on Ω of tame level M_0* is a formal q-expansion

$$F(q) = \sum_{n=1}^{\infty} a_n q^n \in \big(\mathcal{O}(\Omega)\,\widehat{\otimes}_{\mathbb{Q}_p} E\big)[[q]]$$

such that, for all classical weights $k \in \Omega$,

$$F_k(q) := \sum_{n=1}^{\infty} a_n(k) q^n \in E[[q]]$$

is the q-expansion of an eigenform in $S^{\dagger}_{k+2}(\Gamma_0(pM_0))$.

In this introduction, we will consider modular forms with trivial nebentype and level divisible by p but not by p^2 for ease of exposition. In the rest of the paper, we will work in more generality.

1.2 The Jacquet–Langlands Correspondence

Suppose $M_0 = DM$, where D is squarefree with an even number of prime factors and $(D, M) = 1$. Let B be the indefinite quaternion $\mathbb{Q}$-algebra ramified precisely at the primes dividing D. Associated with these data is a space $S_{k+2}(\Gamma_0^D(pM))$ of eigenforms on a Shimura curve associated with a choice of Eichler order of level pM in B. Let $\mathcal{T}^1_+ = \mathcal{T}^1(pM_0, 1)_+$ and $\mathcal{T}^D_+ = \mathcal{T}^D(pM, 1)_+$ be the double-coset Hecke algebras described in Section 7.1. There is a natural map $\mathcal{T}^1_+ \to \operatorname{End}_{\mathbb{C}} S_{k+2}(\Gamma_0(pM_0))$. The D-new subspace $S_{k+2}(\Gamma_0(pM_0))^{D\text{-new}} \subset S_{k+2}(\Gamma_0(pM_0))$ is $\mathcal{T}^1_+$-stable, and we can set $T_k^{1,D\text{-new}} = \operatorname{im}(\mathcal{T}^1_+ \to \operatorname{End}_{\mathbb{C}} S_{k+2}(\Gamma_0(pM_0))^{D\text{-new}})$. Define

$$T_k^D = \operatorname{im}\big(\mathcal{T}^D_+ \to \operatorname{End}_{\mathbb{C}} S^D_{k+2}\big(\Gamma_0(pM)\big)\big).$$

Theorem 1.2 (Jacquet–Langlands correspondence) *There is a canonical isomorphism $T_k^{1,D\text{-new}} \xrightarrow{\sim} T_k^D$.*

Thus, if $f \in S_{k+2}(\Gamma_0(pM_0))^{D\text{-new}}$ is an normalized eigenform, then there is an eigenform $f^B \in S_{k+2}(\Gamma_0^D(pM))$ with the same system of Hecke eigenvalues.

Under certain conditions, f can be fit into a p-adic analytic family. Let $a_p(f)$ be the U_p-eigenvalue of f.

Theorem 1.3 (Hida, Coleman) *If $\operatorname{ord}_p a_p(f) < k+1$ and $a_p(f)^2 \neq p^{k+1}$, then there is an affinoid $\Omega \subset \mathcal{X}$ with $k \in \Omega$ and a p-adic analytic family F of eigenforms on Ω of tame level M_0 such that $F_k = f$.*

At the expense of possibly shrinking Ω around k, we assume that for all classical weights $w \in \Omega$, the specialization F_w of the family F is D-new. Thus, F_w admits a

Jacquet–Langlands lift $F_w^B \in S_{w+2}(\Gamma_0^D(pM))$. It is natural to ask if the F_w can be interpolated by a "p-adic analytic family." The quotation marks in the last sentence are used because we have not yet defined a notion of p-adic family for modular forms on Shimura curves. The corresponding notion for elliptic modular forms does not generalize, as modular forms on B do not admit Fourier expansions. Note also that the absence of Fourier expansions precludes a simple notion of "normalized" for modular forms on Shimura curves; the F_w^B are only well defined up to scalar multiple.

1.3 Cohomological Modular Forms

There are two ways to resolve these issues that allow for a useful notion of p-adic family in the context of Shimura curves. The method we pursue in this paper involves replacing the forms F_v^B by their associated Eichler–Shimura cohomology classes.[2] It is these cohomology classes that we interpolate. The Eichler–Shimura theorem furnishes us with a canonical isomorphism

$$\mathrm{ES}^{\pm}\colon S_{k+2}\big(\Gamma_0^D(Mp)\big) \xrightarrow{\sim} H^1\big(\Gamma_0^D(Mp), V_k(\mathbb{C})\big)^{\pm},$$

where V_k is the highest weight k representation of $\mathbf{GL}_{2/\mathbb{Q}}$ and $\Gamma_0^D(Mp)$ acts on $V_k(\mathbb{C})$ through a choice of splitting $B \otimes \mathbb{C} \cong M_2(\mathbb{C})$. For either choice of sign, there is a natural action of $\mathcal{T}_+^D$ on the right hand side and $\mathrm{ES}^{\pm}$ is $\mathcal{T}_+^D$-equivariant. It follows that the image of $\mathcal{T}_+^D$ acting on this cohomology group is identified with $\mathcal{T}_k^D$.

There is a canonical isomorphism

$$H^1\big(\Gamma_0^D(Mp), V_k(\mathbb{Q})\big) \otimes \mathbb{C} \xrightarrow{\sim} H^1\big(\Gamma_0^D(Mp), V_k(\mathbb{C})\big).$$

The $\pm$-decomposition is defined over $\mathbb{Q}$, as are the Hecke operators. The above map respects $\pm$-decompositions and is Hecke-equivariant. It follows from the Eichler–Shimura theorem that if $\mathbb{Q}(f^B)$ is the number field obtained by adjoining the Hecke eigenvalues of f^B, then

$$\dim_{\mathbb{Q}(f^B)} H^1(\Gamma_0^D(Mp), V_k(\mathbb{Q}(f^B)))^{\pm, f^B} = 1,$$

where the superscript f^B denotes the associated $\mathcal{T}_+^D$-eigenspace. If E is a p-adic field containing $\mathbb{Q}(f^B)$, then

$$\dim_E H^1\big(\Gamma_0^D(Mp), V_k(E)\big)^{\pm, f^B} = 1.$$

1.4 p-adic Families of Cohomological Modular Forms

Remarkably, the representations $V_k(\mathbb{Q}_p)$ themselves can be p-adically interpolated. Consider the subsemigroup

$$\Sigma_0(p\mathbb{Z}_p) = \Big\{ \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in \mathbf{M}_2(\mathbb{Z}_p) \cap \mathbf{GL}_2(\mathbb{Q}_p) : p \nmid a \text{and } p \mid c \Big\}.$$

of $\mathbf{GL}_2(\mathbb{Q}_p)$. Then there is a universal highest weight module/vector pair $(\mathcal{D}, \delta)$ for linear representations of $\Sigma_0(p\mathbb{Z}_p)$ on locally convex $\mathcal{O}(\mathcal{X})$-vector spaces. Let $k \in \mathcal{X}$.

[2] Andreatta, Iovita, and Stevens [2] have developed an elegant and powerful arithmetico-geometric method of studying p-adic families.

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If $v \in V_k(\mathbb{Q}_p)$ is a highest weight vector, then by the universal property of $(\mathcal{D}, \delta)$, there is a unique $\mathcal{O}(\mathcal{X})$-linear, $\Sigma_0(p\mathbb{Z}_p)$-equivariant weight k specialization map

$$\rho_k : \mathcal{D} \longrightarrow V_k(\mathbb{Q}_p) \tag{1.1}$$

such that $\rho_k(\delta) = v$. Here, $\mathcal{O}(\mathcal{X})$ acts on $V_k(\mathbb{Q}_p)$ through the evaluation-at-k map $\mathcal{O}(\mathcal{X}) \to \mathbb{Q}_p$.

If $\Omega \subset \mathcal{X}$, write $\mathcal{D}_\Omega$ for $\mathcal{D} \widehat{\otimes}_{\mathcal{O}(\mathcal{X})} \mathcal{O}(\Omega)$. We define the *space of p-adic families of quaternionic modular forms parametrized by* Ω to be the space $H^1(\Gamma_0^D(Mp), \mathcal{D}_\Omega)^\pm$. The Hecke algebra $\mathcal{T}_+^D$ acts naturally on $H^1(\Gamma_0^D(Mp), \mathcal{D}_\Omega)^\pm$. If $k \in \Omega$ is a classical weight, then the specialization map (1.1) induces an $\mathcal{O}(\Omega)$-linear, $\mathcal{T}_+^D(pM, N)$-equivariant map

$$\rho_k : H^1\big(\Gamma_0^D(pM), \mathcal{D}_\Omega\big)^\pm \longrightarrow H^1\big(\Gamma_0^D(Mp), V_k(\mathbb{Q}_p)\big)^\pm. \tag{1.2}$$

The specializations of such a family $\Phi \in H^1(\Gamma_0^D(Mp), \mathcal{D}_\Omega)^\pm$ are simply the classes $\rho_k(\Phi) \in H^1(\Gamma_0^D(Mp), V_k)^\pm$, where $k \in \Omega$ is a classical weight.

But the principal question remains: to what extent do p-adic families of quaternionic eigenforms exist? Our answer is given by the following theorem.

Theorem 1.4 *Suppose $\phi_k \in H^1(\Gamma_0^D(Mp), V_k(E))^\pm$ is a $\mathcal{T}_+^D$-eigenvector with U_p-eigenvalue $a_p(\phi_k)$ such that $\operatorname{ord}_p a_p(\phi_k) < k+1$ and $a_p^2 \neq p^{k+1}$. Then there is a $\mathcal{T}_+^D$-eigenvector $\Phi \in H^1(\Gamma_0^D(Mp), \mathcal{D}_\Omega)^\pm$ such that $\rho_{k*}(\Phi) = \phi_k$, and Φ is unique up to multiplication by an element $\alpha \in \mathcal{O}(\Omega)^\times$ with $\alpha(k) = 1$.*

The distribution module $\mathcal{D}$ to be considered in this paper is the "classical one" considered in [22]. The relation with those considered in [4, Theorem 3.7.3] when the reductive group is isomorphic to $\mathbf{GL}_2$ over $\mathbb{Q}_p$ is the following. Let ω be the character of the Borel subgroup of upper triangular matrices that sends g to its upper left entry. Ash and Stevens define a space of distribution $\mathcal{D}_\omega(X)$ supported on a suitable three dimensional manifold X modelled on the "big cell" of $\mathbf{GL}_2$. This space is endowed with the action of a semigroup $\Sigma_p \supsetneq \Sigma_0(p\mathbb{Z}_p)$ and has a highest weight vector $\delta \in \mathcal{D}_\omega(X)$. The universal property of $(\mathcal{D}, \delta)$ yields a unique $\Sigma_0(p\mathbb{Z}_p)$-equivariant morphism $(\mathcal{D}, \delta) \to (\mathcal{D}_\omega(X), \delta)$. However, we will not need and will not exploit the universal property of $(\mathcal{D}, \delta)$: the specialization maps will have a more concrete description.

1.5 Slope $\leq h$ Decompositions

Key to the proof of Theorem 1.4 is the fact that, for sufficiently small Ω, the space $H^1(\Gamma_0^D(Mp), \mathcal{D}_\Omega)^\pm$ admits a slope $\leq h$ decomposition with respect to U_p, *i.e.*, a $\mathcal{T}_+^D$-equivariant decomposition

$$H^1\big(\Gamma_0^D(Mp), \mathcal{D}_\Omega\big)^\pm = H^1\big(\Gamma_0^D(Mp), \mathcal{D}_\Omega\big)^{\pm, \leq h} \oplus H^1\big(\Gamma_0^D(Mp), \mathcal{D}_\Omega\big)^{\pm, > h}$$

such that $H^1(\Gamma_0^D(Mp),\mathcal{D}_\Omega)^{\pm,\le h}$ is a finitely generated $\mathcal{O}(\Omega)$-module on which U_p acts with slope $\le h$ and is maximal with respect to this property (see [3]). The Eichler–Shimura cohomology group $H^1(\Gamma_0^D(Mp),V_k(\mathbb{Q}_p))$ also admits a slope $\le h$ decomposition. (This is a linear algebraic result, lacking the functional analytic depth of the existence of slope $\le h$ decompositions over affinoid algebras.)

Theorem 1.5 *There is an affinoid $\Omega\subset\mathcal{X}$ with $k\in\Omega$ such that the following hold.*

(i) $H^1(\Gamma_0^D(Mp),\mathcal{D}_\Omega)^{\pm}$ *admits a $\mathcal{T}_+^D$-equivariant slope $\le h$ decomposition with respect to U_p.*

(ii) $H^1(\Gamma_0^D(Mp),\mathcal{D}_\Omega)^{\pm,\le h}$ *is free of finite rank over $\mathcal{O}(\Omega)$.*

(iii) *The specialization map (1.2) induces an isomorphism*

$$H^1\big(\Gamma_0^D(Mp),\mathcal{D}_\Omega\big)^{\pm,\le h}\otimes_{\mathcal{O}(\Omega)}E\xrightarrow{\sim}H^1\big(\Gamma_0^D(Mp),V_k(E)\big)^{\pm,\le h},$$

where the $\mathcal{O}(\Omega)$-algebra structure on E is given by the evaluation-at-k-map.

Theorem 1.5(i) and (iii) are applications of the Ash–Stevens theory of slope decompositions for the arithmetic cohomology developed in [3,4] of slope decompositions of arithmetic cohomology. Much of the first part of the paper is devoted to establishing the functional analytic properties of the modules $\mathcal{D}_\Omega$ required for application of Ash–Stevens machinery. Part (ii) is not a formal consequence of the Ash–Stevens theory in that it uses the one-dimensionality of $\mathcal{X}$ in an essential way.

Theorem 1.5 is the main input for the proof of Theorem 1.4. Before taking up the existence of eigenvectors, we discuss the problem of lifting systems of Hecke eigenvalues or eigenpackets. Note that liftability of eigenvectors implies liftability of eigenvalues, but these are not equivalent in general.

It is convenient to use geometric language when considering systems of Hecke eigenvalues. Write $T_k^{D,\pm,\le h}$ and $T_\Omega^{D,\pm,\le h}$ for the image of the images of the Hecke algebras in the endomorphism rings of

$$H^1\big(\Gamma_0^D(Mp),V_k(E)\big)^{\pm,\le h}\quad\text{and}\quad H^1\big(\Gamma_0^D(Mp),\mathcal{D}_\Omega\big)^{\pm,\le h},$$

respectively. There is a structural morphism

$$\operatorname{Sp}T_k^{D,\pm,\le h}\longrightarrow\operatorname{Sp}E\qquad(\text{resp. }\operatorname{Sp}T_\Omega^{D,\pm,\le h}\longrightarrow\Omega).$$

Specialization in weight k induces a morphism $\operatorname{Sp}T_k^{D,\pm,\le h}\to\operatorname{Sp}T^{D,\pm,\le h}$ such that

$$\begin{array}{ccc}\operatorname{Sp}T_k^{D,\pm,\le h} & \longrightarrow & \operatorname{Sp}T_\Omega^{D,\pm,\le h}\\ \big\downarrow & & \big\downarrow\\ \operatorname{Sp}E & \longrightarrow & \Omega\end{array}$$

commutes, where $\operatorname{Sp}E\to\Omega$ is induced by the evaluation-at-k map $\mathcal{O}(\Omega)\to E$. We view the eigenpacket of $\phi_k\in H^1(\Gamma_0^D(Mp),V_k(E))^{\pm,\le h}$ as an E-valued point x_k of $\operatorname{Sp}T_k^{D,\pm,\le h}$. Our prospective lift of this eigenpacket is a section of x_Ω of $\operatorname{Sp}T_\Omega^{D,\pm,\le h}\to$

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Ω such that

$$\begin{array}{ccc} \mathrm{Sp}\,T_k^{D,\pm,\le h} & \longrightarrow & \mathrm{Sp}\,T_\Omega^{D,\pm,\le h} \\ {\scriptstyle x_k}\uparrow & & \uparrow{\scriptstyle x_\Omega} \\ \mathrm{Sp}\,E & \longrightarrow & \Omega \end{array}$$

commutes.

Corollary 1.6 *Under the above assumptions, the section x_Ω exists at the expense of possibly shrinking Ω around k.*

In addition to its reliance on Theorem 1.5, this corollary depends crucially on the *multiplicity-one property*:

$$\dim_E H^1(\Gamma_0^D(Mp), V_k(E))^{\pm,x_k} = 1.$$

From here, the fact that the eigenspace associated with x_Ω is free of rank one over $\mathcal{O}(\Omega)$ follows from some commutative algebra. This establishes Theorem 1.4. For details, see Corollary 11.4.

1.6 Eigencurves

The preceding discussion can be globalized. The compatibility of slope $\le h$ decompositions with flat base change implies that there is a unique coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{T}_{\mathcal{X}}^{D,\pm,\le h}$ such that $\Gamma(\Omega, \mathcal{T}_{\mathcal{X}}^{D,\pm,\le h}) = T_\Omega^{D,\pm,\le h}$ whenever $H^1(\Gamma_0^D(Mp), \mathcal{D}_\Omega)^\pm$ admits a slope $\le h$ decomposition. We define the *slope $\le h$ eigencurve* $\mathcal{C}^{D,\pm,\le h}$ by

$$\mathcal{C}^{D,\pm,\le h} = \mathrm{Sp}_{\mathcal{O}_{\mathcal{X}}}\,\mathcal{T}_{\mathcal{X}}^{D,\pm,\le h}.$$

By construction, it admits a structural "weight" map $\mathrm{wt} : \mathcal{C}^{D,\pm,\le h} \to \mathcal{X}$. As indicated above, the fiber of wt over a classical weight k with $k+1 > h$ is identified with the set of slope $\le h$ systems of $\mathcal{T}_+^D$-eigenvalues occuring in $H^1(\Gamma_0(Mp), V_k(E))^{\pm,\le h}$:

$$\mathcal{C}_k^{D,\pm,\le h} = \mathrm{Sp}\,T_k^{D,\pm,\le h}.$$

A similar approach, built on overconvergent modular forms rather than Eichler–Shimura cohomology groups, yields the more classical eigencurves wt: $\mathcal{C}_{\mathrm{CMB}}^{\le h} \to \mathcal{X}$ of Coleman–Mazur and Buzzard; see Section 12.1 for details.

1.7 Compatibility with the Jacquet–Langlands Correspondence

We can reword Theorem 1.2 as follows. By the Hecke-equivariance of slope $\le h$ decompositions, there is a canonical isomorphism

$$\mathrm{Sp}\,T_k^{1,D\text{-new},\le h} \xrightarrow{\sim} \mathrm{Sp}\,T_k^{D,\pm,\le h}.$$

On the other hand, by construction, the fiber over a classical weight k with $k > h-1$ of $\mathcal{C}_{\mathrm{CMB}}^{D\text{-new},\le h}$ is $\mathrm{Sp}\,T_k^{1,\le h}$. Consequently,

$$\mathcal{C}_{\mathrm{CMB},k}^{D\text{-new},\le h} \xrightarrow{\sim} \mathcal{C}_k^{D,\pm,\le h}.$$

It follows from Chenevier's lemma (Proposition 10.9) that there is a unique morphism

$$\mathcal{C}^{D\text{-new},\le h}_{\mathrm{CMB,red}}/\mathcal{X}^{\mathrm{fl}} \xrightarrow{\sim} \mathcal{C}^{D,\pm,\le h}_{\mathrm{red}}/\mathcal{X}^{\mathrm{fl}}$$

whose restriction to fibers above k with $k > h-1$ are as described above. The subscript "red" means that we need to take the associated reduced curve, and the isomorphism is over a suitable flat locus $\mathcal{X}^{\mathrm{fl}} \subset \mathcal{X}$ as defined in Section 10 containing all the arithmetic weights. As a consequence, we prove that, loosely speaking, D-new p-adic families have Jacquet–Langlands lifts.

