COMPOSITIO MATHEMATICA

Quivers with loops and generalized crystals

Tristan Bozec

Compositio Math. **152** (2016), 1999–2040.

doi:10.1112/S0010437X1600751X

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 152 (2016) 1999–2040
doi:10.1112/S0010437X1600751X

Quivers with loops and generalized crystals

Tristan Bozec

Abstract

In the context of varieties of representations of arbitrary quivers, possibly carrying loops, we define a generalization of Lusztig Lagrangian subvarieties. From the combinatorial study of their irreducible components arises a structure richer than the usual Kashiwara crystals. Along with the geometric study of Nakajima quiver varieties, in the same context, this yields a notion of generalized crystals, coming with a tensor product. As an application, we define the semicanonical basis of the Hopf algebra generalizing quantum groups, which was already equipped with a canonical basis. The irreducible components of the Nakajima varieties provide the family of highest weight crystals associated to dominant weights, as in the classical case.

Introduction

Lusztig defined in [Lus91] Lagrangian subvarieties of the cotangent stack to the moduli stack of representations of a quiver associated to any Kac–Moody algebra. The proof of the Lagrangian character of these varieties was obtained via the study of some natural stratifications of each irreducible component, and then proceeding by induction. The particular combinatorial structure thus attached to the set of irreducible components made it possible for Kashiwara and Saito in [KS97] to relate this variety to the usual quantum group associated to Kac–Moody algebras, via the notion of *crystals*. This later led Lusztig in [Lus00] to define a *semicanonical basis* of this quantum group, indexed by the irreducible components of these Lagrangian varieties.

There is more and more evidence of the relevance of the study of quivers with loops. A particular class of such quivers are the comet-shaped quivers, which have recently been used by Hausel *et al.* in their study of the topology of character varieties, where the number of loops at the central vertex is the genus of the considered curve (see [HR08, HLR13]). We can also see quivers with loops appearing in a work of Nakajima relating quiver varieties with branching (see [Nak09]), as in the work of Maulik and Okounkov about quantum cohomology (see [MO12]).

Kang *et al.* generalized these varieties in the framework of generalized Kac–Moody algebras in [KKS09], using quivers with loops. In this case, one has to impose a somewhat unnatural restriction on the regularity of the maps associated to the loops.

In this article we define a generalization of such Lagrangian varieties in the case of arbitrary quivers, possibly carrying loops. As opposed to the Lagrangian varieties constructed by Lusztig, which consisted in nilpotent representations, we have to consider here slightly more general representations. That this is necessary is already clear from the Jordan quiver case. Note that our Lagrangian variety is strictly larger than the one considered in [KKS09] and has many more irreducible components. Our proof of the Lagrangian character is also based on induction, but

Received 20 January 2015, accepted in final form 15 December 2015, published online 20 July 2016.
2010 Mathematics Subject Classification 16G20, 17B37 (primary).
Keywords: quiver varieties, quantum groups, Kashiwara crystals.

with nontrivial first steps, consisting in the study of quivers with one vertex but possible loops. From our proof emerges a new combinatorial structure on the set of irreducible components, which is more general than the usual crystals, in that there are now more operators associated to a vertex with loops; see Proposition 1.11.

In a second section we study Nakajima varieties, still in the context of arbitrary quivers. We construct Lagrangian subvarieties, and generalize the notion of a tensor product of their irreducible components, introduced by Nakajima in [Nak01]. The geometric statements obtained in the two first sections give the intuition of the way crystals and their tensor product should be generalized, which is done in a third section. The algebraic definition and study of the crystal $\mathcal{B}(\infty)$ enable us to define a semicanonical basis for the positive part of the generalized quantum group U^+ defined in [Boz15], where it is already equipped with a canonical basis, built via the theory of Lusztig perverse sheaves associated to quivers with loops. We finally use our study of Nakajima quiver varieties to produce a geometric realization of the generalized crystals $\mathcal{B}(\lambda)$.

1. Lusztig quiver varieties

Let Q be a quiver, defined by a set of vertices I and a set of oriented edges $\Omega = \{h : s(h) \to t(h)\}$. We denote by $\bar{h} : t(h) \to s(h)$ the opposite arrow of $h \in \Omega$, and $\bar{Q}$ the quiver $(I, H = \Omega \sqcup \bar{\Omega})$, where $\bar{\Omega} = \{\bar{h} \mid h \in \Omega\}$: each arrow is replaced by a pair of arrows, one in each direction, and we set $\epsilon(h) = 1$ if $h \in \Omega$, $\epsilon(h) = -1$ if $h \in \bar{\Omega}$. Note that the definition of $\bar{h}$ still makes sense if $h \in \bar{\Omega}$. We denote by $\Omega(i)$ the set of loops of Ω at i, and call i *imaginary* if $\omega_i = |\Omega(i)| \geqslant 1$, *real* otherwise. Denote by I^{im} (respectively, I^{re}) the set of imaginary vertices (respectively, real vertices). Finally, set $H(i) = \Omega(i) \sqcup \bar{\Omega}(i)$.

We work over the field of complex numbers $\mathbb{C}$.

For any pair of I-graded $\mathbb{C}$-vector spaces $V = (V_i)_{i\in I}$ and $V' = (V'_i)_{i\in I}$, we set

$$\bar{E}(V, V') = \bigoplus_{h\in H} \operatorname{Hom}(V_{s(h)}, V'_{t(h)}).$$

For any dimension vector $\nu = (\nu_i)_{i\in I}$, we fix an I-graded $\mathbb{C}$-vector space V_ν of dimension ν, and put $\bar{E}_\nu = \bar{E}(V_\nu, V_\nu)$. The space $\bar{E}_\nu = \bar{E}(V_\nu, V_\nu)$ is endowed with a symplectic form

$$\omega_\nu(x, x') = \sum_{h\in H} \operatorname{Tr}(\epsilon(h) x_h x'_{\bar{h}}),$$

which is preserved by the natural action of $G_\nu = \prod_{i\in I} \mathrm{GL}_{\nu_i}(\mathbb{C})$ on $\bar{E}_\nu$. The associated moment map $\mu_\nu : \bar{E}_\nu \to \mathfrak{g}_\nu = \bigoplus_{i\in I} \operatorname{End}(V_\nu)_i$ is given by

$$\mu_\nu(x) = \sum_{h\in H} \epsilon(h) x_{\bar{h}} x_h.$$

Here we have identified $\mathfrak{g}_\nu^*$ with $\mathfrak{g}_\nu$ via the trace pairing.

Definition 1.1. An element $x \in \bar{E}_\nu$ is said to be *seminilpotent* if there exists an I-graded flag $\mathsf{W} = (\mathsf{W}_0 = V_\nu \supset \cdots \supset \mathsf{W}_r = \{0\})$ of V_ν such that for all k the vector space $\mathsf{W}_k/\mathsf{W}_{k+1}$ is concentrated on one vertex and

$$\begin{aligned} x_h(\mathsf{W}_\bullet) &\subseteq \mathsf{W}_{\bullet+1} && \text{if } h \in \Omega,\\ x_h(\mathsf{W}_\bullet) &\subseteq \mathsf{W}_\bullet && \text{if } h \in \bar{\Omega}. \end{aligned}$$

We put $\Lambda(\nu) = \{x \in \mu_\nu^{-1}(0) \mid x \text{ seminilpotent}\}$.

Lemma 1.1. *The variety $\Lambda(\nu)$ is isotropic.*

Proof. We use the following general fact (see e.g. [KS94, § 8.3]).

Proposition 1.2. *Let X be a smooth algebraic variety, Y a projective variety and Z a smooth closed algebraic subvariety of $X \times Y$. Consider the Lagrangian subvariety $\Lambda = T^*_Z(X \times Y)$ of $T^*(X \times Y)$. Then the image of the projection $q : \Lambda \cap (T^*X \times T^*_Y Y) \to T^*X$ is isotropic.*

We apply this result to $X = \bigoplus_{h\in\Omega} \mathrm{End}(V_{\nu_{s(h)}}, V_{\nu_{t(h)}})$, Y the I-graded flag variety of V_ν and

$$Z = \{(x, \mathsf{W}) \in X \times Y \mid x(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1}\}.$$

In this case, we get

$$\begin{aligned} T^*X &= \bar{E}_\nu, \\ T^*Y &= \{(\mathsf{W}, \xi) \in Y \times \mathfrak{g}_\nu \mid \xi(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1}\}, \\ \Lambda &= \left\{ (x, \mathsf{W}, \xi) \;\middle|\; \begin{array}{l} \xi = \sum_{h\in H} \epsilon(h) x_{\bar{h}} x_h \\ \forall h \in \Omega, x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1} \text{ and } x_{\bar{h}}(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet \end{array} \right\}, \\ \operatorname{Im} q &= \left\{ x \in \bar{E}_\nu \;\middle|\; \begin{array}{l} \mu_\nu(x) = 0 \text{ and there exists } \mathsf{W} \in Y \text{ such that} \\ \forall h \in \Omega, x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1} \text{ and } x_{\bar{h}}(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet \end{array} \right\} \end{aligned}$$

and hence $\Lambda(\nu) \subseteq \operatorname{Im} q$, which proves the lemma. □

1.1 The case of the Jordan quiver

This case is very well known. For $\nu \in \mathbb{N}$, we have

$$\Lambda(\nu) = \{(x, y) \in (\mathrm{End}\,\mathbb{C}^\nu)^2 \mid x \text{ nilpotent and } [x, y] = 0\} = \bigcup_\lambda T^*_{\mathcal{O}_\lambda}(\mathrm{End}\,\mathbb{C}^\nu),$$

where $\mathcal{O}_\lambda$ is the nilpotent orbit associated to the partition λ of ν (we write $|\lambda| = \nu$). Therefore, $\Lambda(\nu)$ is a Lagrangian subvariety of $(\mathrm{End}\,\mathbb{C}^\nu)^2$, and its irreducible components are the closures of the conormal bundles to the nilpotent orbits.

1.2 The case of the quiver with one vertex and $g \geqslant 2$ loops

For $\nu \in \mathbb{N}$, $\Lambda(\nu)$ is the subvariety of $(\mathrm{End}\,\mathbb{C}^\nu)^{2g}$ with elements $(x_i, y_i)_{1\leqslant i\leqslant g}$ such that:

- ▷ there exists a flag W of $\mathbb{C}^\nu$ such that $x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1}$ and $y_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet$;
- ▷ $\sum_{1\leqslant i\leqslant g} [x_i, y_i] = 0$.

We will denote by $\mathcal{C}_\nu$ the set of compositions of ν, i.e. tuples $\mathsf{c} = (\mathsf{c}_1, \dots, \mathsf{c}_r)$ of $\mathbb{N}_{>0}$ such that

$$|\mathsf{c}| = \sum_{1\leqslant k\leqslant r} \mathsf{c}_k = \nu.$$

We will also often forget the index $1 \leqslant i \leqslant g$ in the rest of this section, which is dedicated to the proof of the following theorem.

Theorem 1.2. *The subvariety $\Lambda(\nu) \subseteq (\mathrm{End}\,\mathbb{C}^\nu)^{2g}$ is Lagrangian, its irreducible components being parametrized by $\mathcal{C}_\nu$.*

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Notation 1.3. For $(x_i, y_i) \in \Lambda(\nu)$, we define $\mathsf{W}_0(x_i, y_i) = \mathbb{C}^\nu$; then by induction $\mathsf{W}_{k+1}(x_i, y_i)$ is the smallest subspace of $\mathbb{C}^\nu$ containing $\sum x_i(\mathsf{W}_k(x_i, y_i))$ and stable by (x_i, y_i). By seminilpotency, we can define r to be the first power such that $\mathsf{W}_r(x_i, y_i) = \{0\}$. Although r depends on (x_i, y_i), we do not write it explicitly.

Let $\mathsf{c}(x_i, y_i)$ denote the composition associated to the flag $\mathsf{W}_\bullet(x_i, y_i)$:

$$\mathsf{c}_k(x_i, y_i) = \dim \frac{\mathsf{W}_{k-1}(x_i, y_i)}{\mathsf{W}_k(x_i, y_i)}.$$

For every $\mathsf{c} \in \mathcal{C}_\nu$, we define a locally closed subvariety

$$\Lambda(\mathsf{c}) = \left\{ (x_i, y_i) \in \Lambda(\nu) \;\middle|\; \dim \frac{\mathsf{W}_{\bullet-1}(x_i, y_i)}{\mathsf{W}_\bullet(x_i, y_i)} = \mathsf{c} \right\} \subseteq \Lambda(\nu).$$

Then, if $\delta = (\delta_1, \ldots, \delta_{r-1}) \in \mathbb{N}^{r-1}$, let $\Lambda(\mathsf{c})_\delta \subseteq \Lambda(\mathsf{c})$ be the locally closed subvariety defined by

$$\dim\left(\bigcap_{1 \leqslant i \leqslant g} \ker\{\xi \mapsto y_i^{(k)}\xi - \xi y_i^{(k+1)}\} \right) = \delta_k,$$

where

$$y_i^{(k)} \in \mathrm{End}\left(\frac{\mathsf{W}_{k-1}(x_i, y_i)}{\mathsf{W}_k(x_i, y_i)} \right)$$

is induced by y_i and

$$\xi \in \mathrm{Hom}\left(\frac{\mathsf{W}_k(x_i, y_i)}{\mathsf{W}_{k+1}(x_i, y_i)}, \frac{\mathsf{W}_{k-1}(x_i, y_i)}{\mathsf{W}_k(x_i, y_i)} \right).$$

Set $l = \mathsf{c}_1$; then

$$\check{\Lambda}(\mathsf{c})_\delta = \left\{ (x_i, y_i, \mathfrak{X}, \beta, \gamma) \;\middle|\; \begin{array}{l} (x_i, y_i) \in \Lambda(\mathsf{c})_\delta, \\ \mathsf{W}_1(x_i, y_i) \oplus \mathfrak{X} = \mathbb{C}^\nu, \\ \beta : \mathsf{W}_1(x_i, y_i) \xrightarrow{\sim} \mathbb{C}^{\nu - l} \text{ and } \gamma : \mathfrak{X} \xrightarrow{\sim} \mathbb{C}^l \end{array} \right\}$$

and

$$\pi_{\mathsf{c},\delta} \;\left|\; \begin{array}{l} \check{\Lambda}(\mathsf{c})_\delta \to \Lambda(\mathsf{c}^-)_{\delta^-} \times (\mathrm{End}\,\mathbb{C}^l)^g \\ (x_i, y_i, \mathfrak{X}, \beta, \gamma) \mapsto (\beta_*(x_i, y_i)_{\mathsf{W}_1}, \gamma_*(y_i)_{\mathfrak{X}}) \end{array} \right. ,$$

where $\mathsf{c}^- = (\mathsf{c}_2, \ldots, \mathsf{c}_r)$ and $\delta^- = (\delta_2, \ldots, \delta_{r-1})$. Finally, let $(\Lambda(\mathsf{c}^-)_{\delta^-} \times (\mathrm{End}\,\mathbb{C}^l)^g)_{\mathsf{c},\delta}$ denote the image of $\pi_{\mathsf{c},\delta}$.