Theorem 1.7 *Let F be a p-adic family of eigenforms on Ω of tame level M, and slope $\le h$. Then there is a system of Hecke eigenvalues $x_\Omega \in \mathrm{Sp}\,\mathcal{T}^{D,\pm,\le h}_{\Omega}$ and a Hecke eigenvector $\Phi^\pm \in H^1(\Gamma_0^D(Mp), \mathcal{D}_\Omega)^{\pm,x_\Omega}$ such that for all classical weights $k \in \Omega$ with $k > h-1$, $\rho_k(\Phi^\pm) \in H^1(\Gamma_0^D, V_k(E))^{\pm,x_k}$ is the image under the Eichler–Shimura map of a Jacquet–Langlands correspondent of F_k.*

1.8 Other Approaches

We use the cohomological machinery of Ash and Stevens to establish p-adic Jacquet–Langlands correspondences, it being well-adapted to the applications to p-adic L-functions we have in mind; see [13, 20]. There are two other approaches to such p-adic Jacquet–Langlands correspondences in the literature, both of a more geometric nature than that of this paper. In [14], Newton establishes p-adic Jacquet–Langlands correspondences using the vanishing cycles functor on integral models of Shimura curves in conjunction with Emerton's completed cohomology theory and corresponding eigencurve construction [11]. Presumably, one could also approach the results of [13, 20] using Newton's method in conjunction with Emerton's completed cohomology based construction of p-adic L-functions [10]. The second alternative approach, due to Andreatta, Iovita, and Stevens [2] uses their notion of (families of) p-adic overconvergent sheaves and corresponding eigencurve construction. The theory developed in [2] has two very attractive features. First, it gives us a conceptual, geometric way to talk about the space of p-adic families of overconvergent modular forms. This theory would allow for a much cleaner statement of Theorem 1.7. Second, it seems to generalize extremely well to higher dimensional situations; see [1].

1.9 Organization of the paper

We begin the paper by describing the weight space—the parameter space for the p-adic families we want to study in Section 2. In the subsequent Sections 3–5, we attach polynomial, locally polynomial and locally analytic weight modules to points $\Bbbk \in \mathcal{X}(R)$ valued in certain affinoid algebras. As we describe their construction, we establish functional analytic properties of these modules required for the existence of slope decompositions on their cohomology. Our local study of weight modules being finished, we move on to describing the Ash–Stevens machinery that gives rise to slope decompositions of arithmetic cohomology groups with coefficients in the weight modules studied in the previous sections: this is the content of Section 6.

We analyze closely the case of the cohomology of unit groups arising from indefinite quaternion algebras in Section 7–9. This analysis, together with an formalism for eigencurves developed in the subsequent Section 10, yields our main results concerning the existence of p-adic families of classes in the cohomology groups of quaternionic unit groups, proved in Section 11, and the compatibility of their existence with the Jacquet–Langlands correspondence, proved in Section 12.

2 Weight Characters

We fix E, a p-adic field, as our working field. Let $N \in \mathbb{N}$ prime to p and let $\mathcal{X}_N$ be the rigid analytic variety over $\mathbb{Q}_p$ such that, for every affinoid E-algebra R,

$$\mathcal{X}_N(R) = \mathrm{Hom}_{\mathrm{cts}}\big(\mathbb{Z}_{p,N}^\times, R^\times\big),$$

where $\mathbb{Z}_{p,N}^\times := \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$. We will refer to elements of $\mathcal{X}_N(R)$ as *R-valued weights*.

Write $\mathcal{D}(\mathbb{Z}_{p,N}^\times)$ for the space of locally analytic distributions on $\mathbb{Z}_{p,N}^\times$. If $\mu \in \mathcal{D}(\mathbb{Z}_{p,N}^\times)$, we can define a function $\widehat{\mu}$ on $\mathcal{X}_N$ by the rule

$$\widehat{\mu}(\lambda) = \int_{\mathbb{Z}_{p,N}^\times} t^\lambda d\mu(t).$$

Theorem 2.1 (Amice–Velu) *The map $\mu \mapsto \widehat{\mu}$ is a topological E-linear isomorphism of $\mathcal{D}(\mathbb{Z}_{p,N}^\times)$ onto $\mathcal{O}(\mathcal{X}_N)$.*

Definition 2.2

- A weight $\epsilon \in \mathcal{X}_N(E)$ has *level* r if it factors through $(\mathbb{Z}/p^rN\mathbb{Z})^\times$. The minimal such r is called the *conductor of* ϵ.
- A weight $\kappa \in \mathcal{X}_N(E)$ is *arithmetic of level* (resp. *conductor*) r if $t^\kappa = t^k\epsilon(t)$ for some $k \in \mathbb{N}$ and $\epsilon\colon \mathbb{Z}_{p,N}^\times \to E^\times$ of level (resp. conductor) r.

If $\kappa \in \mathcal{X}_N(E)$ is arithmetic of level r, we write $\kappa = (k,\epsilon)$. This slight abuse is justified by the fact that κ is uniquely determined by (k,ϵ). More precisely, setting $\Delta_N := \mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^\times, E^\times)$ and assuming that $\mu_{p^rN} \subset E$, the set of E-valued arithmetic weights of level r is identified with $\mathbb{N} \times \Delta_{p^rN} \hookrightarrow \mathcal{X}_N(E)$ by the rule $(k,\epsilon) \mapsto (\cdot)^k\epsilon$.

We set $\mathcal{X} := \mathcal{X}_1$ so that, over $E \supset \mu_N$,

$$\mathcal{X}_N = \bigsqcup_{\epsilon_N \in \Delta_N} \mathcal{X}_{\epsilon_N} \quad \text{and} \quad \mathcal{X} \simeq \mathcal{X}_{\epsilon_N},$$

the identification being given by the rule $\kappa \mapsto \kappa\epsilon_N$.

Definition 2.3 When $\mu_{p^rN} \subset E$ we define $\mathbb{N}_{r,N} \subset \mathcal{X}_N(E)$ to be the image of $\mathbb{N}\times\Delta_{p^rN}$, *i.e.,* the set of arithmetic weights of level r.

With this notation we have, when $\mu_{p^rN} \subset E$,

$$\mathbb{N}_{r,N} = \bigsqcup_{\epsilon_N \in \Delta_N} \mathbb{N}_{r,\epsilon_N}, \quad \mathbb{N}_{r,\epsilon_N} := \mathbb{N}_{r,N} \cap \mathcal{X}_{\epsilon_N} \quad \text{and} \quad \mathbb{N}_r := \mathbb{N}_{r,1} \simeq \mathbb{N}_{r,\epsilon_N}.$$

The canonical inclusion $\mathbb{Z}_p^\times \subset \mathbb{Z}_{p,N}^\times$ induces a finite and étale morphism $\mathcal{X}_N \to \mathcal{X}$ under which $\mathbb{N}_{r,N}$ (resp. $\mathbb{N}_{r,\epsilon_N}$) maps to $\mathbb{N}_r$. Note that $\mathbb{N}_1 \subset \mathcal{X}(\mathbb{Q}_p)$. For later use we define

$$\mathbb{N}_{r,N}^{>\lambda} := \left\{ \kappa = (k,\epsilon) \in \mathbb{N}_{r,N} : k > \lambda \right\}$$

for $\lambda \in \mathbb{R}$ as well as the sets $\mathbb{N}_{r,\epsilon_N}^{>\lambda}$ and $\mathbb{N}_r^{>\lambda}$ defined in a similar way.

Remark 2.4 If $\Omega \subset \mathcal{X}$ is an affinoid subset, we write $\Omega_N \subset \mathcal{X}_N$ for its inverse image under the map $\mathcal{X}_N \to \mathcal{X}$ identifying each $\mathcal{X}_{\epsilon_N} \subset \mathcal{X}_N$ with $\mathcal{X}$. We also set $\Omega_{\epsilon_N} = \Omega_N \cap \mathcal{X}_{\epsilon_N}$. Furthermore, if M is an $\mathcal{O}(\Omega_N)$-module, it will be locally free if and only if the ϵ_N-component is locally free over $\mathcal{O}(\Omega)$-module for every ϵ_N. (This follows from the fact that $\mathcal{X}_N \to \mathcal{X}$ is finite and étale.)

3 Arithmetic Weight Modules

3.1 Algebraic Highest Weight Modules

For an integer $k \geq 0$, let $\mathbf{P}_k$ be the space of polynomials in x of degree at most k. The algebraic group $\mathbf{GL}_2$ acts on $\mathbf{P}_k$ from the left by the rule

(3.1) $$(g_k f)(x,y) = j_k(g,x) f(xg),$$

where

(3.2) $$j_k(g,x) = (a+cx)^k \quad \text{and} \quad xg = \frac{b+dx}{a+cx}, \quad \text{with } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Let $\mathbf{V}_k$ be the space of linear functionals on $\mathbf{P}_k$. Then $\mathbf{V}_k$ is a right $\mathbf{GL}_2$-module under the dual action: $(\mu|_k g)(f) = \mu(g_k f)$. The $\mathbf{P}_k$ (resp. $\mathbf{V}_k$) form a complete list of the irreducible left (resp. right) representations of the algebraic group $\mathbf{GL}_2$, up to twists by powers of the determinant.

3.2 Locally Polynomial Weight Modules

For integers $n \geq 0$, let $P_{k,n} = P_{k,n}[\mathbb{Z}_p]$ be the space of functions $f\colon \mathbb{Z}_p \to E$ such that, for each disk $B[a,p^{-n}] = a + p^n\mathbb{Z}_p$ of radius p^{-n} in $\mathbb{Z}_p$, the restriction $f|_{a+p^n\mathbb{Z}_p}$ is a polynomial function of degree at most k. There are obvious inclusions $i_{n,m}\colon P_{k,n} \hookrightarrow P_{k,m}$ for $m \geq n$. Define the semigroup

$$\Sigma_0(p^n\mathbb{Z}_p) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{GL}_2(\mathbb{Q}_p) \cap \mathbf{M}_2(\mathbb{Z}_p) : a \in \mathbb{Z}_p^\times, \quad c \in p^n\mathbb{Z}_p \right\}.$$

Then $\Sigma_0(p\mathbb{Z}_p)$ acts on $\mathbb{Z}_p$ from the right by the rule $(x,\sigma) \mapsto x\sigma$ as in (3.2).

Lemma 3.1 *If $\sigma \in \Sigma_0(p\mathbb{Z}_p)$ and $a \in \mathbb{Z}_p$, then*

$$B[a,p^{-n}]\sigma \subset B\left[a\sigma, p^{-n}|\det(\sigma)|\right]$$

Equality holds when $\det(\sigma) \in \mathbb{Z}_p^\times$, i.e., $\sigma \in \Gamma_0(p\mathbb{Z}_p)$.

Let $\Bbbk = (k,\epsilon_p)$ be an arithmetic weight with ϵ_p of conductor r. Then for $\sigma \in \Sigma_0(p\mathbb{Z}_p)$, the function $j_\Bbbk(\sigma,\cdot)$ defined by

$$j_\Bbbk(\sigma,x) = (a+cx)^\Bbbk = \epsilon_p(a+cx) j_k(\sigma,x), \qquad \sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),$$

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belongs to $P_{k,r-1}$ (where $P_{k,-1} := P_{k,0}$). Therefore, (3.1) with k replaced by $\Bbbk$ defines a left *weight* $\Bbbk$ *action* of $\Sigma_0(p\mathbb{Z}_p)$ on $P_{k,n}$ for $n \geq r-1$. We can actually make a more precise statement. It follows from Lemma 3.1 that if $\sigma \in \Sigma_0(p\mathbb{Z}_p)$, $\sigma_\Bbbk$ is as in (3.1) and

$$n_\sigma := \operatorname{ord}_p(\det \sigma),$$

then there is a (unique) E-linear map $\widetilde{\sigma}_\Bbbk \colon P_{k,n+n_\sigma} \to P_{k,n}$ such that $\sigma_\Bbbk = i_{n,n+n_\sigma} \circ \widetilde{\sigma}_\Bbbk$ for all $n \geq r-1$.

Note also that the natural map

$$\mathbf{P}_k \xrightarrow{\sim} P_{k,0} \tag{3.3}$$

is a $\Sigma_0(p\mathbb{Z}_p)$-equivariant isomorphism.

Let $V_{k,n}$ be the E-dual of $P_{k,n}$. This space is equipped with a right weight $\Bbbk$ action of $\Sigma_0(p\mathbb{Z}_p)$ by duality for $n \geq r-1$; write $V_{\Bbbk,n}$ for the corresponding module. Abusing notation, write $\sigma_\Bbbk$ for the endomorphism of $V_{k,n}$ dual of the endomorphism $\sigma_\Bbbk$ of $P_{k,n}$. Dual to $\widetilde{\sigma}_\Bbbk$, there is a unique map $\widetilde{\sigma}_\Bbbk : V_{k,n} \to V_{k,n+n_\sigma}$ such that $\sigma_\Bbbk = p_{n+n_\sigma,n} \circ \widetilde{\sigma}_\Bbbk$, where $p_{m,n}$ is dual to $i_{n,m}$:

$$\begin{array}{ccc} & & V_{\Bbbk,n+n_\sigma} \\ & \overset{\widetilde{\sigma}_\Bbbk}{\nearrow} & \downarrow \\ V_{\Bbbk,n} & \xrightarrow[\sigma_\Bbbk]{} & V_{\Bbbk,n}. \end{array}$$

The projection $V_{k,0} \to \mathbf{V}_k$ dual to (3.3) is a $\Sigma_0(p\mathbb{Z}_p)$-equivariant isomorphism.

Let $\mathcal{P}_k = \mathcal{P}_k(\mathbb{Z}_p)$ be the space of functions on $\mathbb{Z}_p$ that are locally polynomial of degree at most k:

$$\mathcal{P}_k = \varinjlim \mathcal{P}_{k,n}.$$

The maps $i_{n,m} \colon P_{\Bbbk,n} \to P_{\Bbbk,n}$ being of $\Sigma_0(p\mathbb{Z}_p)$ equivariant, we obtain a left weight $\Bbbk$ action of $\Sigma_0(p\mathbb{Z}_p)$ on $\mathcal{P}_k$; write $\mathcal{P}_\Bbbk$ for the corresponding module. Define $\mathcal{V}_k = \mathcal{V}_k(\mathbb{Z}_p)$ to be the strong dual of $\mathcal{P}_k$. Equipping $\mathcal{V}_k$ with the weight $\Bbbk$ action dual to that on $\mathcal{P}_\Bbbk$, we have a canonical $\Sigma_0(p\mathbb{Z}_p)$-equivariant topological isomorphism:

$$\mathcal{V}_\Bbbk \longrightarrow \varprojlim_n V_{\Bbbk,n}.$$

In the special case $\sigma \in \Sigma_0(p^r\mathbb{Z}_p)$, we have

$$j_\Bbbk(\sigma, x) = \epsilon_p(a+cx)(a+cx)^k = \epsilon_p(a)(a+cx)^k \in P_{k,0}.$$

Thus, by the same reasoning as above, we obtain a weight $\Bbbk$-action of $\Sigma_0(p^r\mathbb{Z}_p)$ on $P_{k,n}$ for all $n \geq 0$. Write $P_{\Bbbk,n}$ for the corresponding $\Sigma_0(p^r\mathbb{Z}_p)$-module. It is clear that if $n \geq r-1$, then this weight $\Bbbk$ action of $\Sigma_0(p^r\mathbb{Z}_p)$ agrees with the weight $\Bbbk$ action of $\Sigma_0(p^r\mathbb{Z}_p)$ defined above, justifying the overlapping notation. Dually, we have $\Sigma_0(p^r\mathbb{Z}_p)$-modules $V_{\Bbbk,n}$ for all $n \geq 0$, notationally consistent with those introduced previously for $n \geq r-1$.

3.3 Twisting

If V is an $(E,\Sigma_0(p^r\mathbb{Z}_p))$-module, we write $V\{\epsilon_p\}$ for the $\Sigma_0(p^r\mathbb{Z}_p)$ module with underlying set V but with action twisted by $\epsilon_p\colon(v,\sigma)\mapsto\epsilon(a_\sigma)v\sigma$. With this convention, the identity maps $P_{k,n}\{\epsilon_p\}\xrightarrow{\sim}P_{\Bbbk,n}$ and $\mathcal{P}_k\{\epsilon_p\}\xrightarrow{\sim}\mathcal{P}_{\Bbbk}$ are $\Sigma_0(p^r\mathbb{Z}_p)$-equivariant isomorphisms. Dualizing, we obtain canonical identifications $V_{k,n}\{\epsilon_p\}\xrightarrow{\sim}V_{\Bbbk,n}$ and $\mathcal{V}_k\{\epsilon_p\}\xrightarrow{\sim}\mathcal{V}_{\Bbbk,n}$.

4 Locally Analytic Weight Modules

Let Ω be an E-affinoid variety and suppose $\Bbbk\colon\Omega\to\mathcal{X}$ is a morphism of E-rigid spaces.

Assumption 4.1 *We assume throughout that Ω is absolutely reduced so that the norm on the F-algebra*

$$\mathcal{O}(\Omega\times_{\mathrm{Sp}\,E}\mathrm{Sp}\,F)=\mathcal{O}(\Omega)\mathbin{\widehat{\otimes}_E}F$$

is multiplicative for any p-adic field F.

(Our primary cases of interest are when Ω is a closed disk in $\mathcal{X}$ or an E-valued point.) Set $R=\mathcal{O}(\Omega)$ and let $A_n[R]=A_n[\mathbb{Z}_p,R]$ be the R-Banach module for the sup-norm $|\cdot|_n$ of functions $f\colon\mathbb{Z}_p\to R$ that can be represented by convergent power series on each disk $B[a,p^{-n}]$. When there is no danger of ambiguity, we will use the shorthand

$$A_n=A_n[E].$$

In fact, we have a canonical isomorphism

$$A_n\mathbin{\widehat{\otimes}_E}R\xrightarrow{\sim}A_n[R]. \tag{4.1}$$

Since $|\cdot|_n$ is the sup-norm, we have the following lemma.

Lemma 4.2 *Let $\sigma\in\Sigma_0(p\mathbb{Z}_p)$, let $f\in A_n[R]$, and let $\sigma f\in A_n[R]$ be defined by $(\sigma f)(x)=f(x\sigma)$. Then $|\sigma f|_n\le|f|_n$.*

It is well known that the continuous homomorphism $\mathbb{Z}_p^\times\to R^\times$ associated with $\Bbbk$ is necessarily locally analytic. In other words, there is an $n=n_\Bbbk$ such that $\Bbbk\in A_{n_\Bbbk+1}[\mathbb{Z}_p^\times,R]$. It follows that for each $\sigma\in\Sigma_0(p\mathbb{Z}_p)$, we have $j_\Bbbk(\sigma,\cdot)\in A_{n_\Bbbk}[R]$, where $j_\Bbbk(\sigma,x)$ is defined as in (3.2). Combining this with Lemma 3.1, we conclude that

$$(\sigma_\Bbbk f)(x):=j_\Bbbk(\sigma,x)f(x\sigma)$$

defines an left *weight* $\Bbbk$ *action* of $\Sigma_0(p\mathbb{Z}_p)$ on $A_n[R]$ for all $n\ge n_\Bbbk$; write $A_{\Bbbk,n}$ for the corresponding module. (We drop explicit mention of the coefficient ring R, since it is encoded in the weight $\Bbbk$.)

It follows from Assumption 4.1 that $|j_\Bbbk(\sigma,\cdot)|_{n_\Bbbk}\le 1$ for all $\sigma\in\Sigma_0(p\mathbb{Z}_p)$. Together with Lemma 4.2, this observation implies

$$|\sigma|_{A_n[R]}\le 1. \tag{4.2}$$

Here, $|\cdot|_{A_n[R]}$ is the operator norm on the space $\mathcal{L}_R(A_n[R],A_n[R])$ of bounded R-linear endomorphisms of $A_n[R]$. As in the locally polynomial case, the mapping $\sigma_\Bbbk$

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increases radius of convergence: there is a unique $\widetilde{\sigma}_{\Bbbk}: A_{k,n+n_\sigma}[\mathbb{Z}_p] \to A_{k,n}[\mathbb{Z}_p]$ such that $\sigma_k = i_{n,n+n_\sigma} \circ \widetilde{\sigma}_{\Bbbk}$, where $i_{n,m}$ is the inclusion of $A_n[\mathbb{Z}_p]$ into $A_m[\mathbb{Z}_p]$ for $n \geq m$.