Proposition 1.4. *The morphism $\pi_{\mathsf{c},\delta}$ is smooth over its image, with connected fibers of dimension $\nu^2 + (2g-1)l(\nu - l) + \delta_1$ whenever $\Lambda(\mathsf{c})_\delta \neq \emptyset$.*

Proof. Let $(x_i, y_i, z_i) \in (\Lambda(\mathsf{c}^-)_{\delta^-} \times (\mathrm{End}\,\mathbb{C}^l)^g)_{\mathsf{c},\delta}$. Let $\mathfrak{W}$ and $\mathfrak{X}$ be two supplementary subspaces of $\mathbb{C}^\nu$ such that $\dim \mathfrak{X} = l$, together with two isomorphisms

$$\beta : \mathfrak{W} \xrightarrow{\sim} \mathbb{C}^{\nu - l} \quad \text{and} \quad \gamma : \mathfrak{X} \xrightarrow{\sim} \mathbb{C}^l.$$

We identify x_i, y_i and z_i with $\beta^*(x_i, y_i)$ and $\gamma^* z_i$, and define an element (X_i, Y_i) in the fiber of (x_i, y_i, z_i) by setting

$$\begin{aligned} (X_i, Y_i)_{\mathfrak{W}} &= (x_i, y_i), \\ (X_i, Y_i)_{\mathfrak{X}} &= (0, z_i), \\ (X_i, Y_i)_{|\mathfrak{X}}^{|\mathfrak{W}} &= (u_i, v_i) \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{W})^{2g}. \end{aligned}$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Then

$$\mu_\nu(X_i,Y_i)=0 \Leftrightarrow \phi(u_i,v_i)=\sum_{i=1}^{g}(x_iv_i+u_iz_i-y_iu_i)=0$$

and, for $\xi\in \operatorname{Hom}(\mathfrak{W},\mathfrak{X})$,

$$\begin{aligned}
\forall (u_i,v_i),\quad \operatorname{Tr}(\xi\phi(u_i,v_i))=0 &\Leftrightarrow \begin{cases}\forall i,\forall u_i, & \operatorname{Tr}(\xi(u_iz_i-y_iu_i))=0,\\ \forall i,\forall v_i, & \operatorname{Tr}(\xi x_iv_i)=0,\end{cases}\\
&\Leftrightarrow \begin{cases}\forall i,\forall u_i, & \operatorname{Tr}((z_i\xi-\xi y_i)u_i)=0,\\ \forall i,\forall v_i, & \operatorname{Tr}(\xi x_iv_i)=0,\end{cases}\\
&\Leftrightarrow \begin{cases}\forall i, & z_i\xi=\xi y_i,\\ \forall i, & \xi x_i=0,\end{cases}\\
&\Leftrightarrow \begin{cases}\mathsf{W}_1(x_i,y_i)\subseteq \ker\xi,\\ \forall i, \quad z_i\xi^{(1)}=\xi^{(1)}y_i^{(1)},\end{cases}
\end{aligned}$$

where $\xi^{(1)}$ denotes the map $\mathfrak{W}/\mathsf{W}_1(x_i,y_i)\to\mathfrak{X}$ induced by ξ. Since (x_i,y_i,z_i) is in the image of $\pi_{\mathsf{c},\delta}$, the image of ϕ is of codimension δ_1, and thus its kernel is of dimension $(2g-1)l(\nu-l)+\delta_1$.

Moreover, if we denote by $u_i^{(1)}$ the map $\mathfrak{X}\to\mathfrak{W}/\mathsf{W}_1(x_i,y_i)$ induced by u_i, $\mathsf{W}_1(X_i,Y_i)=\mathfrak{W}$ if and only if the space spanned by the action of $(y_i^{(1)})_i$ on $\sum_i \operatorname{Im} u_i^{(1)}$ is $\mathfrak{W}/\mathsf{W}_1(x_i,y_i)$. This condition defines an open subset of $\ker\phi$.

We end the proof by noticing that the set of elements $(\mathfrak{W},\mathfrak{X},\beta,\gamma)$ is isomorphic to $\mathrm{GL}_\nu(\mathbb{C})$. □

Proposition 1.5. *The variety $\Lambda(\mathsf{c})_0$ is not empty.*

Proof. Fix W of dimension c and define x_1 such that

$$x_1(\mathsf{W}_\bullet)\subseteq \mathsf{W}_{\bullet+1},$$
$$x_1{}^{|\mathsf{W}_k/\mathsf{W}_{k+1}}_{|\mathsf{W}_{k-1}/\mathsf{W}_k}\neq 0.$$

We define inductively an element y_1 stabilizing W such that:

- ▷ the action of $y_1{}^{(k+1)}$ on $\operatorname{Im}(x_1{}^{|\mathsf{W}_k/\mathsf{W}_{k+1}}_{|\mathsf{W}_{k-1}/\mathsf{W}_k})$ spans $\mathsf{W}_k/\mathsf{W}_{k+1}$;
- ▷ $\operatorname{Spec} y_1{}^{(k)}\cap \operatorname{Spec} y_1{}^{(k+1)}=\emptyset$.

We finally set $x_2=-x_1, y_2=y_1$ and $x_i=y_i=0$ for $i>2$. This yields an element (x_i,y_i) in $\Lambda(\mathsf{c})_0$. □

Corollary 1.6. *For any $\mathsf{c}\in\mathcal{C}_\nu$, $\Lambda(\mathsf{c})_0$ is irreducible of dimension $g\nu^2$.*

Proof. We argue by induction on r. If $\mathsf{c}=(\nu)$, we have $\Lambda(\mathsf{c})_0=\Lambda(\mathsf{c})=(\operatorname{End}\mathbb{C}^\nu)^g$, which is irreducible of dimension $g\nu^2$. For the induction step, Propositions 1.4 and 1.5 assure us that $\check{\Lambda}(\mathsf{c})_0$ is irreducible of dimension

$$\nu^2+(2g-1)l(\nu-l)+\dim(\Lambda(\mathsf{c}^-)_0\times(\operatorname{End}\mathbb{C}^l)^g)_{\mathsf{c},0}=\nu^2+(2g-1)l(\nu-l)+g(\nu-l)^2+gl^2,$$

since $(\Lambda(\mathsf{c}^-)_0\times(\operatorname{End}\mathbb{C}^l)^g)_{\mathsf{c},0}$ is a nonempty subvariety of $\Lambda(\mathsf{c}^-)_0\times(\operatorname{End}\mathbb{C}^l)^g$, irreducible of dimension $g(\nu-l)^2+gl^2$ by our induction hypothesis. Moreover,

$$\check{\Lambda}(\mathsf{c})_0\to\Lambda(\mathsf{c})_0$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

being a principal bundle with fibers of dimension $\nu^2 - l(\nu - l)$, we get that $\Lambda(\mathsf{c})_0$ is irreducible of dimension

$$\nu^2 + (2g-1)l(\nu-l) + g(\nu-l)^2 + gl^2 - \nu^2 + l(\nu-l) = g\nu^2. \qquad \square$$

Lemma 1.7. *Let V and W be two vector spaces, and $k \geqslant 0$. For any $(u,v) \in \operatorname{End} V \times \operatorname{End} W$, we set*

$$\mathcal{C}(u,v) = \{x \in \operatorname{Hom}(V,W) \mid xu = vx\},$$
$$(\operatorname{End} V \times \operatorname{End} W)_k = \{(u,v) \in \operatorname{End} V \times \operatorname{End} W \mid \dim \mathcal{C}(u,v) = k\}.$$

Then we have

$$\operatorname{codim}(\operatorname{End} V \times \operatorname{End} W)_k \geqslant k.$$