Let $D_n[R] = D_n[\mathbb{Z}_p, R]$ be the strong dual of $A_n[\mathbb{Z}_p, R]$, *i.e.*,

$$D_n[R] = D_n[\mathbb{Z}_p, R] := \mathcal{L}_R(A_n[\mathbb{Z}_p, R], R).$$

When confusion is unlikely to result, we can write $D_n = D_n[E]$. When $D_n[R]$ is equipped with the dual action of $\Sigma_0(p\mathbb{Z}_p)$ written $(\mu, \sigma) \mapsto \mu|_{\Bbbk}\sigma$, we obtain an $(R, \Sigma_0(p\mathbb{Z}_p))$-module denoted $D_{\Bbbk,n}$. Dualizing the situation for locally analytic functions, for each $n \geq n_{\Bbbk}$ there is a unique map $\widetilde{\sigma}_{\Bbbk}$ making the diagram

$$\begin{array}{ccc} & & D_{n+n_\sigma}[R] \\ & \overset{\widetilde{\sigma}_{\Bbbk}}{\nearrow} & \downarrow \\ D_n[R] & \xrightarrow[\sigma_{\Bbbk}]{} & D_n[R] \end{array} \tag{4.3}$$

commute. It follows from (4.2) that $|\sigma|_{D_n[R]} \leq 1$, where $|\cdot|_{D_n[R]}$ is the operator norm on the space $\mathcal{L}_R(D_n[R], D_n[R])$.

Let $\mathcal{A}(R) = \mathcal{A}(\mathbb{Z}_p, R)$ be the space of locally analytic R-valued functions on $\mathbb{Z}_p$:

$$\mathcal{A}(R) = \mathcal{A}(\mathbb{Z}_p, R) := \varinjlim A_n[\mathbb{Z}_p, R].$$

When there is no danger of confusion, we will use the shorthand $\mathcal{A} = \mathcal{A}(E)$. Since completed tensor products commute with direct limits (because R is normed), we have a canonical isomorphism

$$\mathcal{A} \mathbin{\widehat{\otimes}} R \xrightarrow{\sim} \mathcal{A}(R). \tag{4.4}$$

The inclusion maps $A_n[R] \hookrightarrow A_{n+1}[R]$ being $\Sigma_0(p\mathbb{Z}_p)$-equivariant, we obtain a left weight $\Bbbk$ action of $\Sigma_0(p\mathbb{Z}_p)$ on $\mathcal{A}(\mathbb{Z}_p, R)$, giving rise to a left $\Sigma_0(p\mathbb{Z}_p)$-module that we denote $\mathcal{A}_{\Bbbk}$.

The space of *locally analytic distributions* $\mathcal{D}(R) = \mathcal{D}(\mathbb{Z}_p, R)$ is, by definition, the strong dual of $\mathcal{A}(R)$:

$$\mathcal{D}(R) = \mathcal{D}(\mathbb{Z}_p, R) := \mathcal{L}_R(\mathcal{A}(\mathbb{Z}_p, R), R).$$

When confusion is unlikely, we will use the shorthand $\mathcal{D} = \mathcal{D}(E)$. The natural map

$$\mathcal{D}(R) \longrightarrow \varprojlim_n D_n[R]$$

is a topological isomorphism. The we write $\mathcal{D}_{\Bbbk}$ for the module $\mathcal{D}(R)$ equipped with the right $\Sigma_0(p\mathbb{Z}_p)$-action arising from duality.

Let $\mathcal{L}_E(V, W)$ be the space of bounded, E-linear functionals between E-Banach modules V and W and write $W \mathbin{\widehat{\otimes}} V$ for $W \mathbin{\widehat{\otimes}}_E V$. Let V' be the strong E-dual of V. Then the canonical map $W \mathbin{\widehat{\otimes}} V' \to \mathcal{L}_E(V, W)$ identifies $W \mathbin{\widehat{\otimes}} V'$ with the closed subspace $\mathcal{C}_E(V, W) \subset \mathcal{L}_E(V, W)$ of completely continuous maps. We have

$$R \mathbin{\widehat{\otimes}} D_n \xrightarrow{\sim} \mathcal{C}_E(A_n, R) \subset \mathcal{L}_E(A_n, R) \xrightarrow{\sim} \mathcal{L}_R(A_n[R], R) = D_n[R]. \tag{4.5}$$

(The natural map $\mathcal{L}_R(A_n[R], R) \xrightarrow{\sim} \mathcal{L}_E(A_n, R)$ obtained from (4.1), always a continuous bijection, is a homeomorphism by the Open Mapping Theorem.) Thus, $R \mathbin{\widehat{\otimes}} D_n$ is identified with an R-submodule of $D_n[R]$, proper if R has infinite dimension over

E. On the other hand, after taking inverse limits, we get a canonical isomorphism [16, Proposition 18.2]

$$R\,\widehat{\otimes}\,\mathcal{D} \xrightarrow{\sim} \mathcal{D}(R). \tag{4.6}$$

Lemma 4.3 *The image of $\mathcal{C}_E(A_n, R)$ in $D_n[R]$ is stable under the weight $\Bbbk$-action of $\Sigma_0(p\mathbb{Z}_p)$.*

Proof Consider the diagram

$$\begin{array}{ccc} R\,\widehat{\otimes}\,\mathcal{D} & \xrightarrow{\ j\ } & \mathcal{D}(R) \\ {\scriptstyle 1\widehat{\otimes} p_n}\downarrow & & \downarrow{\scriptstyle p_n} \\ R\,\widehat{\otimes}\,D_n & \xrightarrow[j_n]{} & D_n[R] \end{array} \tag{4.7}$$

where j_n and j are given by (4.5) and(4.6), respectively. We must show that $\mathrm{im}(j_n) \subset \mathcal{D}_n[R]$ is stable under the weight $\Bbbk$-action of $\Sigma_0(p\mathbb{Z}_p)$. Since j_n is a topological isomorphism of $R\,\widehat{\otimes}\,D_n$ onto the closed subspace $\mathcal{C}_E(A_n, R)$ of $\mathcal{L}_E(A_n, R) = D_n[R]$, it follows that j_n induces an isomorphism

$$\text{closure of } \mathrm{im}(1\,\widehat{\otimes}\,p_n) \text{in } R\,\widehat{\otimes}\,D_n \xrightarrow{\sim} \text{closure of } j_n(\mathrm{im}(1\,\widehat{\otimes}\,p_n)) \text{in } D_n[R].$$

It is a standard fact that $p_n : \mathcal{D} \to D_n$ has dense image, implying that $1\,\widehat{\otimes}\,p_n$ does too. Therefore, the closure of $\mathrm{im}(1\,\widehat{\otimes}\,p_n)$ in $R\,\widehat{\otimes}\,D_n$ is equal to $R\,\widehat{\otimes}\,D_n$. Thus, it remains to show that the closure of $j_n(\mathrm{im}(1\widehat{\otimes}p_n))$ is stable under the weight $\Bbbk$ action of $\Sigma_0(p\mathbb{Z}_p)$. By the commutativity of (4.7) and the fact that j is an isomorphism, $j_n(\mathrm{im}(1\,\widehat{\otimes}\,p_n)) = \mathrm{im}(p_n)$. But p_n is equivariant for the weight $\Bbbk$ action of Σ, making $\mathrm{im}(p_n)$ stable. The stability of the closure follows from the continuity of the endomorphism $\sigma_{\Bbbk}$ of $D_n[R]$ for all $\sigma \in \Sigma_0(p\mathbb{Z}_p)$. ■

Thus, we can define $C_{\Bbbk,n} \subset D_{\Bbbk,n}$ to be the $\Sigma_0(p\mathbb{Z}_p)$-submodule with underlying space $\mathcal{C}_E(A_n, R)$. We have established the following lemma.

Lemma 4.4 *There is a canonical $(R, \Sigma_0(p\mathbb{Z}_p))$-equivariant isomorphism*

$$\mathcal{D}_{\Bbbk} \xrightarrow{\sim} \varprojlim C_{\Bbbk,n}.$$

4.1 δ-distributions

For each $x \in \mathbb{Z}_p$, we define

$$\delta_x = \delta_x^R \in \mathcal{D}(R) \qquad (\text{resp. } \delta_x = \delta_x^R \in D_n[R]) \qquad \text{by} \qquad \delta_x(f) = f(x),$$

where $f \in \mathcal{A}(R)$ (resp. $f \in A_n[R]$). It is obvious that the projections $p_n : \mathcal{D}(R) \to D_n[R]$ send δ_x to δ_x, justifying the overlap in notation.

Lemma 4.5 *The distributions δ_x^R, $x \in \mathbb{Z}_p$, topologically generate $\mathcal{D}(R)$ over R.*

Proof By [17, Lemma 3.1], the E-span of $\{\delta_x^E : x \in \mathbb{Z}_p\}$ is dense in $\mathcal{D}$. One can verify that the isomorphism (4.6) maps $1\,\widehat{\otimes}\,\delta_x^E$ to δ_x^R, and the result follows. ■

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4.2 Specialization

Let $\phi: R \to R'$ be an R-algebra homomorphism and let $\Bbbk' \in \mathcal{X}(R')$ be an R'-valued weight character. If $\Bbbk' = \phi \circ \Bbbk$, we say that $\Bbbk'$ *is a specialization of* $\Bbbk$ or, more specifically, that ϕ specializes $\Bbbk$ to $\Bbbk'$.

Lemma 4.6 *Suppose ϕ specializes $\Bbbk$ to $\Bbbk'$. There are canonical $(R, \Sigma_0(p\mathbb{Z}_p))$-equivariant maps*

$$\phi_n: D_{\Bbbk,n} \longrightarrow D_{\Bbbk',n} \qquad \text{and} \qquad \phi: \mathcal{D}_{\Bbbk} \longrightarrow \mathcal{D}_{\Bbbk'}.$$

They satisfy the obvious compatibilities with the projections

$$\mathcal{D}_{\Bbbk} \longrightarrow D_{\Bbbk,n+1} \longrightarrow D_{\Bbbk,n} \qquad \text{and} \qquad \mathcal{D}_{\Bbbk'} \longrightarrow D_{\Bbbk',n+1} \longrightarrow D_{\Bbbk',n}.$$

The map R-linear map $\phi: \mathcal{D}(R) \to \mathcal{D}(R')$ underlying $\phi: \mathcal{D}_{\Bbbk} \to \mathcal{D}_{\Bbbk'}$ is given by

$$\mathcal{D}(R) = R \mathbin{\widehat{\otimes}} \mathcal{D} \xrightarrow{\phi \widehat{\otimes} 1} R' \mathbin{\widehat{\otimes}} \mathcal{D} = \mathcal{D}(R').$$

Proof Let $\phi_n: D_n(R) \to D_n(R')$ be the composite

$$(4.8) \qquad D_n[R] = \mathcal{L}_R(A_n[R], R) = \mathcal{L}_E(A_n, R) \xrightarrow{\phi} \mathcal{L}_E(A_n, R') = \mathcal{L}_{R'}(A_n[R'], R') = D_n[R'].$$

Here, we have used (4.1) twice, once for R and once for R'. By (4.4) we have the continuous bijection $\mathcal{L}_R(\mathcal{A}(E), R) \xrightarrow{\sim} \mathcal{L}_E(\mathcal{A}, R)$, which is a homeomorphism by the Open Mapping Theorem. Hence, we can define the map ϕ in a similar way. The compatibilities alluded to in the statement of the lemma are obvious.

We now establish the $\Sigma_0(p\mathbb{Z}_p)$-equivariance of ϕ_n and ϕ. Since the projections $p_n: \mathcal{D}(R) \to D_n[R]$ and $p'_n: \mathcal{D}(R') \to D_n[R']$ have dense image and $\phi_n \circ p_n = p'_n \circ \phi$, the $\Sigma_0(p\mathbb{Z}_p)$-equivariance of ϕ implies that of the ϕ_n. By Lemma 4.5, it suffices to show that $\phi(\delta_x^R|_{\Bbbk}\sigma) = \phi(\delta_x^R)|_{\Bbbk'}\sigma$. Tracing through the isomorphisms in the analogue of (4.8) defining ϕ, one easily checks that $\phi(\delta_x^R) = \delta_x^{R'}$. Since $\mathcal{A} = \mathcal{A}(E)$ generates $\mathcal{A}(R')$ as an R'-module, it suffices to show that $\phi(\delta_x^R(\sigma_{\Bbbk} f)) = \delta_x^{R'}(\sigma_{\Bbbk'} f)$ for $f \in \mathcal{A}(E)$. This last identity follows from the specialization relation $\Bbbk' = \phi \circ \Bbbk$. ∎

Remark 4.7 Setting $b_z = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$, we have $\delta_z^R = \delta_0^R b_z$. Hence, by the density of the δ-distributions, ϕ is the *unique* $(R, \Sigma_0(p\mathbb{Z}_p))$-equivariant map $\mathcal{D}_{\Bbbk} \to \mathcal{D}_{\Bbbk'}$ such that $\phi(\delta_0) = \delta_0$.

5 Specializing Locally Analytic to Locally Polynomial

Let $\Bbbk: \mathbb{Z}_p^\times \to E^\times$ be an arithmetic weight. In this case, we have

$$C_{\Bbbk,n} = D_{\Bbbk,n} = D_{\Bbbk,n}[\mathbb{Z}_p].$$

Write $\Bbbk = (k, \epsilon_p)$ with ϵ_p of conductor r. Then for $n \geq n_{\Bbbk} = r - 1$, we can define $Y_{\Bbbk,n} = Y_{\Bbbk,n}[\mathbb{Z}_p]$ by the following $(E, \Sigma_0(p\mathbb{Z}))$-equivariant exact sequence:

$$0 \longrightarrow Y_{\Bbbk,n}[\mathbb{Z}_p] \longrightarrow D_{\Bbbk,n}[\mathbb{Z}_p] \xrightarrow{\rho} V_{\Bbbk,n}[\mathbb{Z}_p] \longrightarrow 0,$$

where $\rho = \rho_{\Bbbk,n}$ is the dual of the natural inclusion $P_{\Bbbk,n}[\mathbb{Z}_p] \hookrightarrow A_{\Bbbk,n}[\mathbb{Z}_p]$. Set

$$\mathcal{Y}_{\Bbbk} = \mathcal{Y}_{\Bbbk}(\mathbb{Z}_p) = \varprojlim Y_{\Bbbk,n},$$

where the transition maps are the restrictions of those in the projective system of the $D_{\Bbbk,n}[\mathbb{Z}_p]$. Then $\mathcal{Y}_k(\mathbb{Z}_{\Bbbk})$ fits into the exact sequence of $(E, \Sigma_0(p\mathbb{Z}_p))$-modules

$$0 \longrightarrow \mathcal{Y}_{\Bbbk}(\mathbb{Z}_p) \longrightarrow \mathcal{D}_{\Bbbk}(\mathbb{Z}_p) \xrightarrow{\rho} \mathcal{V}_{\Bbbk}(\mathbb{Z}_p) \longrightarrow 0, \tag{5.1}$$

The map $\widetilde{\sigma}_{\Bbbk}$ appearing in (4.3), specialized to the case $\Bbbk = k$ and $R = E$, sends $Y_{k,n}[\mathbb{Z}_p] \subset D_n[\mathbb{Z}_p, E]$ into $Y_{k,n+n_\sigma}[\mathbb{Z}_p] \subset D_{n+n_\sigma}[\mathbb{Z}_p, E]$, yielding the commutative diagram

$$\begin{array}{ccc} & & Y_{k,n+n_\sigma}[\mathbb{Z}_p] \\ & \overset{\widetilde{\sigma}_{\Bbbk}}{\nearrow} & \downarrow \\ Y_{k,n}[\mathbb{Z}_p] & \xrightarrow[\sigma_{\Bbbk}]{} & Y_{k,n}[\mathbb{Z}_p] \end{array} \qquad (n \geq n_{\Bbbk}).$$

Remark 5.1 If we restrict the weight $\Bbbk$ action to $\Sigma_0(p^r\mathbb{Z}_p)$, then all of the above considerations hold for all $n \geq 0$. In particular, $Y_{\Bbbk,n}[\mathbb{Z}_p]$ is defined for all $n \geq 0$ as a $\Sigma_0(p^r\mathbb{Z}_p)$-module.

5.1 An Operator Norm Calculation

Let $\sigma \in \Sigma_0(p\mathbb{Z}_p)$ and write $|\sigma_{\Bbbk}|_{Y_{k,n}[\mathbb{Z}_p]}$ for the operator norm of $\sigma_{\Bbbk}$ as an endomorphism of the E-Banach space $Y_{\Bbbk,n}[\mathbb{Z}_p]$.

Lemma 5.2 *If $\sigma \in \Sigma_0(p\mathbb{Z}_p)$, we have $|\sigma_{\Bbbk}|_{Y_{k,n}[\mathbb{Z}_p]} = p^{-(k+1)n_\sigma}$ for all $n \geq n_{\Bbbk}$. If $\sigma \in \Sigma_0(p^r\mathbb{Z}_p)$, then the same holds for all $n \geq 0$.*

Proof Since $|\sigma_{(k,\epsilon_p)}|_{Y_{k,n}[\mathbb{Z}_p]} = |\sigma_k|_{Y_{k,n}[\mathbb{Z}_p]}$, it suffices to prove the result for $\Bbbk = k \geq 0$. We first remark that, for every $\sigma \in \Sigma_0(p\mathbb{Z})$, there exist $\gamma, \gamma' \in \Gamma_0(p\mathbb{Z})$ such that $\sigma = \gamma \operatorname{diag}(1, p^{n_\sigma})\gamma'$, and it follows that $|\sigma|_{Y_{k,n}[\mathbb{Z}_p]} = |\operatorname{diag}(1, p^{n_\sigma})|_{Y_{k,n}[\mathbb{Z}_p]}$, and we can assume that $\sigma = \operatorname{diag}(1, p^{n_\sigma})$. In the following discussion we can unambiguously write $|\cdot|$ for all the norms involved.