Proof. The restriction of an endomorphism a to a generalized eigenspace associated to an eigenvalue η will be denoted by $a_\eta = \eta\,\mathrm{id} + \tilde{a}_\eta$. As usual, the nilpotent orbit associated to a partition ξ will be denoted by $\mathcal{O}_\xi$. We have

$$\begin{aligned}
&\operatorname{codim}(\operatorname{End} V \times \operatorname{End} W)_k\\
&\quad = \operatorname{codim}\left\{(u,v) \,\middle|\, \sum_{\alpha,\beta} \dim \mathcal{C}(u_\alpha, v_\beta) = k\right\}\\
&\quad = \operatorname{codim}\left\{(u,v) \,\middle|\, \sum_{\alpha \in \operatorname{Spec} u \cap \operatorname{Spec} v} \dim \mathcal{C}(u_\alpha, v_\alpha) = k\right\}\\
&\quad = \operatorname{codim}\left\{(u,v) \,\middle|\, \sum_{\alpha} \dim \mathcal{C}(\tilde{u}_\alpha, \tilde{v}_\alpha) = k\right\}\\
&\quad = \operatorname{codim}\left\{(u,v) \,\middle|\, \begin{array}{l} (\tilde{u}_\alpha, \tilde{v}_\alpha) \in \mathcal{O}_{\lambda_\alpha} \times \mathcal{O}_{\mu_\alpha} \\ \sum_\alpha \sum_j (\lambda'_\alpha)_j (\mu'_\alpha)_j = k \end{array}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\operatorname{codim}(\operatorname{End} V \times \operatorname{End} W)_k \geqslant k\\
&\quad \Leftrightarrow \sum_\alpha (\operatorname{codim} \mathcal{O}_{\lambda_\alpha} + \operatorname{codim} \mathcal{O}_{\mu_\alpha} - 1) \geqslant \sum_\alpha \sum_j (\lambda'_\alpha)_j (\mu'_\alpha)_j\\
&\quad \Leftrightarrow \sum_\alpha \left(\sum_j (\lambda'_\alpha)_j^2 + \sum_j (\mu'_\alpha)_j^2 - 1 \right) \geqslant \sum_\alpha \sum_j (\lambda'_\alpha)_j (\mu'_\alpha)_j,
\end{aligned}$$

which is clear. $\square$

Proposition 1.8. *If $\delta \neq 0$, we have $\dim \Lambda(\mathsf{c})_\delta < g\nu^2$.*

Proof. It is enough to show that if $\delta_1 > 0$, we have

$$\dim(\Lambda(\mathsf{c}^-)_{\delta^-} \times (\operatorname{End} \mathbb{C}^l)^g)_{\mathsf{c},\delta} + \delta_1 < \dim(\Lambda(\mathsf{c}^-)_0 \times (\operatorname{End} \mathbb{C}^l)^g).$$

This is a consequence of the previous lemma (recall that $g \geqslant 2$). Indeed, if we set

$$((\operatorname{End} V)^g \times (\operatorname{End} W)^g)_k = \{(u_i, v_i) \mid \dim \cap_i \mathcal{C}(u_i, v_i) = k\},$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

we have

$$((\operatorname{End} V)^g \times (\operatorname{End} W)^g)_k \subseteq \prod_{i=1}^{g} (\operatorname{End} V \times \operatorname{End} W)_{k_i}$$

for some $k_i \geqslant k$, and thus

$$\operatorname{codim}((\operatorname{End} V)^g \times (\operatorname{End} W)^g)_k \geqslant \sum_i \operatorname{codim}(\operatorname{End} V \times \operatorname{End} W)_{k_i} \geqslant \sum_i k_i \geqslant gk > k. \qquad \square$$

The following proposition concludes the proof of Theorem 1.2.

PROPOSITION 1.9. *Every irreducible component of $\Lambda(\mathsf{c})$ is of dimension larger than or equal to $g\nu^2$.*

Proof. We first prove the result for the following variety:

$$\tilde{\Lambda}(\mathsf{c}) = \{((x_i, y_i), \mathsf{W}) \in \Lambda(\nu) \times Y_{\mathsf{c}} \mid x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1} \text{ and } y_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet\},$$

where Y_{c} denotes the variety of flags of $\mathbb{C}^\nu$ of dimension c. We use the following notation, analogous to Lemma 1.1:

$$\begin{aligned} X &= \{(x_i)_{1\leqslant i\leqslant g} \in (\operatorname{End} \mathbb{C}^\nu)^g\},\\ Z &= \{((x_i)_{1\leqslant i\leqslant g}, \mathsf{W}) \mid x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1}\} \subseteq X \times Y_{\mathsf{c}}. \end{aligned}$$

We get

$$\begin{aligned} T^*X &= \{(x_i, y_i)_{1\leqslant i\leqslant g} \in (\operatorname{End} \mathbb{C}^\nu)^{2g}\},\\ T^*Y_{\mathsf{c}} &= \{(\mathsf{W}, K) \in Y_{\mathsf{c}} \times \operatorname{End} \mathbb{C}^\nu \mid K(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1}\} \end{aligned}$$

and

$$T^*_Z(X \times Y_{\mathsf{c}}) = \left\{ ((x_i, y_i), \mathrm{F}, K) \;\middle|\; \begin{array}{l} \sum_{1\leqslant i\leqslant g} [x_i, y_i] = K, \\ x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet+1} \text{ and } y_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet \end{array} \right\},$$

which is a pure Lagrangian subvariety of $T^*(X \times Y_{\mathsf{c}})$, of dimension $g\nu^2 + \dim Y_{\mathsf{c}}$. Since T^*Y_{c} is irreducible of dimension $2\dim Y_{\mathsf{c}}$, the irreducible components of the fibers of $T^*_Z(X \times Y_{\mathsf{c}}) \to T^*Y_{\mathsf{c}}$ are of dimension larger than or equal to $g\nu^2 - \dim Y_{\mathsf{c}}$. We denote by $\tilde{\Lambda}_{\mathsf{W}}$ the fiber above $(\mathsf{W}, 0)$, and by P the stabilizer of W in G_ν. Since G_ν and P are irreducible, we get that the components of

$$\tilde{\Lambda}(\mathsf{c}) = G_\nu \times_P \tilde{\Lambda}_{\mathsf{W}}$$

are of dimension larger than or equal to $\dim Y_{\mathsf{c}} + (g\nu^2 - \dim Y_{\mathsf{c}}) = g\nu^2$.

We extend this result to $\Lambda(\mathsf{c})$, noticing that

$$\begin{aligned} \Lambda(\mathsf{c}) &\hookrightarrow \tilde{\Lambda}(\mathsf{c})\\ (x_i, y_i) &\mapsto (x_i, y_i, \mathsf{W}_\bullet(x_i, y_i)) \end{aligned}$$

defines an open embedding. $\square$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

1.3 The general case

Denote by $a_{i,j}$ the number of edges of Ω such that $s(h) = i$ and $t(h) = j$, and denote by

$$C = (2\delta_{i,j} - a_{i,j} - a_{j,i})$$

the Cartan matrix of Q. For every $\nu, \beta \in \mathbb{N}^I$ and $j \in I$, we put

$$\langle \nu, \beta \rangle = \sum_{i \in I} \nu_i \beta_i,$$
$$e_j = (\delta_{i,j})_{i \in I}.$$

Definition 1.3. For every subset $i \in I$ and every $x \in \Lambda(\nu)$, we denote by $\mathfrak{I}_i(x)$ the subspace of V_ν spanned by the action of x on $\bigoplus_{j \neq i} V_j$. Then, for $l > 0$, we set

$$\Lambda(\nu)_{i,l} = \{x \in \Lambda(\nu) \mid \operatorname{codim} \mathfrak{I}_i(x) = l e_i\}.$$

Remark 1.10. By the definition of seminilpotency, we have

$$\Lambda(\nu) = \bigcup_{i \in I, l \geqslant 1} \Lambda(\nu)_{i,l}.$$

Indeed, if $x \in \Lambda(\nu)$, there exists an I-graded flag $(\mathsf{W}_0 \supset \cdots \supset \mathsf{W}_r)$ such that (x, W) satisfies Definition 1.1. Therefore, there exist $i \in I$ and $l > 0$ such that $\mathsf{W}_0/\mathsf{W}_1 \simeq V_{le_i}$, and thus $x \in \bigcup_{k \geqslant l} \Lambda(\nu)_{i,k}$.

Proposition 1.11. *There exist a variety $\check{\Lambda}(\nu)_{i,l}$ and a diagram*

$$\begin{tikzcd} & \check{\Lambda}(\nu)_{i,l} \arrow[dl, "q_{i,l}"'] \arrow[dr, "p_{i,l}"] & \\ \Lambda(\nu)_{i,l} & & \Lambda(\nu - le_i)_{i,0} \times \Lambda(le_i) \end{tikzcd}$$

such that $p_{i,l}$ and $q_{i,l}$ are smooth with connected fibers, inducing a bijection

$$\operatorname{Irr} \Lambda(\nu)_{i,l} \xrightarrow{\sim} \operatorname{Irr} \Lambda(\nu - le_i)_{i,0} \times \operatorname{Irr} \Lambda(le_i).$$

Proof. In this proof we will denote by $I(V, V')$ the set of I-graded isomorphisms between two I-graded spaces V and V' of the same I-graded dimension. We set

$$\check{\Lambda}(\nu)_{i,l} = \left\{ (x, \mathfrak{X}, \beta, \gamma) \;\middle|\; \begin{array}{l} x \in \Lambda(\nu)_{i,l}, \\ \mathfrak{X} \text{ } I\text{-graded and } \mathfrak{I}_i(x) \oplus \mathfrak{X} = V_\nu, \\ \beta \in I(\mathfrak{I}_i(x), V_{\nu - le_i}) \text{ and } \gamma \in I(\mathfrak{X}, V_{le_i}) \end{array} \right\},$$

and

$$p_{i,l} \left| \begin{array}{l} \check{\Lambda}(\nu)_{i,l} \to \Lambda(\nu - le_i)_{i,0} \times \Lambda(le_i) \\ (x, \mathfrak{X}, \beta, \gamma) \mapsto (\beta_*(x_{\mathfrak{I}_i(x)}), \gamma_*(x_{\mathfrak{X}})). \end{array} \right.$$

We study the fibers of $p_{i,l}$: take $y \in \Lambda(\nu - le_i)_{i,0}$ and $z \in \Lambda(le_i)$ and consider $\mathfrak{I}$ and $\mathfrak{X}$, two supplementary I-graded subspaces of V_ν, such that $\dim \mathfrak{X} = le_i$, together with two isomorphisms

$$\beta \in I(\mathfrak{I}, V_{\nu - le_i}) \quad \text{and} \quad \gamma \in I(\mathfrak{X}, V_{le_i}).$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

We identify y and z with $\beta^* y$ and $\gamma^* z$, and we define a preimage x by setting $x_{|\mathfrak{I}}^{|\mathfrak{I}} = y$, $x_{|\mathfrak{X}}^{|\mathfrak{X}} = z$ and $x_{|\mathfrak{X}}^{|\mathfrak{I}} = \eta \in \bar{E}(\mathfrak{X}, \mathfrak{I})$. In order to get $\mu_\nu(x) = 0$, η must satisfy the following relation:

$$\phi(\eta) = \sum_{h \in H : s(h) = i} \epsilon(h)(y_{\bar{h}}\eta_h + \eta_{\bar{h}} z_h) = 0.$$

We need to show that ϕ is surjective to conclude the proof. Consider $\xi \in \operatorname{Hom}(\mathfrak{I}_i, \mathfrak{X}_i)$ such that $\operatorname{Tr}(\phi(\eta)\xi) = 0$ for every η. For every edge h such that $s(h) = i \neq j = t(h)$ and every η_h, we have

$$\begin{aligned} 0 &= \operatorname{Tr}(y_{\bar{h}}\eta_h\xi) \\ &= \operatorname{Tr}(\xi y_{\bar{h}}\eta_h). \end{aligned}$$

Hence, $\xi y_{\bar{h}} = 0$, and $\operatorname{Im} y_{\bar{h}} \subseteq \ker \xi$. Now consider a loop $h \in H(i)$. For every η_h, we have

$$\begin{aligned} 0 &= \operatorname{Tr}((\eta_h z_{\bar{h}} - y_{\bar{h}}\eta_h)\xi) \\ &= \operatorname{Tr}(\eta_h(z_{\bar{h}}\xi - \xi y_{\bar{h}})). \end{aligned}$$

Hence, $\xi y_{\bar{h}} = z_{\bar{h}}\xi$ and therefore $\ker \xi$ is stable by $y_{\bar{h}}$. As $\operatorname{codim} \mathfrak{I}_i(y) = 0$, we get $\xi = 0$, which finishes the proof. □

We can now state the following theorem, which answers a question asked in [Li].

THEOREM 1.4. *The subvariety $\Lambda(\nu)$ of $\bar{E}_\nu$ is Lagrangian.*

Proof. Since this subvariety is isotropic by Lemma 1.1, we just have to show that the irreducible components of $\Lambda(\nu)$ are of dimension $\langle \nu, (1 - C/2)\nu \rangle$. We proceed by induction on $|\nu| = \sum_i \nu_i$, the first step corresponding to the one-vertex quiver case which has already been treated: we have seen that $\Lambda(le_i)$ is of dimension $\langle le_i, (1 - C/2)le_i \rangle$.

Next consider $C \in \operatorname{Irr} \Lambda(\nu)$ for some ν. By Remark 1.10, there exist $i \in I$ and $l \geqslant 1$ such that $C \cap \Lambda(\nu)_{i,l}$ is dense in C. Let $\check{C} = (C_1, C_2)$ be the couple of irreducible components corresponding to C via the bijection obtained in Proposition 1.11:

$$\operatorname{Irr} \Lambda(\nu)_{i,l} \xrightarrow{\sim} \operatorname{Irr} \Lambda(\nu - le_i)_{i,0} \times \operatorname{Irr} \Lambda(le_i).$$

We also know by the proof of Proposition 1.11 that the fibers of $p_{i,l}$ are of dimension

$$\langle \nu, \nu \rangle + \langle \nu - le_i, (1 - C)le_i \rangle.$$

Since $q_{i,l}$ is a principal bundle with fibers of dimension $\langle \nu, \nu \rangle - \langle le_i, \nu - le_i \rangle$, we get

$$\dim C = \dim \check{C} + \langle \nu - le_i, (2 - C)le_i \rangle.$$

But $\Lambda(\nu - le_i)_{i,0}$ is open in $\Lambda(\nu - le_i)$, so we can use our induction hypothesis and the first step to write

$$\dim \check{C} = \langle \nu - le_i, (1 - C/2)(\nu - le_i) \rangle + \langle le_i, (1 - C/2)le_i \rangle$$

and thus obtain

$$\dim C = \langle \nu, (1 - C/2)\nu \rangle. \qquad \square$$

1.4 Constructible functions

Following [Lus00], we denote by $\mathcal{M}(\nu)$ the $\mathbb{Q}$-vector space of constructible functions $\Lambda(\nu) \to \mathbb{Q}$, which are constant on any G_ν-orbit. Put $\mathcal{M} = \bigoplus_{\nu \geqslant 0} \mathcal{M}(\nu)$, which is a graded algebra once equipped with the product $*$ defined in [Lus00, § 2.1].

For $Z \in \operatorname{Irr} \Lambda(\nu)$ and $f \in \mathcal{M}(\nu)$, we put $\rho_Z(f) = c$ if $Z \cap f^{-1}(c)$ is an open dense subset of Z.

If $i \in I^{\mathrm{im}}$ and (l) denotes the trivial composition or partition of l, we denote by $1_{i,l}$ the characteristic function of the associated irreducible component $Z_{i,(l)} \in \operatorname{Irr} \Lambda(le_i)$ (the component of elements x such that $x_h = 0$ for any loop $h \in \Omega(i)$). If $i \notin I^{\mathrm{im}}$, we just denote by 1_i the function mapping to 1 the only point in $\Lambda(e_i)$.