We begin with a key calculation. For $c \in \{0, \ldots, p^n - 1\}$, $r \geq 0$, and $|x| \leq 1$, define

$$b_{c,r}(x) = \begin{cases} p^{-nr}(x-c)^r & \text{if } |x-c| \leq p^{-n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the $b_{c,r}$ form an orthonormal basis of the E-Banach space $A_n[\mathbb{Z}_p]$. Therefore, if $\mu \in D_n[\mathbb{Z}_p]$, we have

$$|\mu| = \sup_{c,r} |\mu(b_{c,r})|.$$

Let $\sigma = \operatorname{diag}(1, p^d)$. Then

$$(\sigma_k b_{c,r})(x) = \begin{cases} p^{-nr}(p^d x - c)^r & \text{if } |p^d x - c| \leq p^{-n}, \\ 0 & \text{otherwise.} \end{cases}$$

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Suppose $n \leq d$. Now $|p^d x - c| \leq p^{-n}$ means that $p^d x = c + p^n t$ for some $t \in \mathbb{Z}_p$. Therefore, $p^n | c$ as $|x| \leq 1$, $|c| \leq 1$, and $n \leq d$. But since $0 \leq c \leq p^n - 1$, this happens only when $c = 0$. Thus, $n \leq d$ implies $\sigma_k b_{c,r} = 0$ if $c \neq 0$, while $\sigma_k b_{0,r}(x) = p^{-nr}(p^d x)^r$ for every $x \in \mathbb{Z}_p \subset p^{n-d}\mathbb{Z}_p$. We write

$$\mathbb{Z}_p = \bigsqcup_{c=0}^{p^n-1} c + p^n \mathbb{Z}_p$$

and we note that, for $x \in c + p^n \mathbb{Z}_p$,

$$\sigma_k b_{0,r}(x) = p^{-nr}(p^d x)^r = p^{dr} p^{-nr}(x - c + c)^r = p^{dr} \sum_{s=0}^{r} \binom{r}{s} c^{r-s} p^{n(s-r)} b_{c,s}(x).$$

It follows that

$$\sigma_k b_{0,r} = p^{dr} \sum_{\substack{s=0,\dots,r \\ c=0,\dots,p^n-1}} \binom{r}{s} c^{r-s} p^{n(s-r)} b_{c,s}.$$

When $r \geq k+1$, we define the k-truncation of $\sigma_k b_{0,r}$ by the rule

$$\begin{aligned} T_k(\sigma_k b_{0,r}) &= p^{dr} \sum_{\substack{s=k+1,\dots,r \\ c=0,\dots,p^n-1}} \binom{r}{s} c^{r-s} p^{n(s-r)} b_{c,s} \\ &= p^{d(k+1)} \sum_{\substack{s=k+1,\dots,r \\ c=0,\dots,p^n-1}} \binom{r}{s} c^{r-s} p^{n(s-r)+d(r-(k+1))} b_{c,s}. \end{aligned}$$

We remark that $n(s-r) + d(r-(k+1)) \geq (r-s)(d-n) \geq 0$ for every $k+1 \leq s$ and $n \leq d$, so that

$$|T_k(\sigma_k b_{0,r})| \leq p^{-d(k+1)}|\mu| \qquad (n \leq d, r \geq k+1). \tag{5.2}$$

Suppose now that $d \leq n$. Then

$$|p^d x - c| \leq p^{-n} \Longleftrightarrow |x - p^{-d}c| \leq p^{d-n} \leq 1.$$

There exist $x \in \mathbb{Z}_p$ satisfying these inequalities only if $p^d | c$, *i.e.*,

$$c = p^d c', \quad c' \in \{0, \dots, p^{n-d} - 1\},$$

in which case

$$(\sigma_k b_{c,r})(x) = (\sigma_k b_{p^d c',r})(x) = \begin{cases} p^{(d-n)r}(x - c') & \text{if } |x - c'| \leq p^{d-n}, \\ 0 & \text{otherwise.} \end{cases}$$

We write $\mathbb{Z}_p = \bigsqcup_{c'=0}^{p^{n-d}-1} c' + p^{n-d}\mathbb{Z}_p$ and $\mathbb{Z}_p = \bigsqcup_{c''=0}^{p^d-1} c'' + p^d \mathbb{Z}_p$. Therefore, we have

$$c' + p^{n-d}\mathbb{Z}_p = \bigsqcup_{c''=0}^{p^d-1} c' + p^{n-d}(c'' + p^d \mathbb{Z}_p) = \bigsqcup_{c''=0}^{p^d-1} c' + p^{n-d}c'' + p^n \mathbb{Z}_p.$$

If $c'' \in \{0, \dots, p^d - 1\}$ and $|x - (c' + p^d c'')| \leq p^{-n}$, then one computes that

$$(\sigma_k b_{c,r})(x) = p^{dr} \sum_{s=0}^{r} \binom{r}{s} c''^{r-s} p^{(n-d)(r-s)+n(s-r)} b_{c'+p^{n-d}c'',s}(x).$$

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It follows that

$$\sigma_k b_{c,r} = p^{dr} \sum_{\substack{s=0,\dots,r \\ c''=0,\dots,p^d-1}} \binom{r}{s} c''^{r-s} p^{(n-d)(r-s)+n(s-r)} b_{c'+p^{n-d}c'',s}.$$

If $r \geq k+1$, we define the k-truncation of $\sigma_k b_{c,r}$ by the rule

$$\begin{aligned} T_k(\sigma_k b_{c,r}) &= p^{dr} \sum_{\substack{s=k+1,\dots,r \\ c''=0,\dots,p^d-1}} \binom{r}{s} c''^{r-s} p^{(n-d)(r-s)+n(s-r)} b_{c'+p^{n-d}c'',s} \\ &= p^{d(k+1)} \sum_{\substack{s=k+1,\dots,r \\ c''=0,\dots,p^d-1}} \binom{r}{s} c''^{r-s} p^{d(s-(k+1))} b_{c'+p^{n-d}c'',s}. \end{aligned}$$

We have $d(s-(k+1)) \geq 0$ for $s \geq k+1$, so that

$$|T_k(\sigma_k b_{c,r})| \leq p^{-d(k+1)}|\mu| \qquad (d \leq n, r \geq k+1). \tag{5.3}$$

Suppose now that $\sigma = \operatorname{diag}(1, p^d)$ and $\mu \in Y_{n,k}[\mathbb{Z}_p]$. We have $\mu\sigma_k \in Y_{n,k}[\mathbb{Z}_p]$ so that $(\mu\sigma_k)(b_{c,r}) = 0$ if $r \leq k+1$, and, when $r \geq k+1$, by (5.2) and (5.3), we have

$$\big|(\mu\sigma_k)(b_{c,r})\big| = \big|\mu(T_k(\sigma_k b_{c,r}))\big| \leq p^{-d(k+1)}|\mu|,$$

implying

$$|\mu\sigma_k| = \sup_{c,r}\big|(\mu\sigma_k)(b_{c,r})\big| \leq p^{-d(k+1)}|\mu|.$$

Therefore, σ acts on $Y_{n,k}[\mathbb{Z}_p]$ with norm $\leq p^{-d(k+1)} = p^{n_\sigma(k+1)}$. To show that this value is achieved, we must find a distribution $\mu \in Y_{n,k}[\mathbb{Z}_p]$ with $|\mu\sigma_k| = p^{-d(k+1)}|\mu|$. The bounded distribution determined by

$$\mu(b_{c,r}) = \begin{cases} 1 & \text{if } (c,r) = (0,k+1), \\ 0 & \text{otherwise,} \end{cases}$$

works. ∎

6 Slope Decompositions and the Ash–Stevens Machinery

6.1 $\mathcal{S}$-decompositions

Suppose that R is a commutative Noetherian ring, $\mathcal{R}$ is a commutative R-algebra, and $\mathcal{S} \subset \mathcal{R}$ is a multiplicative subset. Let H be an $\mathcal{R}$-module. Ash and Stevens [4] introduce the following key notion.

Definition 6.1 A direct sum decomposition $H = H_{\mathcal{S}} \oplus H'$ is an $\mathcal{S}$-*decomposition* if the following hold:

(i) for every $h \in H_{\mathcal{S}}$, there is an element $s \in \mathcal{S}$ such that $sh = 0$;
(ii) $H_{\mathcal{S}}$ is a finitely generated R-module;
(iii) every element of $\mathcal{S}$ acts invertibly on H'.

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When $\mathcal{S}$-decompositions exist, they are unique.

Let G be a finite group of order n acting on H. Then have the canonical decomposition of H:

$$H = \bigoplus_{\chi} H(\chi)$$

Lemma 6.2 *Suppose the action of G on H commutes with that of $\mathcal{R}$ and that $\mu_n \subset R^\times$. Then each $H(\chi)$ admits an $\mathcal{S}$-decomposition. Moreover, $H(\chi)_{\mathcal{S}} = H_{\mathcal{S}}(\chi)$ and $H(\chi)' = H'(\chi)$.*

Proof It is easy to check that G preserves $H_{\mathcal{S}}$ and H'. Therefore, $H_{\mathcal{S}} = \bigoplus_\chi H_{\mathcal{S}}(\chi)$ and $H' = \bigoplus_\chi H'(\chi)$. Since $H_{\mathcal{S}}(\chi) \oplus H'(\chi) \subset H(\chi)$ and

$$H = H_{\mathcal{S}} \oplus H' = \bigoplus_{\chi} H_{\mathcal{S}}(\chi) \oplus H'(\chi) \subset \bigoplus_{\chi} H(\chi) = H,$$

it follows that $H_{\mathcal{S}}(\chi) \oplus H'(\chi) = H(\chi)$. It is obvious that this decomposition is fact an $\mathcal{S}$-decomposition of $H(\chi)$, *i.e.*, $H(\chi)_{\mathcal{S}} = H_{\mathcal{S}}(\chi)$ and $H(\chi)' = H'(\chi)$. ∎

6.2 Slope $\leq h$ Decompositions

Let E be a p-adic field and let R be an E-Banach algebra. We let $R^{\times m}$ be the group of multiplicative elements in R, *i.e.*, elements $r \in R^\times$ such that $|rx| = |r||x|$ for all $x \in R$.

Definition 6.3 A polynomial $Q(T) \in R[T]$ has *slope* $\leq h$ if the slopes of all segments of comprising its Newton polygon are $\leq h$.

Let V be an R-module. We do not assume that V possesses a topological structure. If $Q(T) \in R[T]$, we set $Q^*(T) := T^{\deg(Q)}Q(1/T)$ and we let $a_Q = Q^*(0)$ be the leading coefficient of Q. Let $u\colon V \to V$ be an R-linear map. For $h \in \mathbb{R}$, $h > 0$, define the semigroup

$$S_h(u) = \big\langle \{Q(u) : Q(T) \text{ has slope } \leq h \text{ and } a_Q \in R^{\times m}\} \big\rangle \subset \operatorname{End}_R V$$

and set

$$V^{\leq h} = \{x \in V : Q^*(u)x = 0 \text{ for some } Q(u) \in S_h(u)\}.$$

Lemma 6.4 ([4, Lemma 7.0.2]) *$V^{\leq h}$ is an R-submodule of V.*

Suppose we have a direct sum decomposition

$$V = V^{\leq h} \oplus V^{>h} \tag{6.1}$$

for some R-submodule $V^{>h}$ of V. We call (6.1) a *slope $\leq h$ decomposition* if it is an $\mathcal{S}$-decomposition with $\mathcal{S} = S_h(u)$.

Among our main goals is showing that, for suitable $\Omega \subset \mathcal{X}$, the space $H^1(\Gamma_0, \mathcal{D}_\Omega)$ admits a slope $\leq h$ decomposition. To prove this, the Ash–Stevens machinery relies on finiteness properties of the arithmetic group Γ_0 as well as on functional analytic properties of the coefficient module $\mathcal{D}_\Omega$.

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6.3 Orthonormalizable Modules

Let G be a group, let $\Gamma \subset G$ be a subgroup, and let $\Sigma \subset G$ be a subsemigroup with $\Gamma \subset \Sigma \subset G$.

Definition 6.5 By an *orthonormalizable Banach* (R, Σ)*-module* we understand a Banach (R, Σ)-module V such that V is an orthonormalizable Banach R-module, as defined in [7, p. 423], and such that each $\sigma \in \Sigma$ acts on V with operator norm at most 1.

The next result follows directly from [18, Corollary to Proposition 1].

Lemma 6.6 *Let* $\Bbbk \in \mathcal{X}(E)$. *Then* $V_{\Bbbk,n}$, $D_{\Bbbk,n}$, *and* $Y_{\Bbbk,n}$ *are orthonormalizable Banach* $(E, \Sigma_0(p\mathbb{Z}_p))$*-modules for all* $n \geq n_{\Bbbk}$. *If* $\Bbbk = (k, \epsilon)$ *is an arithmetic weight of conductor* p^r, *then spaces are orthonormalizable Banach* $(E, \Sigma_0(p^r\mathbb{Z}_p))$*-modules for all* $n \geq 0$.

We now consider projective limits of such objects.

Definition 6.7 An *orthonormalizable* (R, Σ)*-module* is an (R, Σ)-module $\mathcal{D}$ together with an (R, Σ)-module isomorphism

$$\mathcal{D} \xrightarrow{\sim} \varprojlim (D_n, p_{n+1,n}),$$

where $(D_n, p_{n+1,n})_{n \geq n_{\mathcal{D}}}$ is a projective system of orthonormalizable Banach (R, Σ)-modules. When $\Sigma = 1$, we simply say that $\mathcal{D}$ is an orthonormalizable R-module.

Lemma 6.8 *Let R be an affinoid E-algebra and let* $\Bbbk \in \mathcal{X}(R)$. *Then* $\mathcal{D}_{\Bbbk}$ *is an orthonormalizable* $(R, \Sigma_0(p\mathbb{Z}_p))$*-module.*

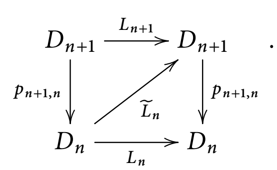
Proof By Lemma 4.4, there is a canonical $(R, \Sigma_0(p\mathbb{Z}_p))$-module isomorphism $\mathcal{D}_{\Bbbk} = \varprojlim C_{\Bbbk,n}$. Each $C_{\Bbbk,n}$ is orthonormalizable thanks to the canonical isomorphism $R \widehat{\otimes}_E D_n = C_{\Bbbk,n}$ and [7, Proposition A1.3], and the lemma follows. ∎

Suppose that $\mathcal{D}$ is an orthonormalizable R-module and $L\colon \mathcal{D} \to \mathcal{D}$ is an R-module homomorphism.

Definition 6.9 We say that L is *completely continuous* if

- $L = \varprojlim L_n$, where $L_n\colon D_n \to D_n$ are completely continuous R-linear morphisms;
- there exists an integer $n_L \geq n_{\mathcal{D}}$ and, for every $n \geq n_L$, a commutative diagram

(6.2)



Remark 6.10 (i) If the $p_{n+1,n}$ are completely continuous, then every continuous R-linear morphism $L_n\colon D_n \to D_n$ making (6.2) commutative is completely continuous. Hence, in this case, we simply require the L_n to be R-linear morphisms in order for $L = \varprojlim L_n$ to be completely continuous.

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(ii) If $\mathcal{D}$ is an orthonormalizable (R,Σ)-module and $L = \varprojlim L_n$ is completely continuous, $\gamma L \gamma' = \varprojlim \gamma L_n \gamma'$ is completely continuous for every $\gamma, \gamma' \in \Gamma$ with $n_{\gamma L \gamma'} = n_L$ (take $\gamma \widetilde{L}_n \gamma'$ in (6.2).

(iii) We always view $\mathcal{D} = \varprojlim (D_n, p_{n+1,n})$ with the initial topology making the projections $p_n \colon \mathcal{D} \to D_n$ continuous (see [16, S 5 D]). In this way $\mathcal{D}$ becomes a Fréchet space endowed with a continuous (R,Σ)-module structure. Conversely, a Fréchet space is the inverse limit $\mathcal{D} = \varprojlim (D_n, p_{n+1,n})$ of Banach spaces such that the $p_{n+1,n}$s have dense image. If $\mathcal{D}$ is endowed with a continuous (R,Σ)-module structure, we can take the D_n to be Banach R-modules on which Σ acts continuously and the $p_{n+1,n}$ to be (R,Σ)-linear.

(iv) Let $\mathcal{D}$ be an orthonormalizable (R,Σ)-module and suppose $R \to R'$ is a contraction. Then

$$R' \widehat{\otimes}_R \mathcal{D} = \varprojlim (R' \widehat{\otimes}_R D_n, 1 \widehat{\otimes}_R p_{n+1})$$

and the $1 \widehat{\otimes}_R p_{n+1,n}$ have dense image if the $p_{n+1,n}$ have dense image. In particular, $R' \widehat{\otimes}_R \mathcal{D}$ is an orthonormalizable (R',Σ)-module.

Remark 6.11 Let V and W be Banach E-spaces on which a group Γ acts continuously by operators of norm ≤ 1. Then for every $\gamma \in \Gamma$ and every $f \in \mathcal{L}(V,W)$, $|f\gamma|_{\mathcal{L}(V,W)} = |\gamma f|_{\mathcal{L}(V,W)} = |f|_{\mathcal{L}(V,W)}$.

Suppose from now on that (Γ,Σ) is a Hecke pair in G. For each $\sigma \in \Sigma$, the double coset $\Gamma\sigma\Gamma$ can be decomposed into finitely many left Γ-cosets:

$$\Gamma\sigma\Gamma = \bigsqcup_{i \in I} \Gamma\sigma_i.$$

When $\mathcal{D} = D$ is an orthonormalizable Banach (R,Σ)-module, Remark 6.11 implies that $|\sigma| = |\Gamma\sigma\Gamma|$ depends only on the double coset $\Gamma\sigma\Gamma$. Thus, we can set $|\Gamma\sigma\Gamma| = |\sigma|$. Justified by Remark 6.10(ii), we give the following definition.

Definition 6.12 If $\mathcal{D}$ is an orthonormalizable (R,Σ)-module, we say that $[\Gamma\sigma\Gamma]$ defines a completely continuous operator if $\sigma' \in \Gamma\sigma\Gamma$ is completely continuous for some or equivalently any $\sigma' \in \Gamma\sigma\Gamma$. In this case, $n_{\Gamma\sigma\Gamma} := n_{\sigma'} \geq n_{\mathcal{D}}$ is well defined.

We now assume that R is an absolutely irreducible affinoid E-algebra (endowed with the supremum norm) and write $\Omega := \mathfrak{Sp}(R)$. If $\Omega' = \mathfrak{Sp}(R') \subset \Omega$ is an open affinoid domain, the associated morphism $R \to R'$ is a contraction, and we set

$$D_{n,\Omega'} := R' \widehat{\otimes}_R D_n, \quad p_{n+1,n,\Omega} = 1 \widehat{\otimes}_R p_{n+1,n}, \quad \text{and} \quad \mathcal{D}_{\Omega'} := \varprojlim (D_{n,\Omega'}, p_{n+1,n,\Omega'}).$$

In particular, $\mathcal{D}_{\Omega'} = R' \widehat{\otimes}_R \mathcal{D}$ (Remark 6.10(iv)). The following important result is the main result of the theory of Ash and Stevens, in which the usefulness of the purely algebraic notion of slope decomposition manifests itself. This result extends Coleman's results on the existence of slope decompositions for orthonormalizable Banach modules to the cohomology of an inverse limit of such objects.

We need one further notion.

Definition 6.13 We say that Γ is *arithmetic* if and only if there exists $\Gamma' \subset \Gamma$ of finite index such that $\mathbb{Z}$ has a resolution by finitely generated free $\mathbb{Z}[\Gamma']$-modules.

The proof of the following theorem can be extracted from [4].

Theorem 6.14 *Suppose that $\mathcal{D}$ is an orthonormalizable left (R,Σ)-module on which $[\Gamma\sigma\Gamma]$ defines a completely continuous Hecke operator, that (Γ,Σ) is a Hecke pair in G, and that Γ is arithmetic. Let $k\in\Omega$.*

(i) *There is an open affinoid neighbourhood $\Omega'=\operatorname{Sp}R'$ of k in Ω such that, setting $\mathcal{D}_{\Omega'}=R'\widehat{\otimes}_R\mathcal{D}$, the cohomology group $H^i(\Gamma,\mathcal{D}_{\Omega'})$ admits a slope $\le h$ decomposition relative to the Hecke operator $[\Gamma\sigma\Gamma]$. The spaces $H^i(\Gamma,D_{n,\Omega'})$ also admit slope $\le h$ decompositions and the natural maps $\mathcal{D}\to D_n$ induce isomorphisms*

$$H^i(\Gamma,\mathcal{D}_{\Omega'})^{\le h}\xrightarrow{\sim}H^i(\Gamma,D_{n,\Omega'})^{\le h}\qquad(n\ge n_{\mathcal{D}}).$$

(ii) *If $\Omega''=\operatorname{Sp}R''$ is an open affinoid subset of Ω', then the groups $H^i(\Gamma,\mathcal{D}_{\Omega''})$ and $H^i(\Gamma,D_{n,\Omega''})$ also admit slope $\le h$ decompositions and there are canonical isomorphisms*

$$H^i(\Gamma,\mathcal{D}_{\Omega''})^{\le h}\cong R''\widehat{\otimes}_{R'}H^i(\Gamma,\mathcal{D}_{\Omega'})^{\le h},\quad H^i(\Gamma,D_{n,\Omega''})^{\le h}\cong R''\widehat{\otimes}_{R'}H^i(\Gamma,D_{n,\Omega'})^{\le h}.$$

(iii) *If R is a p-adic field, $\mathcal{D}=D$ is an orthonormalizable Banach (R,Σ)-module, and $|\Gamma\sigma\Gamma|<p^{-h}$, then $H^i(\Gamma,\mathcal{D})^{\le h}=0$.*

7 Level Structures

Let B be the (unique up to isomorphism) indefinite quaternion $\mathbb{Q}$-algebra of discriminant D. For a place v let $\mathbf{H}_v$ (resp. $\mathbf{M}_2(\mathbb{Q}_v)$) be the unique (up to isomorphism) division (resp. split) quaternion $\mathbb{Q}_v$-algebra and fix identifications

$$\iota_v\colon B_v\xrightarrow{\sim}\mathbf{H}_v\quad(v\mid D),\qquad\iota_v\colon B_v\xrightarrow{\sim}\mathbf{M}_2(\mathbb{Q}_v)\quad(v\nmid D).$$

Taken together, these induce an identification

$$\iota_{\mathbb{A}}\colon B\otimes\mathbb{A}_{\mathbb{Q}}\xrightarrow{\sim}\prod_{v\mid D}\mathbf{H}_v\times\prod_{v\nmid D}{}'\,\mathbf{M}_2(\mathbb{Q}_v).$$

Note that $\infty\nmid D$ as B is indefinite.