We have $1_{i,l} \in \mathcal{M}(le_i)$ for $i \in I^{\mathrm{im}}$ and $1_i \in \mathcal{M}(e_i)$ for $i \notin I^{\mathrm{im}}$. We denote by $\mathcal{M}_\circ \subseteq \mathcal{M}$ the subalgebra generated by these functions.

Lemma 1.12. *Suppose that Q has one vertex $\circ$ and $g \geqslant 1$ loop(s). For every $Z \in \operatorname{Irr} \Lambda(\nu)$, there exists $f \in \mathcal{M}_\circ(\nu)$ such that $\rho_Z(f) = 1$ and $\rho_{Z'}(f) = 0$ for $Z' \neq Z$.*

Proof. We denote by Z_{c} the irreducible component associated to the partition (respectively, composition) c of ν if $g = 1$ (respectively, $g \geqslant 2$). By convention, if $g = 1$, Z_{c} will denote the component associated to the orbit $\mathcal{O}_{\mathsf{c}}$ defined by

$$x \in \mathcal{O}_{\mathsf{c}} \Leftrightarrow \dim \ker x^i = \sum_{1 \leqslant k \leqslant i} \mathsf{c}_k.$$

If $g \geqslant 2$, we remark that by trace duality, we can assume that Z_{c} is the closure of $\check{\Lambda}_{\mathsf{c}}$ defined by

$$(x_i, y_i)_{1 \leqslant i \leqslant g} \in \check{\Lambda}_{\mathsf{c}} \Leftrightarrow \dim \mathrm{K}_i = \sum_{1 \leqslant k \leqslant i} \mathsf{c}_k,$$

where we define by induction $\mathrm{K}_0 = \{0\}$, then K_{j+1} as the biggest subspace of $\cap_i x_i^{-1}(\mathrm{K}_j)$ stable by (x_i, y_i). From now on, $\mathsf{c} = (\mathsf{c}_1, \ldots, \mathsf{c}_r)$ will denote indistinctly a partition or a composition depending on the value of g. We define an order by

$$\mathsf{c} \preceq \mathsf{c}' \quad \text{if and only if for any } i \geqslant 1 \text{ we have } \sum_{1 \leqslant k \leqslant i} \mathsf{c}_k \leqslant \sum_{1 \leqslant k \leqslant i} {\mathsf{c}'}_k.$$

Therefore, setting $\tilde{1}_{\mathsf{c}} = 1_{\mathsf{c}_r} * \cdots * 1_{\mathsf{c}_1}$, where $1_l = 1_{\circ,l}$, we get

$$x \in Z_{\mathsf{c}}, \quad \tilde{1}_{\mathsf{c}'}(x) \neq 0 \;\Rightarrow\; \mathsf{c}' \preceq \mathsf{c}.$$

For $\mathsf{c} = (\nu)$, we have $\tilde{1}_{\mathsf{c}} = 1_\nu$, which is the characteristic function of Z_{c}, and we put $1_{\mathsf{c}} = \tilde{1}_{\mathsf{c}}$ in this case. Then, by induction,

$$1_{\mathsf{c}} = \tilde{1}_{\mathsf{c}} - \sum_{\mathsf{c}' \prec \mathsf{c}} \rho_{Z_{\mathsf{c}'}}(\tilde{1}_{\mathsf{c}}) 1_{\mathsf{c}'}$$

has the expected property. □

Notation 1.13. ▷ From now on, if c corresponds to an irreducible component of $\Lambda(|\mathsf{c}|e_i)$, we will denote by $1_{i,\mathsf{c}}$ the function corresponding to 1_{c} in the previous proof.

▷ For $Z \in \operatorname{Irr} \Lambda(\nu)_{i,l}$, we denote by $\epsilon_i(Z) \in \operatorname{Irr} \Lambda(le_i)$ the composition of the second projection with the bijection obtained in Proposition 1.11. Note that $|\epsilon_i(Z)| = l$.

PROPOSITION 1.14. *For every $Z \in \operatorname{Irr} \Lambda(\nu)$, there exists $f \in \mathcal{M}_\circ(\nu)$ such that $\rho_Z(f) = 1$ and $\rho_{Z'}(f) = 0$ if $Z' \neq Z$.*

Proof. We proceed as in [Lus00, Lemma 2.4], by induction on $|\nu|$. The first step consists in Lemma 1.12. Then consider $Z \in \operatorname{Irr} \Lambda(\nu)$. There exist $i \in I$ and $l > 0$ such that $Z \cap \Lambda(\nu)_{i,l}$ is dense in Z.

We now proceed by descending induction on l. There is nothing to say if $l > \nu_i$.

Otherwise, let $(Z', Z_{\mathsf{c}}) \in \operatorname{Irr} \Lambda(\nu - le_i)_{i,0} \times \operatorname{Irr} \Lambda(le_i)$ be the pair of components corresponding to Z. By the induction hypothesis on ν, there exists $g \in \mathcal{M}_\circ(\nu - le_i)$ such that $\rho_{\overline{Z'}}(g) = 1$ and $\rho_Y(g) = 0$ if $\overline{Z'} \neq Y \in \operatorname{Irr} \Lambda(\nu - le_i)$.

Then we set $\tilde{f} = 1_{i,\mathsf{c}} * g \in \mathcal{M}_\circ(\nu)$, and get:

- $\rho_Z(\tilde{f}) = 1$;
- $\rho_{Z'}(\tilde{f}) = 0$ if $Z' \in \operatorname{Irr} \Lambda(\nu) \backslash Z$ satisfies $|\epsilon_i(Z')| = l$;
- $\tilde{f}(x) = 0$ if $x \in \Lambda(\nu)_{i,<l}$, so that $\rho_{Z'}(\tilde{f}) = 0$ if $|\epsilon_i(Z')| < l$.

If $|\epsilon_i(Z')| > l$, we use the induction hypothesis on l: there exists $f_{Z'} \in \mathcal{M}_\circ(\nu)$ such that $\rho_{Z'}(f_{Z'}) = 1$ and $\rho_{Z''}(f_{Z'}) = 0$ if $Z'' \in \operatorname{Irr} \Lambda(\nu) \backslash Z'$. We end the proof by setting

$$f = \tilde{f} - \sum_{Z' : |\epsilon_i(Z')| > l} \rho_{Z'}(\tilde{f}) f_{Z'}. \qquad \square$$

2. Nakajima quiver varieties

Fix an I-graded vector space W of dimension $\lambda = (\lambda_i)_{i \in I}$. For any dimension vector $\nu = (\nu_i)_{i \in I}$, we still fix an I-graded $\mathbb{C}$-vector space $V_\nu = ((V_\nu)_i = V_{\nu_i e_i})_{i \in I}$ of dimension ν. We will denote by $(x, f, g) = ((x_h)_{h \in H}, (f_i)_{i \in I}, (g_i)_{i \in I})$ the elements of the following space:

$$E(V, \lambda) = \bar{E}(V, V) \oplus \bigoplus_{i \in I} \operatorname{Hom}(V_i, W_i) \bigoplus_{i \in I} \operatorname{Hom}(W_i, V_i)$$

defined for any I-graded space V, and put $E_{\nu,\lambda} = E(V_\nu, \lambda)$ for any dimension vector ν. This space is endowed with a symplectic form

$$\omega_{\nu,\lambda}((x, f, g), (x', f', g')) = \sum_{h \in H} \operatorname{Tr}(\epsilon(h) x_h x'_{\bar{h}}) + \sum_{i \in I} \operatorname{Tr}(g_i f'_i - g'_i f_i),$$

which is preserved by the natural action of $G_\nu = \prod_{i \in I} \mathrm{GL}_{\nu_i}(\mathbb{C})$ on $E_{\nu,\lambda}$. The associated moment map $\mu_{\nu,\lambda} : E_{\nu,\lambda} \to \mathfrak{g}_\nu = \bigoplus_{i \in I} \operatorname{End}(V_\nu)_i$ is given by

$$\mu_{\nu,\lambda}(x, f, g) = \left(g_i f_i + \sum_{h \in H : s(h) = i} \epsilon(h) x_{\bar{h}} x_h \right)_{i \in I}.$$

Here we have identified $\mathfrak{g}_\nu^*$ with $\mathfrak{g}_\nu$ via the trace pairing. Put

$$\mathsf{M}_\circ(\nu, \lambda) = \mu_{\nu,\lambda}^{-1}(0).$$

DEFINITION 2.1. Set $\chi : G_\nu \to \mathbb{C}^*$, $(g_i)_{i \in I} \mapsto \prod_{i \in I} \det^{-1} g_i$. We denote by

$$\mathfrak{M}_\circ(\nu, \lambda) = \mathsf{M}_\circ(\nu, \lambda) /\!\!/ G_\nu$$
$$\mathfrak{M}(\nu, \lambda) = \mathsf{M}_\circ(\nu, \lambda) /_\chi G_\nu$$

the geometric and symplectic quotients (with respect to χ).

Proposition 2.1. *An element $(x,f,g) \in \mathsf{M}_\circ(\nu,\lambda)$ is stable with respect to χ if and only if the only x-stable subspace of* $\ker f$ *is* $\{0\}$. *Set*

$$\mathsf{M}(\nu,\lambda) = \{(x,f,g) \in \mathsf{M}_\circ(\nu,\lambda) \mid (x,f,g) \text{ stable}\};$$

then $\mathfrak{M}(\nu,\lambda) = \mathsf{M}(\nu,\lambda)/\!\!/G_\nu$.

2.1 A crystal-type structure

Definition 2.2. An element $(x,f,g) \in E_{\nu,\lambda}$ is said to be *seminilpotent* if $x \in \bar{E}_\nu$ is, according to Definition 1.1. We put

$$\mathsf{L}_\circ(\nu,\lambda) = \{(x,f,0) \in \mathsf{M}_\circ(\nu,\lambda) \mid x \text{ seminilpotent}\} \subseteq \mathsf{M}_\circ(\nu,\lambda)$$

and define $\mathsf{L}(\nu,\lambda) \subseteq \mathsf{M}(\nu,\lambda)$ in the same way. Finally, set

$$\begin{aligned} \mathfrak{L}_\circ(\nu,\lambda) &= \mathsf{L}_\circ(\nu,\lambda)/\!\!/G_\nu, \\ \mathfrak{L}(\nu,\lambda) &= \mathsf{L}_\circ(\nu,\lambda)/_\chi G_\nu = \mathsf{L}(\nu,\lambda)/\!\!/G_\nu. \end{aligned}$$

We will simply denote by (x,f) the elements of $\mathsf{L}_\circ(\nu,\lambda)$.

There is an alternative definition of $\mathfrak{L}(\nu,\lambda)$. Define a $\mathbb{C}^*$-action on $\mathfrak{M}(\nu,\lambda)$ by

$$t \diamond [x,f,g] = [t^{(1+\epsilon)/2}x, f, tg].$$

When the only oriented cycles of Q are the potential loops, we have

$$\mathfrak{L}(\nu,\lambda) = \Big\{[x,f,g] \mid \exists \lim_{t\to\infty} t \diamond [x,f,g]\Big\}.$$

By the same arguments as in [Nak94, Theorem 5.8], we have the following result.

Proposition 2.2. *The subvariety $\mathfrak{L}(\nu,\lambda) \subset \mathfrak{M}(\nu,\lambda)$ is Lagrangian.*

When Q has cycles, this is a consequence of §§ 2.1 and 2.2.

Definition 2.3. For every subset $i \in I$ and every $(x,f,g) \in \mathsf{M}_\circ(\nu,\lambda)$, we denote by $\mathfrak{I}_i(x,f,g)$ the subspace of V_ν spanned by the action of $x \oplus g$ on $(\bigoplus_{j\neq i} V_j) \oplus W_i$. Then, for $l \geqslant 0$, we set

$$\mathsf{M}_\circ(\nu,\lambda)_{i,l} = \{x \in \mathsf{M}_\circ(\nu,\lambda) \mid \operatorname{codim} \mathfrak{I}_i(x,f,g) = le_i\}.$$

We define $\mathsf{M}(\nu,\lambda)_{i,l}$, $\mathsf{L}_\circ(\nu,\lambda)_{i,l}$ and $\mathsf{L}(\nu,\lambda)_{i,l}$ in the same way. The quantity $\operatorname{codim} \mathfrak{I}_i(x,f,g)$ being stable on G_ν-orbits, the notation $\mathfrak{M}_\circ(\nu,\lambda)_{i,l}$, $\mathfrak{M}(\nu,\lambda)_{i,l}$, $\mathfrak{L}_\circ(\nu,\lambda)_{i,l}$ and $\mathfrak{L}(\nu,\lambda)_{i,l}$ also make sense.

Remark 2.3. – As in Remark 1.10, we have

$$\mathsf{L}_\circ(\nu,\lambda) = \bigsqcup_{i\in I, l\geqslant 1} \mathsf{L}_\circ(\nu,\lambda)_{i,l}.$$

– Note that $\mathsf{L}_\circ(le_i, 0) = \Lambda(le_i)$.

PROPOSITION 2.4. *There exist a variety* $\check{\mathsf{M}}_\circ(\nu,\lambda)_{i,l}$ *and a diagram*

$$\begin{tikzcd} & \check{\mathsf{M}}_\circ(\nu,\lambda)_{i,l} \arrow[dl, "q_{i,l}"'] \arrow[dr, "p_{i,l}"] & \\ \mathsf{M}_\circ(\nu,\lambda)_{i,l} & & \mathsf{M}_\circ(\nu-le_i,\lambda)_{i,0}\times \mathsf{M}_\circ(le_i,0) \end{tikzcd} \tag{2.5}$$

such that $p_{i,l}$ *and* $q_{i,l}$ *are smooth with connected fibers, inducing a bijection*

$$\operatorname{Irr}\mathsf{M}_\circ(\nu,\lambda)_{i,l} \xrightarrow{\sim} \operatorname{Irr}\mathsf{M}_\circ(\nu-le_i,\lambda)_{i,0}\times\operatorname{Irr}\mathsf{M}_\circ(le_i,0).$$

Proof. In this proof we will denote by $I(V,V')$ the set of I-graded isomorphisms between two I-graded spaces V and V' of the same I-graded dimension. We set

$$\check{\mathsf{M}}_\circ(\nu,\lambda)_{i,l}=\left\{(x,f,g,\mathfrak{X},\beta,\gamma)\ \middle|\ \begin{array}{l}(x,f,g)\in\mathsf{M}_\circ(\nu,\lambda)_{i,l}\\ \mathfrak{X}\ I\text{-graded and }\mathfrak{I}_i(x,f,g)\oplus\mathfrak{X}=V_\nu\\ \beta\in I(\mathfrak{I}_i(x,f,g),V_{\nu-le_i})\\ \gamma\in I(\mathfrak{X},V_{le_i})\end{array}\right\}$$

and

$$p_{i,l}\ \left|\ \begin{array}{l}\check{\mathsf{M}}_\circ(\nu,\lambda)_{i,l}\to\mathsf{M}_\circ(\nu-le_i,\lambda)_{i,0}\times\mathsf{M}_\circ(le_i,0)\\ (x,f,g,\mathfrak{X},\beta,\gamma)\mapsto(\beta_*(xf,g)_{\mathfrak{I}_i(x,f,g)},\gamma_*(x,f,g)_{\mathfrak{X}}).\end{array}\right.$$