For a 2-by-2 matrix g, we define a_g, b_g, c_g, and d_g by

$$g=\begin{pmatrix}a_g&b_g\\c_g&d_d\end{pmatrix}.$$

If $v=\ell\mid D$, let $\mathcal{O}_{\mathbf{H}_\ell}$ be the ring of integers of $\mathbf{H}_v$. If $v=\ell\nmid D\infty$ and $r\ge1$, we write

$$\Sigma_1(\ell^r\mathbb{Z}_\ell)\subset\mathbf{M}_2(\mathbb{Z}_\ell)\cap\mathbf{GL}_2(\mathbb{Q}_\ell)\qquad(\text{resp. }\Sigma_0(\ell^r\mathbb{Z}_\ell)\subset\mathbf{M}_2(\mathbb{Z}_\ell)\cap\mathbf{GL}_2(\mathbb{Q}_\ell))$$

to denote the subsemigroup defined by the conditions $a_g\in1+\ell^r\mathbb{Z}_\ell$ (resp. $a_g\in\mathbb{Z}_\ell^\times$) and $c_g\in\ell^r\mathbb{Z}_\ell$. If $r=0$, we define the semigroup

$$\Sigma(\mathbb{Z}_\ell):=\mathbf{M}_2(\mathbb{Z}_\ell)\cap\mathbf{GL}_2(\mathbb{Q}_\ell).$$

We set

$$\Gamma_*(\ell^r\mathbb{Z}_\ell):=\Sigma_*(\ell^r\mathbb{Z}_\ell)\cap\mathbf{GL}_2(\mathbb{Z}_\ell)$$

for $*=1$, 0, or nothing.

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Let $M, N \in \mathbb{N}$ be such that M, N, and D are pairwise coprime. We consider the semigroup

$$\Sigma_*^D(M,N) = B^\times \cap \iota_{\mathbb{A}}^{-1}\Big(\prod_{\ell\mid D}\mathbf{H}_\ell^\times \times \prod_{\ell^r\| M}\Sigma_0(\ell^r\mathbb{Z}_\ell)\times\prod_{\ell^r\| N}\Sigma_1(\ell^r\mathbb{Z}_\ell) \times \prod_{\ell\nmid DMN\infty}\Sigma(\mathbb{Z}_\ell)\times \mathbf{GL}_2^*(\mathbb{R})\Big),$$

where $*$ is $+$ or nothing and $\mathbf{GL}_2^+(\mathbb{R})$ indicates the subset of matrices with positive determinant and its congruence subgroup

$$\Gamma^D(M,N) = B^\times \cap \iota_{\mathbb{A}}^{-1}\Big(\prod_{\ell\mid D}\mathcal{O}_{\mathbf{H}_\ell}^\times \times \prod_{\ell^r\| M}\Gamma_0(\ell^r\mathbb{Z}_\ell)\times\prod_{\ell^r\| N}\Gamma_1(\ell^r\mathbb{Z}_\ell) \times \prod_{\ell\nmid DMN\infty}\Gamma(\mathbb{Z}_\ell)\times \mathbf{GL}_2^+(\mathbb{R})\Big).$$

7.1 Hecke Algebras

For any commutative ring R, let

$$\mathcal{T}_R^D(M,N) = \mathcal{T}_R\big(\Gamma^D(M,N),\Sigma^D(MN,1)\big),$$
$$\mathcal{T}_R^D(M,N)_+ = \mathcal{T}_R\big(\Gamma^D(M,N),\Sigma_+^D(MN,1)\big)$$

be the double coset R-algebras associated with the Hecke pairs

$$\big(\Gamma^D(M,N),\Sigma^D(MN,1)\big) \quad\text{and}\quad \big(\Gamma^D(M,N),\Sigma_+^D(MN,1)\big),$$

respectively. We point out that, in addition to containing the Hecke operators T_ℓ for $\ell\nmid MN$ and U_ℓ for $\ell\mid MN$, the algebra $\mathcal{T}_R^D(M,N)$ also contains an involution "at infinity": let $g_{-1}\in\Sigma^D(M,N)\subset\Sigma^D(MN,1)$ be an element of reduced norm -1 normalizing $\Gamma^D(M,N)$ and such that $g_{-1}^2=1$ and define the involution

$$W_\infty := \big[\Gamma^D(M,N)g_{-1}\Gamma^D(M,N)\big] = \big[\Gamma^D(M,N)g_{-1}\big]\in\mathcal{T}^D(M,N).$$

In addition, $\mathcal{T}_R^D(M,N)$ contains the diamond operators $\langle d\rangle$ for $d\in(\mathbb{Z}/N\mathbb{Z})^\times$. If $g\in\Gamma^D(MN,1)\subset\Sigma^D(MN,1)$ is such that $a_{\iota_\ell(g)}\in d+N\mathbb{Z}_\ell$ for $\ell\mid N$, then

$$\langle d\rangle := \big[\Gamma^D(M,N)g\Gamma^D(M,N)\big] = \big[\Gamma^D(M,N)g\big]\in\mathcal{T}^D(M,N).$$

If $\mathcal{T}_R^D(M,N)$ acts on V, then $V=V^+\oplus V^-$, where $V^\pm$ is the ±1-eigenspace for the action of the involution W_∞. If $\epsilon_N\in\Delta_N := \mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^\times,R^\times)$, let

$$V(\epsilon_N) := \bigcap_{d\in(\mathbb{Z}/N\mathbb{Z})^\times}\ker(\langle d\rangle-\epsilon_N(d));$$

then we can write $V = V(\epsilon_N)\oplus V^{\epsilon_N}$ for an R-module V^{ϵ_N} such that

$$R'\otimes_R V = \bigoplus_{\epsilon_N'\in\Delta_N,\epsilon_N'\neq\epsilon_N}(R'\otimes_R V)(\epsilon_N')$$

for any R-algebra R' such that $\mu_N\subset R'$. In particular, if $\mu_N\subset R$, we have

$$V = \bigoplus_{\epsilon_N\in\Delta_N,\varepsilon\in\{\pm1\}}V^\varepsilon(\epsilon_N) = \bigoplus_{\epsilon_N\in\Delta_N,\varepsilon\in\{\pm1\}}V(\epsilon_N)^\varepsilon.$$

Since $\langle d\rangle \in \mathcal{T}^D(M,N)$ for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, if v is a $\mathcal{T}^D_R(M,N)$-eigenvector, then $v \in V(\epsilon_N)^\varepsilon$ for some $\epsilon_N \in \Delta_N$ and $\varepsilon \in \{\pm 1\}$. In our applications, R will be a $\mathcal{D}(\mathbb{Z}_p^\times)$-algebra. Then V is naturally a $\mathcal{D}(\mathbb{Z}_{p,N}^\times)$-module via $dv := v\langle d\rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $V^\varepsilon(\epsilon_N)$ (as well as $V(\epsilon_N)$) is naturally a $\mathcal{D}(\mathbb{Z}_{p,N}^\times)$-module via ϵ_N. Via the Amice–Velu theorem we can regard them as $\mathcal{O}(\mathcal{X}_N)$-modules.

Suppose now that D is an $R[\Sigma^D(MN,1)]$-module. Then

$$H^i\big(\Gamma^D(M,N),D\big) = \bigoplus_{\epsilon_N\in\Delta_N,\varepsilon\in\{\pm1\}} H^i\big(\Gamma^D(M,N),D\big)^\varepsilon(\epsilon_N)$$

and, when R is a $\mathbb{Q}$-algebra, restriction gives isomorphism

$$H^i\big(\Gamma^D(MN,1),D\{\epsilon_N^{-1}\}\big) \xrightarrow{\sim} H^i\big(\Gamma^D(M,N),D\{\epsilon_N^{-1}\}\big)^{\Gamma^D(MN,1)/\Gamma^D(M,N)} = H^i\big(\Gamma^D(M,N),D\big)^\varepsilon(\epsilon_N).$$

As remarked above, when R is a $\mathcal{D}(\mathbb{Z}_p^\times)$-algebra,

$$H^i\big(\Gamma^D(M,N),D\big) \quad\text{and}\quad H^i\big(\Gamma^D(M,N),D\big)^\varepsilon(\epsilon_N)$$

are naturally $\mathcal{O}(\mathcal{X}_N)$-modules. Thanks to Lemma 6.2 the same applies to the slope $\le h$ subspaces with respect to suitable Hecke operators.

7.2 Dependence on D

By the local structure theory, there is a canonical isomorphism

$$\begin{aligned}\mathcal{T}^D(M,N)_* &= \mathcal{T}(\mathbf{GL}_2^+(\mathbb{R}),\mathbf{GL}_2^*(\mathbb{R})) \otimes \bigotimes_{\ell\mid D}\mathcal{T}(\mathcal{O}_{\mathbf{H}_\ell}^\times,\mathbf{H}_\ell^\times) \otimes \bigotimes_{\ell^r\| M}\mathcal{T}(\Gamma_0(\ell^r\mathbb{Z}_\ell),\Sigma_0(\ell^r\mathbb{Z}_\ell))\\ &\quad\otimes \bigotimes_{\ell^r\| N}\mathcal{T}\big(\Gamma_1(\ell^r\mathbb{Z}_\ell),\Sigma_0(\ell^r\mathbb{Z}_\ell)\big)\otimes {\bigotimes}'_{\ell\nmid DMN\infty}\mathcal{T}\big(\Gamma(\mathbb{Z}_\ell),\Sigma(\mathbb{Z}_\ell)\big)\\ &=: \mathcal{T}_\infty^*\otimes{\bigotimes}'_{\ell<\infty}\mathcal{T}_\ell^D(M,N),\end{aligned}$$

the symbol $\bigotimes'$ denoting restricted tensor product and $*$ being $+$ or nothing.

Suppose $D = D'M'$ is a factorization of D with D' divisible by an even number of primes. By our running assumption that D is squarefree, D' is squarefree and $(D',M') = 1$. Let R be a commutative ring. For $\ell \nmid M'$, we have $\mathcal{T}_\ell^D(M,N) = \mathcal{T}_\ell^{D'}(MM',N)$, while for $\ell \mid M'$, there are R-algebra isomorphisms

$$R[T] \xrightarrow{\sim} \mathcal{T}_R\big(\Gamma_0(\ell\mathbb{Z}_\ell),\Sigma_0(\ell\mathbb{Z}_\ell)\big) \qquad\text{and}\qquad R[T,T^{-1}] \xrightarrow{\sim} \mathcal{T}_R(\mathbf{H}_\ell^\times,\mathcal{O}_{\mathbf{H}_\ell}^\times)$$

given by $T \mapsto [\Gamma_0(\ell\mathbb{Z}_\ell)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_0(\ell\mathbb{Z}_\ell)] = U_\ell$ and $T \mapsto [\mathcal{O}_{\mathbf{H}_\ell}^\times\pi_\ell\mathcal{O}_{\mathbf{H}_\ell}^\times] = W_\ell$, where $\pi_\ell \in \mathcal{O}_{\mathbf{H}_\ell}^\times$ has reduced norm ℓ. If $\lambda_\ell \in R^\times$, then the map $\mathcal{T}_R(\Gamma_0(\ell\mathbb{Z}_\ell),\Sigma_0(\ell\mathbb{Z}_\ell)) \to \mathcal{T}_R(\mathbf{H}_\ell^\times,\mathcal{O}_{\mathbf{H}_\ell}^\times)$ given by $T \mapsto T$ induces an isomorphism

$$\mathcal{T}_R\big(\Gamma_0(\ell\mathbb{Z}_\ell),\Sigma_0(\ell\mathbb{Z}_\ell)\big)/(U_\ell^2-\lambda_\ell) \xrightarrow{\sim} \mathcal{T}_R(\mathbf{H}_\ell^\times,\mathcal{O}_{\mathbf{H}_\ell}^\times)/(W_\ell^2-\lambda_\ell).$$

Taking the restricted tensor product of the above isomorphisms, we obtain an R-algebra isomorphism

$$(7.1)\quad j^{D,D'}\colon \mathcal{T}_R^{D'}(MM',N)/(U_\ell^2-\lambda_\ell:\ell\mid M') \xrightarrow{\sim} \mathcal{T}_R^D(M,N)/(W_\ell^2-\lambda_\ell:\ell\mid M').$$

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8 The Comparison Theorem

Let p be a prime and suppose that M, N, and D are integers prime to each other and prime to p. Consider the Hecke pair

$$(\Gamma,\Sigma)=\big(\Gamma^D(p^rM,N),\Sigma^D(p^rMN,1)\big)$$

and the associated Hecke algebra $\mathcal{T}^D(\Gamma,\Sigma)$. Let $\Bbbk=(k,\epsilon_p)\in\mathcal{X}(E)$ be an arithmetic weight character of conductor p^r. The $\mathcal{T}(\Gamma,\Sigma)$-equivariant long exact sequence in Γ-cohomology arising from (5.1) takes the form

$$\cdots\longrightarrow H^1(\Gamma,\mathcal{Y}_\Bbbk)\longrightarrow H^1(\Gamma,\mathcal{D}_\Bbbk)\longrightarrow H^1(\Gamma,\mathcal{V}_\Bbbk)\longrightarrow H^2(\Gamma,\mathcal{Y}_\Bbbk)\longrightarrow\cdots.$$

Let $\sigma\in\Sigma$ be such that $n_\sigma\geq 1$. By Theorem 6.14 and Lemma 6.8, we can take slope $\leq h$ components with respect to $\Gamma\sigma\Gamma$ of each term in the above sequence without disturbing the exactness. The following *comparison theorem* now follows from Lemma 5.2 and Theorem 6.14(iii).

Theorem 8.1 (Stevens) *If $h<(k+1)n_\sigma$ then the natural maps $\rho_\Bbbk\colon\mathcal{D}_\Bbbk\to\mathcal{V}_\Bbbk$ and $p\colon\mathcal{V}_\Bbbk\to V_{\Bbbk,0}=V_k\{\epsilon_p\}$ induce $\mathcal{T}(\Sigma,\Gamma)$-equivariant isomorphisms*

$$\rho_\Bbbk\colon H^1(\Gamma,\mathcal{D}_\Bbbk)^{\leq h}\xrightarrow{\sim}H^1(\Gamma,\mathcal{V}_\Bbbk)^{\leq h}\xrightarrow{\sim}H^1(\Gamma,V_k\{\epsilon_p\})^{\leq h}.$$

Remark 8.2 By the discussion in Section 7.1, these isomorphisms respect decomposition into $\pm$-eigenspaces as well as ϵ_N-isotypic components associated with nebentype characters.

9 The Control Theorem

Let R be an absolutely irreducible affinoid E-algebra, let $\Bbbk\in\mathcal{X}(R)$, and suppose that $\phi\colon R\to E$ specializes $\Bbbk$ to $\kappa\in\mathcal{X}(E)$ as in Section 4.2. Let $I_\kappa=\ker\phi_\kappa$ be its kernel, sitting in the exact sequence

$$0\longrightarrow I_\kappa\longrightarrow R\xrightarrow{\phi}E\longrightarrow 0. \tag{9.1}$$

Applying the exact functor $\cdot\,\widehat{\otimes}_E\,\mathcal{D}$ yields the exact sequence

$$0\longrightarrow I_\kappa\,\widehat{\otimes}_E\,\mathcal{D}\longrightarrow R\,\widehat{\otimes}_E\,\mathcal{D}\xrightarrow{\phi\widehat{\otimes}1}E\,\widehat{\otimes}_E\,\mathcal{D}\longrightarrow 0.$$

We have canonical isomorphisms $E\,\widehat{\otimes}_E\,\mathcal{D}=\mathcal{D}$ and, by (4.6), $R\,\widehat{\otimes}_E\,\mathcal{D}=\mathcal{D}(R)$. Moreover,

$$I_\kappa\,\widehat{\otimes}_E\,\mathcal{D}=I_\kappa\,\widehat{\otimes}_R\,R\,\widehat{\otimes}_E\,\mathcal{D}=I_\kappa\,\widehat{\otimes}_R\,\mathcal{D}(R).$$

The image of $I_\kappa\,\widehat{\otimes}_R\,\mathcal{D}(R)$ in $R\,\widehat{\otimes}_R\,\mathcal{D}(R)=\mathcal{D}(R)$ is $I_\kappa\mathcal{D}(R)$. Thus, we obtain the exact sequence

$$0\longrightarrow I_\kappa\mathcal{D}(R)\longrightarrow\mathcal{D}(R)\xrightarrow{\phi}\mathcal{D}(E)\longrightarrow 0.$$

Taking $\Sigma_0(p\mathbb{Z}_p)$-actions into account and observing that if ϕ specializes $\Bbbk$ to κ as in Section 4.2 then the induced map $\phi\colon\mathcal{D}_\Bbbk\to\mathcal{D}_\kappa$ is $\Sigma_0(p\mathbb{Z}_p)$-equivariant by Lemma 4.6, we get the following theorem.

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Theorem 9.1 (*cf.* [4, Theorem 3.7.4]) *If $\phi: R \to E$ specializes $\Bbbk \in \mathcal{X}(R)$ to $\kappa \in \mathcal{X}(E)$, then ϕ induces the following $(R, \Sigma_0(p\mathbb{Z}_p))$-equivariant, topologically exact sequence:*

$$0 \longrightarrow I_\kappa \mathcal{D}_\Bbbk \longrightarrow \mathcal{D}_\Bbbk \xrightarrow{\phi} \mathcal{D}_\kappa \longrightarrow 0. \tag{9.2}$$

Suppose that R is a principal ideal domain, so that we may write $I_\kappa = R\pi_\kappa$. Then (9.1) is identified with the exact sequence

$$0 \longrightarrow R \xrightarrow{\pi_\kappa} R \xrightarrow{\kappa} E \to 0. \tag{9.3}$$

Applying the exact functor $\cdot \mathbin{\widehat{\otimes}_E} \mathcal{D}$ and arguing as above, we obtain an alternate form of (9.2):

$$0 \longrightarrow \mathcal{D}_\Bbbk \xrightarrow{\pi_\kappa} \mathcal{D}_\Bbbk \xrightarrow{\phi} \mathcal{D}_\kappa \longrightarrow 0. \tag{9.4}$$

We continue with the Hecke pair

$$(\Gamma, \Sigma) = \big(\Gamma^D(p^r M, N), \Sigma^D(p^r MN, 1)\big)$$

and the associated Hecke algebra $\mathcal{T}^D(\Gamma, \Sigma)$. Suppose for the remainder of this section that $R = \operatorname{Sp}\Omega$ where Ω is an affinoid subset of $\mathcal{X}$. Let $\sigma \in \Sigma$ be such that $\operatorname{ord}_p \operatorname{nrd}(\sigma) \geq 1$ and so that σ acts completely continuously on $\mathcal{D}_\Bbbk$. Then by Theorem 6.14 and Lemma 6.8, $H^i(\Gamma, \mathcal{D}_\Bbbk)$ admits a slope $\leq h$ decomposition with respect to $[\Gamma\sigma\Gamma]$, after possibly shrinking Ω around $\Bbbk$. (We do not reflect this shrinking in the notation.) Thus, we have a $\mathcal{T}^D(\Gamma, \Sigma)$-equivariant decomposition

$$H^i(\Gamma, \mathcal{D}_\Bbbk) = H^i(\Gamma, \mathcal{D}_\Bbbk)^{\leq h} \oplus H^i(\Gamma, \mathcal{D}_\Bbbk)^{>h}.$$

The cohomology group $H^i(\Gamma, I_\kappa \mathcal{D}_\Bbbk)$ lies in the middle of a five term exact sequence coming from the long exact sequence in Γ-cohomology associated with (9.3). The other terms in this sequence have slope $\leq h$ decompositions for reasons we have already discussed. It follows that $H^i(\Gamma, I_\kappa \mathcal{D}_\Bbbk)$ admits a slope $\leq h$ decomposition with respect to $[\Gamma\sigma\Gamma]$ by Theorem 6.14. Moreover, the long exact sequence associated with (9.3) remains exact after passing to slope $\leq h$ parts:

$$\cdots \longrightarrow H^i(\Gamma, I_\kappa \mathcal{D}_\Bbbk)^{\leq h} \longrightarrow H^i(\Gamma, \mathcal{D}_\Bbbk)^{\leq h} \xrightarrow{\phi_\kappa} H^i(\Gamma, \mathcal{D}_\kappa)^{\leq h} \longrightarrow \cdots.$$

Considering instead the long exact sequence associated with (9.4), we obtain the following long exact sequence of $\mathcal{T}^D(\Gamma, \Sigma)$-modules:

$$\cdots \longrightarrow H^i(\Gamma, \mathcal{D}_\Bbbk)^{\leq h} \xrightarrow{\pi_\kappa} H^i(\Gamma, \mathcal{D}_\Bbbk)^{\leq h} \xrightarrow{\phi_\kappa} H^i(\Gamma, \mathcal{D}_\kappa)^{\leq h} \longrightarrow \cdots. \tag{9.5}$$

Suppose for the remainder of this section that $\kappa = (k, \varepsilon_p)$ is arithmetic of level $r \geq 1$ and that $h < n_\sigma(k+1)$. Then by Theorem 8.1, there is a canonical $\mathcal{T}^D(\Gamma, \Sigma)$-equivariant isomorphism

$$\rho_\kappa : H^i(\Gamma, \mathcal{D}_\kappa)^{\leq h} \xrightarrow{\sim} H^i(\Gamma, V_k\{\epsilon_p\})^{\leq h}.$$

It follows that, abusively writing ρ_κ to denote the composition $\rho_\kappa \circ \phi_\kappa$, (9.5) is identified with

$$\cdots \longrightarrow H^i(\Gamma, \mathcal{D})^{\leq h} \xrightarrow{\pi_\kappa} H^i(\Gamma, \mathcal{D}_\Bbbk)^{\leq h} \xrightarrow{\rho_\kappa} H^i(\Gamma, V_k\{\epsilon_p\})^{\leq h} \longrightarrow \cdots. \tag{9.6}$$

Vanishing criteria for the Γ-cohomology of $V_k\{\epsilon_p\}$ are well known.

Lemma 9.2 *$H^i(\Gamma, V_k\{\varepsilon_p\}) = 0$ if one of the following conditions is satisfied:*

- *$i > 2$,*
- *$i = 0, 2$ and $k > 0$,*
- *$i = 2$, $k = 0$ and Γ is cocompact,* i.e., *$D \neq 1$, where D is the discriminant of B.*

We finally come to the control theorem. Typically, it is applied when $[\Gamma\sigma\Gamma]$ is the U_p-*operator*, *i.e.*, when $\sigma \in \Sigma$ has reduced norm p. The U_p operator has degree p by [21, Proposition 3.33]).

Theorem 9.3 *We can shrink Ω around κ so that*

- *$R = \mathcal{O}(\Omega)$ is a principal ideal domain,*
- *$H^i(\Gamma, \mathcal{D}_{\Bbbk})^{\leq h} = 0$ for $i \neq 1$,*
- *$H^1(\Gamma, \mathcal{D}_{\Bbbk})^{\leq h}$ is free of finite rank over R.*

Moreover, the sequence

$$0 \longrightarrow I_\kappa H^1(\Gamma, \mathcal{D}_{\Bbbk})^{\leq h} \longrightarrow H^1(\Gamma, \mathcal{D}_{\Bbbk})^{\leq h} \xrightarrow{\rho_\kappa} H^1(\Gamma, V_k\{\epsilon_p\})^{\leq h} \longrightarrow 0 \tag{9.7}$$

is exact if one of the following conditions is satisfied:

- *$k > 0$ and $h < n_{\Gamma\sigma\Gamma}(k+1)$;*
- *$k = 0$, $h < n_{\Gamma\sigma\Gamma}$ and $h < \mathrm{ord}_p \deg[\Gamma\sigma\Gamma]$.*

In particular, when $[\Gamma\sigma\Gamma] = U_p$ is the degree p operator U_p with $n_{U_p} = 1$, this is always true when $1 \leq h < k+1$.

Proof Replacing Ω by a closed subdisk containing κ, we can assume that R is a principal ideal domain. By (9.6), the sequence

$$0 \to I_\kappa H^i(\Gamma, \mathcal{D}_{\Bbbk})^{\leq h} \longrightarrow H^i(\Gamma, \mathcal{D}_{\Bbbk})^{\leq h} \longrightarrow H^i(\Gamma, V_k\{\epsilon_p\})^{\leq h}$$

is exact. Set $H^i := H^i(\Gamma, \mathcal{D}_{\Bbbk})^{\leq h}$ and write H^i_κ for the localization of H^i at I_κ. Since I_κ is a maximal ideal, $H^i_\kappa/I_\kappa H^i_\kappa = H^i/I_\kappa H^i$ and $H^i/I_\kappa H^i \subset H^i(\Gamma, V_k\{\varepsilon_p\})^{\leq h}$ by the above exact sequence. If $i \geq 3$, then $H^i(\Gamma, V_k\{\varepsilon_p\}) = 0$ by Lemma 9.2, so that $H^i/I_\kappa H^i = 0$. Since H^i is a finitely generated R-module, by Nakayama's Lemma $H^i_\kappa = 0$, and this means that there exists $s \notin I_\kappa$ such that $sH^i = 0$. Choose $\rho \in K$ such that $0 < |\rho| \leq |s(\kappa)|$. Let $R[s^{-1}]$ (resp. $H^i[s^{-1}]$) be the localization of R (resp. H^i) at $\{s^n\}$. They can be endowed with a seminorm as in [5, p. 233 Prop. 3], that depends on the choice of ρ and makes ρ/s power bounded in $R[s^{-1}]$. Let $R\langle s^{-1}\rangle$ (resp. $H^i\langle s^{-1}\rangle$) be the completion of $R[s^{-1}]$ (resp. $H^i[s^{-1}]$). Then $\Omega' = \mathrm{Sp}\, R\langle s^{-1}\rangle \subset \Omega$ is an open affinoid domain such that $\kappa \in \Omega'$ (it represents the element $x \in \Omega$ such that $|\rho| \leq |s(x)|$, see [5, p. 281, Prop. 4]) and $R\langle s^{-1}\rangle \widehat{\otimes}_R H^i = H^i\langle s^{-1}\rangle = 0$ because the image of $H^i[s^{-1}] = 0$ is dense in $H^i\langle s^{-1}\rangle$. By Theorem 6.14, $R\langle s^{-1}\rangle \widehat{\otimes}_R H^i = H^i(\Gamma, \mathcal{D}_{\Bbbk'})^{\leq h}$ with $\Bbbk'$ associated with $\Omega' \subset \Omega \to \mathcal{X}$, and we are finished with the case $i \geq 3$.

If $k > 0$, Lemma 9.2 together with a similar argument shows that we can further assume $H^i = 0$ for $i = 0, 2$. It follows that (9.6) reduces to (9.7). Localizing (9.7) at κ, we see that H^1_κ is π_κ-torsion free and hence free, since R_κ is a principal ideal domain. Since the property of being free is Zariski-open, there exists $s \notin I_\kappa$ such that

$H^1[s^{-1}]$ is a free $R[s^{-1}]$-module. It follows that $H^1\langle s^{-1}\rangle$ is a free $R\langle s^{-1}\rangle$-module and $\Omega' = \mathrm{Sp}\, R\langle s^{-1}\rangle \subset \Omega$ is an open affinoid domain such that $\kappa \in \Omega'$ and $R\langle s^{-1}\rangle \widehat{\otimes}_R H^1 = H^1\langle s^{-1}\rangle$ is a free $R\langle s^{-1}\rangle$-module. The claim when $k \neq 0$ follows.

Now suppose that $k = 0$, $h < n_{\Gamma\sigma\Gamma}$ and $h < \mathrm{ord}_p(\deg(\Gamma\sigma\Gamma))$. Since the action of $[\Gamma\sigma\Gamma]$ on $H^i(\Gamma, V_k\{\varepsilon_p\})$ is Eisenstein for $i = 0, 2$, *i.e.*, acts through the degree character, we conclude that $H^i(\Gamma, V_k\{\varepsilon_p\})^{\le h} = 0$. An argument similar to that presented above shows that we can assume $H^i = 0$ for $i = 0, 2$, and the rest follows as in the previous paragraph. ∎

Remark 9.4 By the discussion in Subsection 7.1, the maps in (9.7) respect decomposition into ±-eigenspaces as well as ϵ_N-isotypic components associated with nebentype characters. In particular, we get a $\mathcal{T}^D_R(\Gamma, \Sigma)$-equivariant exact sequence

$$0 \longrightarrow I_\kappa H^1(\Gamma, \mathcal{D}_{\Bbbk})(\epsilon_N)^{\pm,\le h} \longrightarrow H^1(\Gamma, \mathcal{D}_{\Bbbk})(\epsilon_N)^{\pm,\le h} \xrightarrow{\rho_\kappa} H^1\big(\Gamma, V_k\{\epsilon_p\}\big)(\epsilon_N)^{\pm,\le h} \longrightarrow 0$$

such that $H^1(\Gamma, \mathcal{D}_{\Bbbk})(\epsilon_N)^{\pm,\le h}$ is a locally free $R = \mathcal{O}(\Omega)$-module in an affinoid neighbourhood Ω of an arithmetic weight $\kappa \in \mathbb{N}_r^{>h/n_\sigma - 1}$.

When $\Bbbk$ corresponds to the inclusion $\Omega \subset \mathcal{X}$ of an affinoid in the weight space in the weight space, we will sometimes write $\mathcal{D}_\Omega$ for $\mathcal{D}_{\Bbbk}$. As explained in Section 7.1, $H^1(\Gamma, \mathcal{D}_\Omega)^{\le h}$ (resp. $H^1(\Gamma, \mathcal{D}_\Omega)(\epsilon_N)^{\pm,\le h}$) is naturally an $\mathcal{O}(\Omega_N)$-module that is a locally free in a suitable neighbourhood of any arithmetic

$$\kappa \in \mathbb{N}_{r,N}^{>h/n_\sigma - 1} \quad (\text{resp. } \kappa \in \mathbb{N}_{r,\epsilon_N}^{>h/n_\sigma - 1} \simeq \mathbb{N}_r^{>h/n_\sigma - 1})$$

(see Remark 2.4).

10 Abstract Eigenvarieties

Suppose that M is a finitely generated module over a noetherian ring A and that $T = T_M \subset \mathrm{End}_A M$ is a commutative A-subalgebra, automatically finitely generated over A. For every (commutative) A-algebra R, we can consider the canonical morphisms of R -algebras

$$R \otimes_A T \longrightarrow R \otimes_A \mathrm{End}_R M \longrightarrow \mathrm{End}_R(R \otimes_A M).$$

We set

$$T_R := R \otimes_A T, \qquad M_R := R \otimes_A M,$$
$$\overline{T}_R := \mathrm{im}\big(T_R \to \mathrm{End}_R(R \otimes_A M)\big), \qquad \mathfrak{K}_R := \ker\big(T_R \twoheadrightarrow \overline{T}_R\big),$$

and define dual modules

$$\mathcal{D}(R) := \mathrm{Hom}_{A-\mathrm{alg}}(T, R) = \mathrm{Hom}_{R-\mathrm{alg}}(T_R, R), \qquad \overline{\mathcal{D}}(R) := \mathrm{Hom}_{R-\mathrm{alg}}(\overline{T}_R, R).$$

If $\lambda \in \mathcal{D}(R)$, we define

$$M_\lambda = M_{R,\lambda} := \big\{ m \in M_R : tm = \lambda(t)m \text{for all } t \in T_R \big\} = \bigcap_{t \in T_R} \ker\big(t - \lambda(t)\colon M_R \to M_R\big).$$

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Elements of the set

$$\mathcal{E}(R) := \{\lambda \in \mathcal{D}(R) : M_\lambda \neq 0\}$$

are called *systems of eigenvalues occurring in* M_R.

Lemma 10.1 *The following facts hold:*

(i) *If* R *is a flat* A*-algebra, then* $T_R = \overline{T}_R$.

(ii) *If* $R = A/\mathfrak{a}$ *is a quotient of* A*, then* $(\mathfrak{K}_{A/\mathfrak{a}})^d = 0$*, with* $d = d_M$ *depending only on the* A*-module* M.

(iii) *The identity* $\mathcal{E}(R) = \overline{\mathcal{D}}(R)$ *holds under either of the following conditions:*

- $R = K$ *is a field.*
- M *is a flat* A*-module and* R *is an integral domain.*

Proof (i) Since M is finitely presented and R is A-flat,

$$R \otimes_A \operatorname{End}_R M = \operatorname{End}_R(R \otimes_A M).$$

Furthermore, since R is a flat A-algebra, the inclusion $T \subset \operatorname{End}_R M$ induces an inclusion $R \otimes_A T \subset R \otimes_A \operatorname{End}_R M$.

(ii) When $R = A/\mathfrak{a}$, we have $T_R = T/\mathfrak{a}T$ and $M_R = M/\mathfrak{a}M$. Suppose that $\bar{t} \in T/\mathfrak{a}T$ is zero in $\operatorname{End}_{A/\mathfrak{a}}(M/\mathfrak{a}M)$ and let $t \in T$ be a lift of $\bar{t}$. To say that the image of $\bar{t}$ is zero means that $tM \subset \mathfrak{a}M$. Suppose that M is generated by d elements. Since $t \in T \subset \operatorname{End}_R M$, Nakayama's Lemma implies that the relation

$$t^d + a_1 t^{d-1} + \cdots + a_{d-1} t + a_d \text{ with } a_i \in \mathfrak{a}^i \subset \mathfrak{a}$$

holds in $\operatorname{End}_R M$. In particular, $t^d \in \mathfrak{a}T$ and $\bar{t}^d = 0$ in T_R. In other words, $(\mathfrak{K}_{A/\mathfrak{a}})^d = 0$.

(iii) If R is a field, then M_R is R-torsion free. Similarly, if R is A-flat, then M_R is R-flat and, hence, R-torsion free. Suppose that $\lambda \in \mathcal{E}(R)$. We first claim that $\lambda\colon T_R \to R$ factors through $\overline{T}_R$. To see this, choose $t \in \mathfrak{K}_R$ and a nonzero $m \in M_\lambda$. Then $0 = tm = \lambda(t)m$, implying $\lambda(t) = 0$ by the R-torsion freeness of M_R. Thus, λ factors through $\overline{T}_R = T_R/\mathfrak{K}_R$, showing that $\mathcal{E}(R) \subset \overline{\mathcal{D}}(R)$.

Let K be the fraction field of R. Since M_R is R-torsion free, $M_R \subset M_K$. The inclusion $R \subset K$ and the identification $K \otimes_R T_R = T_K$ yield

$$\begin{aligned}\overline{\mathcal{D}}(R) &= \operatorname{Hom}_{R\text{-alg}}(\overline{T}_R, R) \subset \operatorname{Hom}_{R\text{-alg}}(\overline{T}_R, K) \subset \operatorname{Hom}_{R\text{-alg}}(T_R, K)\\ &= \operatorname{Hom}_{K\text{-alg}}(T_K, K).\end{aligned}$$

Let $\lambda \in \overline{\mathcal{D}}(R)$ and write λ again to denote its image in $\operatorname{Hom}_{K\text{-alg}}(T_K, K)$. Suppose that $M_{K,\lambda} \neq 0$ and choose a nonzero element $x \in M_{K,\lambda}$. Writing $x = m/s$ with $m \in M_R$ and $0 \neq s \in R$, it easily follows that $m \in M_{R,\lambda}$. Then $0 \neq m = sx \in M_R$ is such that $tm = \lambda(t)m$ for every $t \in T_R$. Thus, it suffices to prove that, given a K-algebra $T \subset \operatorname{End}_K(M)$ acting on a finite dimensional K-vector space M and a K-algebra homomorphism $\lambda\colon T \to K$, we must have $M_\lambda \neq 0$. Being a commutative Artinian algebra over a field, we can write $T = \bigoplus_{\mathfrak{m}} T_{\mathfrak{m}}$ as the direct sum of its localizations at maximal ideals. There is a corresponding T-module decomposition $M = \bigoplus_{\mathfrak{m}} M_{\mathfrak{m}}$. Let $\mathfrak{m} = \ker\lambda$, so that λ factors through $T_{\mathfrak{m}}$. Since $T \subset \operatorname{End}_K(M)$, the equality $M_{\mathfrak{m}} = \mathfrak{m}M_{\mathfrak{m}}$ (yielding $M_{\mathfrak{m}} = 0$ by Nakayama's Lemma) would imply $T_{\mathfrak{m}} = 0$. Hence, we have $\mathfrak{m}M_{\mathfrak{m}} \subsetneq M_{\mathfrak{m}}$ and there is a minimal $n = n_{\mathfrak{m}}$ such that $\mathfrak{m}^{n+1}M_{\mathfrak{m}} = 0$. The action of $T_{\mathfrak{m}}$ on $0 \neq \mathfrak{m}^n M_{\mathfrak{m}} \subset M$ factors through λ and any non-zero $x \in \mathfrak{m}^n M_{\mathfrak{m}}$ yields an eigenvector such that $tx = \lambda(t)x$. ∎

Remark 10.2 Suppose that R is an A-algebra that is the composition of morphisms as in Lemma 10.1(i) and (ii) taken in any order. It is easily checked, using Lemma 10.1(i) and (ii), that $\mathfrak{K}_R^d = 0$ with $d = d_M$ depending only on M as an A-module.

Let $\mathfrak{p}$ be a prime ideal of A and set $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Proposition 10.3 *Suppose that $R = k(\mathfrak{p}) := A_{\mathfrak{p}}/A_{\mathfrak{p}}\mathfrak{p}$ is a residue field at $\mathfrak{p} \in \operatorname{Spec}(A)$ or that M is a flat A-module and that R is an integral domain such that the A-algebra structure on R is the composition of morphisms as in Lemma* 10.1(i) *and* (ii) *taken in any order. Then $\mathcal{E}(R) = \mathcal{D}(R)$.*

Proof If $R = k(\mathfrak{p})$ is a residue field at $\mathfrak{p} \in \operatorname{Spec}(A)$, then $A \to A_{\mathfrak{p}} \to k(\mathfrak{p})$ is the composition of a morphism as in Lemma 10.1(i) followed by a morphism as in (ii). Therefore $T_R \to \overline{T}_R$ has nilpotent kernel in both cases (Remark 10.2). Since R is reduced, $\mathcal{D}(R) = \overline{\mathcal{D}}(R)$ and, by Lemma 10.1(iii), $\overline{\mathcal{D}}(R) = \mathcal{E}(R)$. ∎

Write $f\colon A \to T$ for the structural morphism, and let $R = k(\mathfrak{p})$ with its natural A-module structure. Then $T_{k(\mathfrak{p})} = T_{\mathfrak{p}}/\mathfrak{p}T_{\mathfrak{p}}$ and $M_{k(\mathfrak{p})} = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$. The maximal ideals of $T_{k(\mathfrak{p})}$ are in natural bijection with the primes $\mathfrak{P} \in \operatorname{Spec}(T)$ such that $\mathfrak{p} = f^{-1}(\mathfrak{P})$. We can write $T_{k(\mathfrak{p})}$ as the product of its localization at maximal ideals

$$T_{k(\mathfrak{p})} = \bigoplus_{f^{-1}(\mathfrak{P})=\mathfrak{p}} T_{k(\mathfrak{p}),\mathfrak{P}}.$$

There is a corresponding $T_{k(\mathfrak{p})}$-module decomposition

$$M_{k(\mathfrak{p})} = \bigoplus_{f^{-1}(\mathfrak{P})=\mathfrak{p}} M_{k(\mathfrak{p}),\mathfrak{P}}.$$

Note also that, by Remark 10.2, the maximal ideals of $\overline{T}_{k(\mathfrak{p})}$ are in bijection with those of $T_{k(\mathfrak{p})}$. In particular, we can also write

$$\overline{T}_{k(\mathfrak{p})} = \bigoplus_{f^{-1}(\mathfrak{P})=\mathfrak{p}} \overline{T}_{k(\mathfrak{p}),\mathfrak{P}},$$

where, abusing notation, we denote by $\overline{T}_{k(\mathfrak{p}),\mathfrak{P}}$ the localization of $\overline{T}_{k(\mathfrak{p})}$ at the prime corresponding to $\mathfrak{P}$. Of course, $T_{k(\mathfrak{p}),\mathfrak{P}}$ surjects onto

$$\overline{T}_{k(\mathfrak{p}),\mathfrak{P}} \subset \operatorname{End}_{k(\mathfrak{p})}(M_{k(\mathfrak{p}),\mathfrak{P}}),$$

and the residue fields of $T_{k(\mathfrak{p}),\mathfrak{P}}$ and $\overline{T}_{k(\mathfrak{p}),\mathfrak{P}}$ are identified with $k(\mathfrak{P}) := T_{\mathfrak{P}}/T_{\mathfrak{P}}\mathfrak{P}$.

Suppose that $T_{k(\mathfrak{p})}$ acts semisimply on $M_{k(\mathfrak{p}),\mathfrak{P}}$, *i.e.*, $\overline{T}_{k(\mathfrak{p}),\mathfrak{P}} = k(\mathfrak{P})$ is a field. Then $M_{k(\mathfrak{p}),\mathfrak{P}} = M_{\mathfrak{P}}/\mathfrak{P}M_{\mathfrak{P}}$ and

$$\dim_{k(\mathfrak{p})}(M_{k(\mathfrak{p}),\mathfrak{P}}) = \left[k(\mathfrak{P}):k(\mathfrak{p})\right] \dim_{k(\mathfrak{P})}(M_{\mathfrak{P}}/\mathfrak{P}M_{\mathfrak{P}}). \tag{10.1}$$

The following notion is useful in studying the ramification of the map $A_{\mathfrak{p}} \to T_{\mathfrak{P}}$.

Definition 10.4 $\mathfrak{P} \in \operatorname{Spec}(T)$ is said to be a *multiplicity-one point* when $\overline{T}_{k(\mathfrak{p}),\mathfrak{P}} = k(\mathfrak{P})$ is separable over $k(\mathfrak{p})$ and $\dim_{k(\mathfrak{p})}(M_{k(\mathfrak{p}),\mathfrak{P}}) = [k(\mathfrak{P}) : k(\mathfrak{p})]$.

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Proposition 10.5 *If $A_{\mathfrak{p}} \to T_{\mathfrak{P}}$ is unramified, then $\overline{T}_{k(\mathfrak{p}),\mathfrak{P}} = k(\mathfrak{P})$ is a separable $k(\mathfrak{p})$-algebra. Conversely, if $\mathfrak{P} \in \operatorname{Spec}(T)$ is a multiplicity-one point, then $A_{\mathfrak{p}} \to T_{\mathfrak{P}}$ is unramified.*

Proof Note that $T \subset \operatorname{End}_T(M)$, and M is finitely presented, so $T_{\mathfrak{P}} \subset \operatorname{End}_{T_{\mathfrak{P}}}(M_{\mathfrak{P}})$. By Nakayama's Lemma, $M_{\mathfrak{P}}$ can be generated by

$$d_{\mathfrak{P}/\mathfrak{p}} := \dim_{k(\mathfrak{P})}(M_{\mathfrak{P}}/\mathfrak{P}M_{\mathfrak{P}})$$

elements (as a $T_{\mathfrak{P}}$-module). Set

$$\mathfrak{K}_{\mathfrak{P}/\mathfrak{p}} := \ker(T_{k(\mathfrak{p}),\mathfrak{P}} \to \overline{T}_{k(\mathfrak{p}),\mathfrak{P}}).$$

The proof of Lemma 10.1(ii) with $(A, \mathfrak{a}, T, M) = (T_{\mathfrak{P}}, \mathfrak{p}T_{\mathfrak{P}}, T_{\mathfrak{P}}, M_{\mathfrak{P}})$ shows that $\mathfrak{K}_{\mathfrak{P}/\mathfrak{p}}^{d_{\mathfrak{P}/\mathfrak{p}}} = 0$.

To say that $A_{\mathfrak{p}} \to T_{\mathfrak{P}}$ is unramified is to say that $T_{k(\mathfrak{p}),\mathfrak{P}} = k(\mathfrak{P})$ is a separable $k(\mathfrak{p})$-algebra, so $\overline{T}_{k(\mathfrak{p}),\mathfrak{P}} = T_{k(\mathfrak{p}),\mathfrak{P}}$ is a separable $k(\mathfrak{p})$-algebra. Conversely, let $\mathfrak{P}$ be a multiplicity one point. By (10.1), $d_{\mathfrak{P}/\mathfrak{p}} = 1$, so that $T_{k(\mathfrak{p}),\mathfrak{P}} = \overline{T}_{k(\mathfrak{p}),\mathfrak{P}} = k(\mathfrak{P})$ is separable over $k(\mathfrak{p})$. ∎

Remark 10.6 Suppose $R = k(\mathfrak{p})$ so that $\mathcal{E}(k(\mathfrak{p})) = \mathcal{D}(k(\mathfrak{p}))$ (Proposition 10.3). If $\lambda \in \mathcal{D}(k(\mathfrak{p}))$, then $M_\lambda \neq 0$ and $\mathfrak{P}_\lambda := \ker(\lambda)$ is a prime ideal of $\operatorname{Spec}(T)$ such that $\mathfrak{p} = f^{-1}(\mathfrak{P}_\lambda)$. Since $\lambda\colon T_{k(\mathfrak{p})} \to k(\mathfrak{p})$ is a morphism of $k(\mathfrak{p})$-algebras, we have $k(\mathfrak{P}_\lambda) = k(\mathfrak{p})$. Furthermore, $M_{k(\mathfrak{p}),\mathfrak{P}_\lambda} = M_\lambda$ (see the end of the proof of Lemma 10.1(iii)) and to say that $\mathfrak{P}_\lambda$ is a multiplicity one point is equivalent to saying that $\dim_{k(\mathfrak{p})}(M_\lambda) = 1$.

Suppose now that X is a locally Noetherian rigid analytic space, M is a coherent sheaf of $\mathcal{O}_X$-modules, and T is a sheaf of commutative subalgebras of $\operatorname{End}_{\mathcal{O}_X} M$. Define

$$(C \xrightarrow{w} X) = \operatorname{Sp}_{\mathcal{O}_X} T.$$

Let $X^{M\text{-fl}}$ (resp. $X^{T\text{-fl}}$) be the maximal subspace of X such that $M|_{X^{M\text{-fl}}}$ is a sheaf of flat $\mathcal{O}_{X^{M\text{-fl}}}$-modules (resp. $T|_{X^{T\text{-fl}}}$ is a sheaf of flat, commutative $\mathcal{O}_{X^{T\text{-fl}}}$-algebras). Then $X^{M\text{-fl}}$, $X^{T\text{-fl}}$, and $X^{\text{fl}} := X^{M\text{-fl}} \cap X^{T\text{-fl}}$ are open in X. Setting $C^{*\text{-fl}} = \operatorname{Sp}_{\mathcal{O}_{X^{*\text{-fl}}}} T|_{X^{*\text{-fl}}}$, for $* = M$, T, or nothing, we have a canonical isomorphism $C^{*\text{-fl}} = C \times_X X^{*\text{-fl}}$, and w restricts to a finite, flat map $C^{*\text{-fl}} \to X^{*\text{-fl}}$.

Corollary 10.7

(i) *The formation of the covers $w\colon C \to X$ and $w\colon C^{*\text{-}fl} \to X^{*\text{-}fl}$ commute with flat base change $Y \to X$.*

(ii) *If $x \in X(k(x))$, then there is a canonical bijection between points of the fibre $C_x(k(x))$ and the set of systems of $T_{k(x)}$-eigenvalues occuring in $M_{k(x)}$.*

(iii) *If $\Omega = \operatorname{Sp} R \subset X^{M\text{-}fl}$ is an affinoid and R is an integral domain, then there is a canonical bijection between the set of sections $s : \Omega \to C^{M\text{-}fl}$ of w and the set of systems of $T(\Omega)$-eigenvalues occuring in $M(\Omega)$.*

(iv) *If $y \in C$ (resp. $y \in C^{T\text{-}fl}$) is a multiplicity-one point, then w is unramified (resp. étale) at y.*

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(v) *Let $x \in X^{fl}$ and let $y \in C_x^{fl}(k(x))$ be a multiplicity-one point such that w is integral in a neighbourhood of y. Let λ_y be the system of $T_{k(x)}$-eigenvalues occuring in $M_{k(x)}$ associated with y by* (ii). *Then:*

- *There is an affinoid neighbourhood $\Omega = \operatorname{Sp} R$ of x and a unique section $s\colon \Omega \to C^{fl}$ of w such that $s(x) = y$.*
- *If λ_s is the system of $T(\Omega)$-eigenvalues occuring in $M(\Omega)$ corresponding to s by* (iii), *then λ_y is equal to the composite*

$$T(\Omega) \xrightarrow{\lambda_s} R \xrightarrow{x} k(x),$$

and λ_s is the only system of $T(\Omega)$-eigenvalues with this property.
- *The eigenspace $M(\Omega)_{\lambda_s} \subset M(\Omega)$ is a free R-module of rank one.*

Proof Part (i) follows from Lemma 10.1(i). Parts (ii) and (iii) are consequences of Proposition 10.3. In (iv) we just have to prove that the morphism is unramified, since $w\colon C^{T\text{-fl}} \to X^{T\text{-fl}}$ is flat by construction. To see (iv), let $y \in C_i$, let $A_i \to T_i$ be the morphism induced by w, and let $A_{i,x} \to T_{i,y}$ be its localization. This localization is unramified by Proposition 10.5 and induces $\mathcal{O}_{X,x} \to \mathcal{O}_{C,y}$ on the completions by [5, p. 298 Prop. 3]. Hence, $\mathcal{O}_{X,x} \to \mathcal{O}_{C,y}$ is unramified. The existence and uniqueness of s in (v) follows from the étaleness of w at y (which is (iv)). It also follows from (ii) and (iii) that this is equivalent to the existence and uniqueness of a system of eigenvalues λ_s having the required property. ∎

10.1 Maps Between Eigenvarieties

Let (T^h, M^h), with $h = 1, 2$, be two pairs with M^h a coherent sheaf of locally free $\mathcal{O}_X$-modules and $T^h \subset \operatorname{End}_{\mathcal{O}_X} M^h$ a coherent subsheaf of locally free commutative $\mathcal{O}_X$-subalgebras. We assume that there exists a commutative ring $\mathbf{T}$ and a cover $\mathcal{X} = \bigcup_i \Omega_i$ by open affinoids such that, for every i, there exists a surjective homomorphism of $\mathcal{O}(\Omega_i)$-algebras $\mathcal{O}(\Omega_i) \otimes_{\mathbb{Z}} \mathbf{T} \twoheadrightarrow T^h(\Omega_i)$. We set $(C^h \xrightarrow{w} X) = \operatorname{Sp}_{\mathcal{O}_X} T^h$.

Definition 10.8 A subset $Z \subset X$ is called *Zariski dense* if, whenever $Z \subset Y \subset X$ and Y is an analytic subset, we have $Y = X$ and if, moreover, the same property holds for $Z \cap \Omega \subset \Omega$ when restricting to an open affinoid subdomain $\Omega \subset X$.

If $t \in \mathbf{T}$ and $x \in X$, we let t_x^h be the natural image of t in $\overline{T}_{k(x)}^h \subset \operatorname{End}_{k(x)}(M_{k(x)}^h)$. We leave the proof of the following proposition, an adaptation of [6], to the reader.

Proposition 10.9 *Suppose the Jacobson sheaf of ideals of $\mathcal{O}_X$ is trivial. Let $Z \subset X$ be a Zariski dense subset such that, for every $x \in Z$ and $t \in \mathbf{T}$,*

$$\det(T - t_x^2) \text{ divides } \det(T - t_x^1).$$

in the polynomial ring $k(x)[T]$. Then there is a unique closed immersion $C^2_{\mathrm{red}} \to C^1_{\mathrm{red}}$ of rigid analytic spaces over X such that the diagram

$$\begin{array}{ccc} \mathcal{O}_X(\Omega) \otimes_{\mathbb{Z}} \mathbf{T} & & \\ \downarrow & \searrow & \\ T^1(\Omega)_{\mathrm{red}} & \longrightarrow & T^2(\Omega)_{\mathrm{red}} \end{array}$$

commutes for every open affinoid Ω.

11 The Case of Shimura Curves

Let $h > 0$. We continue working with the Hecke pair

$$(\Gamma, \Sigma) = \big(\Gamma^D(p^r M, N), \Sigma^D(p^r MN, 1)\big).$$

By Theorem 6.14, we can find an admissible cover

$$\mathcal{X} = \bigcup_i \Omega_i, \qquad \Omega_i = \operatorname{Sp} R_i$$

by open affinoids such that the spaces $H^1(\Gamma, \mathcal{D}_{\Omega_i})$ admit slope $\leq h$ decompositions with respect to U_p. As explained at the end of Section 9, $H^1(\Gamma, \mathcal{D}_{\Omega_i})$ is naturally an $\mathcal{O}(\Omega_{i,N})$, where $\Omega_{i,N}$ is the inverse image of Ω_i in $\mathcal{X}_N$ and, by Remark 2.4,

$$\mathcal{X}_N = \bigcup_i \Omega_{i,N}, \qquad \Omega_{i,N} = \operatorname{Sp} R_{i,N},$$

is an admissible cover. Set

$$M_i := H^1(\Gamma, \mathcal{D}_{\Omega_i})^{\leq h},$$
$$T_i = T^D_{R_{i,N}}(p^r M, N)^{\leq h} = \operatorname{im}\big(\mathcal{T}^D_{R_{i,N}}(p^r M, N) \longrightarrow \operatorname{End}_{R_{i,N}}(M_i)\big).$$

Proposition 11.1

(i) *The correspondence $\Omega_{i,N} \mapsto M_i$ extends to a coherent sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{X}_N}$-modules.*
(ii) *The correspondence $\Omega_{i,N} \mapsto T_i$ extends to a coherent sheaf $\mathcal{T} \subset \operatorname{End}_{\mathcal{O}_{\mathcal{X}_N}} \mathcal{M}$ of commutative $\mathcal{O}_{\mathcal{X}_N}$-algebras.*

Proof The glueing conditions [5, S9.3.3] follow directly from the compatibility of the formation of slope $\leq h$ decompositions with flat base change and Lemma 10.1(i). ∎

Definition 11.2 The rigid analytic space

$$\mathcal{C}^D_r(M, N)^{\leq h} := \operatorname{Sp}_{\mathcal{O}_{\mathcal{X}_N}} \mathcal{T}$$

is called the *slope $\leq$ h eigencurve associated with the Hecke pair* (Γ, Σ). It comes equipped with a finite *weight map*

$$\mathrm{wt}\colon \mathcal{C}^D_r(M, N)^{\leq h} \longrightarrow \mathcal{X}_N \longrightarrow \mathcal{X}.$$

Replacing the modules $H^1(\Gamma, \mathcal{D}_{\Omega_i})^{\leq h}$ with the modules $H^1(\Gamma, \mathcal{D}_{\Omega_i})(\epsilon_N)^{\pm, \leq h}$ and assuming that $E \supset \mu_N$, we find in a similar way that

$$\mathrm{wt}\colon \mathcal{C}^D_r(M, \epsilon_N)^{\pm, \leq h} \longrightarrow \mathcal{X}_{\epsilon_N} \simeq \mathcal{X},$$

and we have, over $E \supset \mu_N$,

$$\mathrm{wt}\colon \mathcal{C}_r^D(M,N)^{\le h} = \bigsqcup_{\epsilon_N \in \Delta_N, \varepsilon \in \{\pm 1\}} \mathcal{C}_r^D(M,\epsilon_N)^{\varepsilon,\le h} \longrightarrow \bigsqcup_{\epsilon_N \in \Delta_N} \mathcal{X}_{\epsilon_N} = \mathcal{X}_N.$$

The following result is now a consequence of Theorem 9.3 and Corollary 10.7.

Corollary 11.3 *If $\kappa = (k,\epsilon_p,\epsilon_N) \in \mathbb{N}_{r,N}^{>h-1}$ (resp. $\kappa = (k,\epsilon_p,\epsilon_N) \in \mathbb{N}_{r,\epsilon_N}^{>h-1} \simeq \mathbb{N}_r^{>h-1}$), the set of E-points of the fiber of $\mathcal{C}_r^D(M,N)^{\le h}$ (resp. $\mathcal{C}_r^D(M,N)^{\varepsilon,\le h}$) is in a natural bijection with the E-systems of Hecke eigenvalues occurring in $H^1(\Gamma, V_k\{\epsilon_p\})^{\le h}$ (resp. $H^1(\Gamma, V_k\{\epsilon_p\})(\epsilon_N)^{\varepsilon,\le h} = H^1(\Gamma, V_k\{\epsilon_p\epsilon_N^{-1}\})^{\varepsilon,\le h}$).*

We give the following concrete application of Corollary 10.7.

Corollary 11.4 (Existence of cohomological p-adic families) *Let $\kappa = (k,\epsilon_p,\epsilon_N) \in \mathbb{N}_{r,N}$ be an arithmetic weight. Suppose $\phi \in H^1(\Gamma, V_k\{\epsilon_p\})^{\le h}$ is a nonzero Hecke eigenvector with associated system of eigenvalues $\lambda_\phi\colon \mathcal{T}(\Sigma,\Gamma) \to E$. Suppose, further, that one of the following conditions is satisfied:*

(i) $\mathrm{ord}_p(\lambda_\phi(U_p)) < k+1$ *and* $\dim_E H^1(\Gamma_0^D(p^r N), V_k(E)\{\varepsilon_p\})_{\lambda_\phi} = 1$.

(ii) *$r = 1$ and ϕ is an MN-new cuspidal eigenvector,* $\mathrm{ord}_p(\lambda_\phi(U_p)) < k+1$ *and* $\lambda_\phi(U_p)^2 \ne \lambda_\phi(\langle p\rangle)p^{k+1}$.

Then there exists an open E-affinoid neighbourhood $\Omega_N \subset \mathcal{X}_N$ of κ such that $H^1(\Gamma, \mathcal{D}_\Omega)^{\le h}$ is a free $\mathcal{O}(\Omega_N)$-module of finite rank, and there exists an Hecke eigenvector $\Phi \in H^1(\Gamma, \mathcal{D}_\Omega)^{\le h}$ such that the following hold:

(i) *$\rho_\kappa(\Phi) = \phi$ where ρ_κ is as in (9.7). If $\Phi' \in H^1(\Gamma, \mathcal{D}_\Omega)^{\le h}$ another Hecke eigenvector with $\rho_\kappa(\Phi') = \phi$, then $\Phi' = \alpha\Phi$ with $\alpha \in \mathcal{O}(\Omega_N)$ such that $\alpha(\kappa) = 1$.*

(ii) *If λ_Φ is the system of Hecke eigenvalues attached to Φ, then $H^1(\Gamma, \mathcal{D}_\Omega)_{\lambda_\Phi}^{\le h}$ is a free $\mathcal{O}(\Omega)$-module and*

$$\mathrm{rank}_{\mathcal{O}(\Omega)} H^1(\Gamma, \mathcal{D}_\Omega)_{\lambda_\Phi}^{\le h} = 1.$$

(iii) *If $\kappa' = (k',\epsilon_p',\epsilon_N')$ is an arithmetic weight such that $\kappa' \in \Omega$ and $k' > h-1$, then $\rho_{\kappa'}(\Phi) \ne 0$.*

Remark 11.5 Note that, ϕ being an eigenvector, we have $\phi \in H^1(\Gamma, V_k\{\epsilon_p\})(\epsilon_N)^{\varepsilon,\le h}$ for some (ϵ_N,ε) and $\lambda_\phi(\langle p\rangle) = \epsilon_N(p)$. Then

$$H^1(\Gamma, V_k(E)\{\epsilon_p\})_{\lambda_\phi}^{\le h} = H^1(\Gamma, V_k(E)\{\epsilon_p\})(\epsilon_N)_{\lambda_\phi}^{\varepsilon,\le h},$$
$$H^1(\Gamma, \mathcal{D}_\Omega)_{\lambda_\Phi}^{\le h} = H^1(\Gamma, \mathcal{D}_\Omega)(\epsilon_N)_{\lambda_\Phi}^{\varepsilon,\le h}.$$

Furthermore, λ_ϕ satisfies the second property in Corollary 11.4 if, *e.g.*, ϕ is cuspidal and MNp-new or is a p-stabilization of an N-new cusp form with slope $\le h$.

Proof If $r = 1$ and $\phi \in H^1(\Gamma, V_k\{\epsilon_p\}) =: H$ is an MN-new cuspidal eigenvector such that $\lambda_\phi(U_p)^2 \ne \lambda_\phi(\langle p\rangle)p^{k+1}$, then $\dim_E H_{\lambda_\phi} = 1$ as explained in [7, B.5.7.1].

Now let $r \ge 1$ be arbitrary. Assume that $\dim_E H_{\lambda_\phi} = 1$ and that $\mathrm{ord}_p(\lambda_\phi(U_p)) < k+1$. Then ϕ gives rise to a point over E and, by Remark 10.6, λ_ϕ corresponds to a multiplicity one point. By Corollary 10.7(v), there is a lift λ_Φ of λ_ϕ occurring in

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$H^1(\Gamma,\mathcal{D}_\Omega)^{\le h}$. Let $0 \ne \Phi \in H^1(\Gamma,\mathcal{D}_\Omega)^{\le h}$ be a corresponding eigenvector. Since $\dim_E H_{\lambda_\phi} = 1$, we can assume that $\rho_\kappa(\Phi) = \phi$, scaling Φ if necessary. It follows from Corollary 10.7(v) that $H^1(\Gamma,\mathcal{D}_\Omega)^{\le h}_{\lambda_\Phi}$ is free of rank one. That $\Phi' = \alpha\Phi$ when Φ' is another eigenvector lifting ϕ easily follows from (9.7).

Let e be a basis element of $H^1(\Gamma,\mathcal{D}_\Omega)^{\le h}_{\lambda_\Phi}$, so that $\Phi = \alpha e$ for some $\alpha \in \mathcal{O}(\Omega)$ with $\alpha(k,\epsilon_p) \ne 0$. By the Weierstrass Preparation Theorem we can assume that $\alpha \in \mathcal{O}(\Omega)^\times$ after shrinking Ω in an affinoid neighbourhood of (k,ϵ_p). Then $\rho_{\kappa'}(\Phi) = \alpha(k',\epsilon'_p)\rho_{\kappa'}(e) \ne 0$, because $\rho_{\kappa'}(e) \ne 0$ by (9.7). ∎

In the next section we will focus for simplicity on the case $r = 1$, and we will write $\mathcal{C} = \mathcal{C}_r$. If $\mathcal{C}^{\le h}$ is one of the curves $\mathcal{C}^D(M,N)^{\varepsilon,\le h}$ or $\mathcal{C}^D(M,\epsilon_N)^{\varepsilon,\le h}$ we will write $\mathcal{C}^{\le h,*\text{-fl}}/\mathcal{X}_N^{*\text{-fl}}$ to denote the corresponding flat loci. We remark that, since the weight space $\mathcal{X}_N$ is covered by open affinoid domains $\Omega = \operatorname{Sp} R$ such that R is a principal ideal domain, we have

$$\mathcal{C}^{\le h,\text{fl}}/\mathcal{X}_N^{\text{fl}} = \mathcal{C}^{\le h,M\text{-fl}}/\mathcal{X}_N^{M\text{-fl}} \subset \mathcal{C}^{\le h,T\text{-fl}}/\mathcal{X}_N^{T\text{-fl}} \subset \mathcal{C}^{\le h}/\mathcal{X}_N$$

for our eigencurves.

12 p-adic Jacquet–Langlands Correspondences

12.1 The Eigencurves of Coleman–Mazur and Buzzard

The following theorem, summarizing the construction of eigencurves parametrizing systems of Hecke eigenvalues occurring in spaces of overconvergent p-adic modular forms, is the outgrowth of Coleman's theory of orthonormalizable p-adic Banach algebras and his corresponding functional analytic study of spaces of overconvergent modular forms over affinoids.

Theorem 12.1 (Coleman–Mazur, Buzzard)

(i) *There is an admissible, open, affinoid cover of $\mathcal{X} = \bigcup \Omega_i$ such that the spaces*

$$M_i := S^\dagger_{\Omega_i}\big(\Gamma_0(MD)\cap\Gamma_1(N)\big)^{D\text{-new}}$$

of D-new p-adic families of overconvergent cusp forms over Ω_i admit slope $\le h$ decompositions

$$S^\dagger_{\Omega_i}\big(\Gamma_0(MD)\cap\Gamma_1(N)\big)^{D\text{-new}} = S^\dagger_{\Omega_i}\big(\Gamma_0(MD)\cap\Gamma_1(N)\big)^{D\text{-new},\le h} \oplus S^\dagger_{\Omega_i}\big(\Gamma_0(MD)\cap\Gamma_1(N)\big)^{D\text{-new},>h}$$

with respect to U_p such $M_i := S^\dagger_{\Omega_i}(\Gamma_0(MD)\cap\Gamma_1(N))^{D\text{-new},\le h}$ is an $R_{i,N} = \operatorname{Sp}(\Omega_{i,N})$-module of finite rank, $\Omega_{i,N}$ being inverse image of Ω_i in $\mathcal{X}_N$.

(ii) *Let*

$$\begin{aligned} T_i &= T^1_{R_{i,N}}(pMD,N)^{D\text{-new},\le h}\\ &= \operatorname{im}\Big(\mathcal{T}_{R_{i,N}}\big(\Gamma^1(pMD,N),\Sigma^1_+(pMND,1)\big) \longrightarrow \operatorname{End}_{R_{i,N}}(M_i)\Big).\end{aligned}$$

Then the correspondence $\Omega_{i,N} \mapsto T_i$ extends to a coherent sheaf $\mathcal{T}$ of commutative $\mathcal{O}_{\mathcal{X}_N}$-algebras.

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(iii) *The slope* $\leq h$ Coleman–Mazur–Buzzard eigencurve

$$\mathcal{C}_{\mathrm{CMB}}(MD,N)^{D\text{-}new,\leq h} := \mathrm{Sp}_{\mathcal{O}_{\mathcal{X}_N}} \mathcal{T}$$

comes equipped with a finite, flat weight map

$$\mathrm{wt}\colon \mathcal{C}_{\mathrm{CMB}}(MD,N)^{D\text{-}new,\leq h} \longrightarrow \mathcal{X}_N \longrightarrow \mathcal{X}.$$

When $D = 1$, we abbreviate this to $\mathcal{C}_{\mathrm{CMB}}(MD,N)^{\leq h}$. In a similar way, one can consider, for an admissible cover $\mathcal{X} = \bigcup_i \Omega_i$ by open affinoids Ω_i, the spaces $M^{\dagger}_{\Omega_i}(MD,N)$ of p-adic families of overconvergent p-adic (not necessarily cuspidal) modular forms. A similar construction applies, giving rise to a weight map

$$\mathrm{wt}\colon \mathcal{M}_{\mathrm{CMB}}(MD,N)^{\leq h} \longrightarrow \mathcal{X}_N \longrightarrow \mathcal{X}.$$

When $E \supset \mu_N$, we can define the ϵ_N-components

$$\mathcal{C}_{\mathrm{CMB}}(MD,\epsilon_N)^{\leq h} \quad \text{and} \quad \mathcal{M}_{\mathrm{CMB}}(MD,\epsilon_N)^{\leq h}$$

for $\epsilon_N \in \Delta_N$ and the above curves decomposes as the disjoint union of these ϵ_N-eigencurves.

We apply Proposition 10.9 as follows. Thanks to Theorem 9.3 (see also the discussion at the end of Section 9) we know that $\mathbb{N}_{1,N} \subset \mathcal{X}_N^{\mathrm{fl}}$, where $\mathcal{X}_N^{\mathrm{fl}} = \mathcal{X}_{N,h,M,D}^{\mathrm{fl}}$ is the flat locus defined by any one of the eigencurves $\mathcal{C}^D(M,N)^{\pm,\leq h}/\mathcal{X}_N$, depending on h, M, and D. Let $h > 0$ and let $Z := \mathbb{N}_{1,N}^{>h-1}$. Then $Z \cap \Omega \subset \Omega$ is infinite for every open affinoid $\Omega \subset \mathcal{X}_N^{\mathrm{fl}}$. Since $\mathcal{O}_{\mathcal{X}_N}(\Omega)$ is a principal ideal domain, an easy application of the Weierstrass Preparation Theorem shows that $Z \subset \mathcal{X}_N^{\mathrm{fl}}$ is Zariski dense. Let $\mathcal{C}_{\mathrm{CMB}}(MD,N)^{D\text{-new},\leq h,\mathrm{fl}}/\mathcal{X}_N^{\mathrm{fl}}$ (resp. $\mathcal{M}_{\mathrm{CMB}}(MD,N)^{\leq h,\mathrm{fl}}/\mathcal{X}_N^{\mathrm{fl}}$) be the pull-back of $\mathcal{C}_{\mathrm{CMB}}(MD,N)^{D\text{-new},\leq h}/\mathcal{X}_N$ (resp. $\mathcal{M}_{\mathrm{CMB}}(MD,N)^{\leq h}/\mathcal{X}_N$) to this flat locus. By construction, the conditions required for the application of Proposition 10.9 are fulfilled over $\mathcal{X}_N^{\mathrm{fl}}$. More precisely, take $\mathbf{T} := \mathcal{T}^1(pMD,N)$, so that $\mathcal{O}_{\mathcal{X}_N}(\Omega) \otimes_{\mathbb{Z}} \mathbf{T} = \mathcal{T}^1_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pMD,N)$. We have the following surjections.

(a) By definition,

$$\mathcal{O}_{\mathcal{X}_N}(\Omega) \otimes_{\mathbb{Z}} \mathbf{T} \twoheadrightarrow T^1_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pMD,N)^{D\text{-new},\leq h}.$$

(b) Let $\Bbbk_\Omega\colon \mathbb{Z}_{p,N}^{\times} \to \mathcal{O}_{\mathcal{X}_N}(\Omega)^{\times}$ be the weight corresponding to $\Omega \subset \mathcal{X}_N$. Noticing that $\Bbbk_\Omega(l) \in \mathcal{O}_{\mathcal{X}_N}(\Omega)^{\times}$ and that $W_l^2 = \Bbbk_\Omega(l)$ on $H^i(\Gamma^D(pM,N),\mathcal{D}_\Omega)$ because $\pi_l^2 = l$ and $\mathcal{D}_\Omega$ has central character $\Bbbk_\Omega$ for every $l \mid D$, we have

$$\mathcal{O}_{\mathcal{X}_N}(\Omega) \otimes_{\mathbb{Z}} \mathbf{T} \twoheadrightarrow \frac{\mathcal{T}^1_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pMD,N)}{(T_l^2 - \Bbbk_\Omega(l) \colon l \mid D)} \overset{j^{D,1}}{\to} \frac{\mathcal{T}^1_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pMD,N)}{(W_l^2 - \Bbbk_\Omega(l) \colon l \mid D)} \twoheadrightarrow T^D(pM,N)^{\leq h},$$

where the isomorphism $j^{D,1}$ is given by (7.1).

Suppose first that $D \neq 1$. The comparison Theorem 9.3, together with analogous comparison results for the Coleman-Mazur-Buzzard eigencurve $\mathcal{C}_{\mathrm{CMB}}(MD,N)^{D\text{-new},\leq h}$ imply that the fibers over $(k,\epsilon_p,\epsilon_N) \in Z$ are identified, respectively, with

$$H^1(\Gamma^D(pM,N),V_k(E))(\epsilon_p^{-1}\epsilon_N)^{\pm,\leq h} \text{ and } S_{k+2}(\Gamma^1(pMD,N),E)(\epsilon_p^{-1}\epsilon_N)^{D\text{-new},\leq h}.$$

By the Jacquet–Langlands correspondence, they are isomorphic as $\mathbf{T}$-modules and Proposition 10.9 applies. Suppose now that $D = 1$. In this case, we replace $\mathcal{C}_{\mathrm{CMB}}(MD,N)^{\leq h}$ with $\mathcal{M}_{\mathrm{CMB}}(MD,N)^{\leq h}$ to take into account the presence of Eisenstein cohomology in the fiber of $\mathcal{C}^D(M,N)^{\pm,\leq h}$ over $(k,\epsilon_p,\epsilon_N)$. Then we replace the Jacquet–Langlands correspondence with the Eichler–Shimura isomorphism. Summarizing, we have proved the following result.

Theorem 12.2 *If $D \neq 1$, there is a canonical $\mathcal{X}_N^{fl} = \mathcal{X}_{N,h,M,D}^{fl}$-isomorphism of rigid analytic spaces*

$$\mathcal{C}^D(M,N)_{\mathrm{red}}^{\pm,\leq h,fl} \simeq \mathcal{C}_{CMB}(MD,N)_{\mathrm{red}}^{D\text{-new},\leq h,fl},$$
$$\mathcal{C}^D(M,\epsilon_N)_{\mathrm{red}}^{\pm,\leq h,fl} \simeq \mathcal{C}_{CMB}(MD,\epsilon_N)_{\mathrm{red}}^{D\text{-new},\leq h,fl}.$$

If $D = 1$ there is a canonical $\mathcal{X}_N^{fl}$-isomorphism of rigid analytic spaces

$$\mathcal{C}^1(M,N)_{\mathrm{red}}^{\pm,\leq h,fl} \simeq \mathcal{M}_{CMB}(MD,N)_{\mathrm{red}}^{\leq h,fl}, \quad \mathcal{C}^1(M,\epsilon_N)_{\mathrm{red}}^{\pm,\leq h,fl} \simeq \mathcal{M}_{CMB}(MD,\epsilon_N)_{\mathrm{red}}^{\leq h,fl}.$$

As a concrete manifestation of the Jacquet–Langlands correspondence stated in Theorem 12.2, we give the following result.

Corollary 12.3 *Let $\mathbf{F}(q) = \sum_{n\geq 1} \mathbf{a}_n q^n \in S_\Omega^\dagger(MD,\epsilon_N)^{D\text{-new},\leq h}$ be a D-new Ω-eigenfamily of cuspidal forms with system of eigenvalues $\lambda_{\mathbf{F}}$ such that D is squarefree and divisible by an even numer of primes. Set $\Omega^{fl} := \Omega \cap \mathcal{X}_N^{fl}$, where $\mathcal{X}_N^{fl} = \mathcal{X}_{N,h,M,D}^{fl}$. There exists and eigenfamily $\Phi \in H^1(\Gamma^D(pM,N),\mathcal{D}_{\Omega^{fl}})(\epsilon_N)^{\pm,\leq h}$ with system of eigenvalues $\lambda_\Phi = \lambda_{\mathbf{F}}$.*

Proof We assume $D \neq 1$ for simplicity. Since Ω^{fl} is reduced,

$$\mathcal{C}^D(M,\epsilon_N)^{\pm,\leq h,\mathrm{fl}}(\Omega^{\mathrm{fl}}) = \mathcal{C}^D(M,\epsilon_N)_{\mathrm{red}}^{\pm,\leq h,\mathrm{fl}}(\Omega^{\mathrm{fl}}),$$
$$\mathcal{C}_{\mathrm{CMB}}(MD,\epsilon_N)^{D\text{-new},\leq h}(\Omega^{\mathrm{fl}}) = \mathcal{C}_{\mathrm{CMB}}(MD,\epsilon_N)_{\mathrm{red}}^{D\text{-new},\leq h}(\Omega^{\mathrm{fl}}).$$

Hence, by Theorem 12.2,

$$\mathcal{C}^D(M,\epsilon_N)^{\pm,\leq h,\mathrm{fl}}(\Omega^{\mathrm{fl}}) = \mathcal{C}_{\mathrm{CMB}}(MD,\epsilon_N)^{D\text{-new},\leq h}(\Omega^{\mathrm{fl}}).$$

The eigenvalue $\lambda_{\mathbf{F}}$ gives rise to a section of the weight map, hence an Ω^{fl}-point in $\mathcal{C}_{\mathrm{CMB}}(MD,\epsilon_N)^{D\text{-new},\leq h}(\Omega^{\mathrm{fl}})$. The claim follows from Corollary 10.7(v). ∎

12.2 Moving Between Cohomological Families

Suppose $D = D'M'$ is a factorization of D with D' divisible by an even number of primes. By our running assumption that D is squarefree, D' is too and $(D',M') = 1$. We consider the groups/semigroups:

$$\Gamma = \Gamma^D(pM,N), \qquad \Gamma' = \Gamma^{D'}(pMM',N),$$
$$\Sigma = \Sigma^D(pMN,1), \qquad \Sigma' = \Sigma^{D'}(pMM'N,1).$$

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Let $\mathcal{C}^{D'}(MM',N)^{M'\text{-new},\le h} \to \mathcal{X}_N$ be the eigencurve obtained from the procedure described in Section 11 with $H^1(\Gamma',\mathcal{D}_{\Omega_i})^{M'\text{-new},\le h}$ and

$$T_{R_{i,N}}(pMM',N)^{M'\text{-new},\le h} := \operatorname{im}(\mathcal{T}_{R_{i,N}}(pMN',N) \to \operatorname{End}_{R_{i,N}} H^1(\Gamma',\mathcal{D}_{\Omega_i})^{M'\text{-new}}$$

in place of $H^1(\Gamma',\mathcal{D}_{\Omega_i})^{\le h}$ and $T_{R_{i,N}}(pMM',N)^{\le h}$, respectively. Take

$$\mathbf{T} := \mathcal{T}^{D'}(pMM',N)$$

so that $\mathcal{O}_{\mathcal{X}_N}(\Omega)\otimes_{\mathbb{Z}}\mathbf{T} = \mathcal{T}^{D'}_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pMM',N)$. We have the following surjections.

(a) By definition,

$$\mathcal{O}_{\mathcal{X}_N}(\Omega)\otimes_{\mathbb{Z}}\mathbf{T} \twoheadrightarrow T^1_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pMM',N)^{M'\text{-new},\le h}.$$

(b) Let $\Bbbk_\Omega\colon \mathbb{Z}^\times_{p,N} \to \mathcal{O}_{\mathcal{X}_N}(\Omega)^\times$ be the weight corresponding to $\Omega \subset \mathcal{X}_N$. As above we have

$$\mathcal{O}_{\mathcal{X}_N}(\Omega)\otimes_{\mathbb{Z}}\mathbf{T} \twoheadrightarrow \frac{\mathcal{T}^{D'}_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pMM',N)}{(T_l^2-\Bbbk_\Omega(l) : l \mid M')} \xrightarrow{j^{D,D'}} \frac{\mathcal{T}^{D}_{\mathcal{O}_{\mathcal{X}_N}(\Omega)}(pM,N)}{(W_l^2-\Bbbk_\Omega(l) : l \mid M')} \twoheadrightarrow T^D(pM,N)^{\le h},$$

where the isomorphism $j^{D,D'}$ is given by (7.1).

Let $\mathcal{X}_N^{\mathrm{fl}}$ be the intersection of the flat loci defined by the eigencurves $\mathcal{C}^D(M,N)^{\le h}$ and $\mathcal{C}^{D'}(MM',N)^{M'\text{-new},\le h}$, depending on (h,N,M,M',D,D') and containing the arithmetic weights $\mathbb{N}^{>h-1}_{1,N}$. The following p-adic Jacquet–Langlands correspondence now follows from Proposition 10.9 and the above discussion.

Theorem 12.4 *There is a canonical $\mathcal{X}_N^{fl}$-isomorphism of rigid analytic spaces*

$$\mathcal{C}^D(M,N)^{\le hfl} \simeq \mathcal{C}^{D'}(MM',N)^{M'\text{-}new,\le h,fl}.$$

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