We study the fibers of $p_{i,l}$: take $(x,f,g)\in\mathsf{M}_\circ(\nu-le_i,\lambda)_{i,0}$ and $(z,0,0)\in\mathsf{M}_\circ(le_i,0)$ and consider $\mathfrak{I}$ and $\mathfrak{X}$, two supplementary I-graded subspaces of V_ν, such that $\dim\mathfrak{X}=le_i$, together with two isomorphisms

$$\beta\in I(\mathfrak{I},V_{\nu-le_i})\quad\text{and}\quad\gamma\in I(\mathfrak{X},V_{le_i}).$$

We identify (x,f,g) and z with $\beta^*(x,f,g)$ and γ^*z, and we define a preimage (X,F,G) by setting $(X,F,G)^{|\mathfrak{I}\oplus W}_{|\mathfrak{I}\oplus W}=(x,f,g)$, $X^{|\mathfrak{X}}_{|\mathfrak{X}}=z$ and

$$(X,F)^{|\mathfrak{I}\oplus W}_{|\mathfrak{X}}=(\eta,\theta)\in\bar{E}(\mathfrak{X},\mathfrak{I})\oplus\operatorname{Hom}(\mathfrak{X}_i,W_i).$$

In order to get $\mu_{\nu,\lambda}(X,F,G)=0$, (η,θ) must satisfy the following relation:

$$\psi(\eta,\theta)=\sum_{h\in H:s(h)=i}\epsilon(h)(x_{\bar{h}}\eta_h+\eta_{\bar{h}}z_h)+g_i\theta_i=0.$$

We need to show that ψ is surjective to conclude the proof. Consider $\xi\in\operatorname{Hom}(\mathfrak{I}_i,\mathfrak{X}_i)$ such that $\operatorname{Tr}(\psi(\eta,\theta)\xi)=0$ for every (η,θ). Then we have for every edge $h\in H$ such that $s(h)=i\neq j=t(h)$ and for every η_h,

$$\begin{aligned}0&=\operatorname{Tr}(x_{\bar{h}}\eta_h\xi)\\&=\operatorname{Tr}(\eta_h\xi x_{\bar{h}}).\end{aligned}$$

Hence, $\xi x_{\bar{h}}=0$ and $\operatorname{Im}x_{\bar{h}}\subseteq\ker\xi$. We also have $\operatorname{Tr}(g_i\theta_i\xi)=0$ for every θ_i, so we similarly get $\operatorname{Im}g_i\subseteq\ker\xi$. Now consider a loop $h\in H$ at i. We have for every η_h

$$\begin{aligned}0&=\operatorname{Tr}((x_{\bar{h}}\eta_h-\eta_h z_{\bar{h}})\xi)\\&=\operatorname{Tr}(\eta_h(\xi x_{\bar{h}}-z_{\bar{h}}\xi))\end{aligned}$$

and hence $\xi x_{\bar{h}}=z_{\bar{h}}\xi$ and therefore $\ker\xi$ is stable by $x_{\bar{h}}$. Since $(x,f,g)\in\mathsf{M}_\circ(\nu-le_i,\lambda)_{i,0}$, we get $\xi=0$, which finishes the proof. $\square$

Corollary 2.6. *We also have a bijection*

$$\mathsf{I}_\circ(\nu,\lambda)_{i,l} : \operatorname{Irr}\mathsf{L}_\circ(\nu,\lambda)_{i,l} \xrightarrow{\sim} \operatorname{Irr}\mathsf{L}_\circ(\nu-le_i,\lambda)_{i,0}\times\operatorname{Irr}\mathsf{L}_\circ(le_i,0).$$

Proof. The image of a seminilpotent element by $p_{i,l}$ is a pair of seminilpotent elements, and the fiber of $p_{i,l}$ over a pair of seminilpotent elements consists in seminilpotent elements. □

2.2 Extension to the stable locus

We will often use the following well-known fact.

Lemma 2.7. *Consider $y \in \operatorname{End}\mathfrak{I}$ and $z \in \operatorname{End}\mathfrak{X}$ such that $\operatorname{Spec} y \cap \operatorname{Spec} z = \emptyset$. If $\mathbb{C}[y]\cdot v = \mathfrak{I}$ and $\mathbb{C}[z]\cdot v' = \mathfrak{X}$ for some $v \in \mathfrak{I}$ and $v' \in \mathfrak{X}$, then $\mathbb{C}[y\oplus z]\cdot v\oplus v' = \mathfrak{I}\oplus\mathfrak{X}$.*

Notation 2.8. Let i be imaginary and put $\Omega(i) = \{b_1,\dots,b_{\omega_i}\}$. For every $(x,f)\in\mathsf{L}_\circ(\nu,\lambda)$, we set $\sigma_i(x) = x^*_{\bar{b}_1}$, where $*$ stands for the duality

$$\begin{aligned}\operatorname{End} V &\to \operatorname{End} V^* = \operatorname{End}(\operatorname{Hom}(V,\mathbb{C}))\\ u &\mapsto u^* = [\phi\mapsto\phi\circ u]\end{aligned}$$

for every $\mathbb{C}$-vector space V.

Lemma 2.9. *For every $C \in \operatorname{Irr}\Lambda(le_i)$, there exists $x\in C$ such that*

$$\exists\psi\in V^*_{le_i}, \mathbb{C}[\sigma_i(x)]\cdot\psi = V^*_{le_i}.$$

Proof. It is a consequence of §§ 1.1 and 1.2. If $\omega_i = 1$ and λ is a partition of l, denote by μ the conjugate partition of λ. Let $x\in\mathcal{O}_\lambda$ be defined in a base

$$e = (e_{1,1},\dots,e_{1,\mu_1},\dots,e_{r,1},\dots,e_{r,\mu_r})$$

by

$$x^*_{b_1} = \begin{pmatrix} J_{\mu_1} & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & J_{\mu_r}\end{pmatrix} \quad\text{and}\quad x^*_{\bar{b}_1} = \begin{pmatrix} t_1I_{\mu_1}+J_{\mu_1} & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & t_rI_{\mu_r}+J_{\mu_r}\end{pmatrix}$$

where the t_i are all distinct and nonzero, and

$$J_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & 0 & 0\end{pmatrix}.$$

It is enough to consider ψ with nonzero coordinates relatively to $(e_{1,\mu_1},\dots,e_{r,\mu_r})$ to get $\mathbb{C}[\sigma_i(x)]\cdot\psi = V^*_{le_i}$. If $\omega_i \geqslant 2$, we use the proof of Proposition 1.5: in any irreducible component we can define x such that there exists v such that $\mathbb{C}[x_{\bar{b}_1}]\cdot v = V_{le_i}$ ($x_{\bar{b}_i}$ corresponds to y_i in the aforementioned proof, x_{b_i} to x_i). We get the result by duality. □

Remark 2.10. Note that the case $\omega_i = 1$ is very well known, since it corresponds to the case of the Hilbert scheme of points in the plane.

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Definition 2.4. Set

$$\mathsf{L}(\lambda) := \bigcup_\nu \mathsf{L}(\nu,\lambda) \subseteq \bigcup_\nu \mathsf{L}_\circ(\nu,\lambda) =: \mathsf{L}_\circ(\lambda)$$

and define $\mathrm{B}(\lambda)$ as the smallest subset of $\operatorname{Irr}\mathsf{L}_\circ(\lambda)$ containing the only element of $\operatorname{Irr}\mathsf{L}_\circ(0,\lambda)$, and stable by $\mathsf{l}_\circ(\nu,\lambda)_{i,l}^{-1}(-,\operatorname{Irr}\Lambda(le_i))$ for ν,i,l such that:

- ▷ $\langle e_i, \lambda - C\nu\rangle \geqslant -l$ if $i\in I^{\mathrm{re}}$;
- ▷ $\lambda_i + \sum_{h\in H_i}\nu_{t(h)} > 0$ if $i\in I^{\mathrm{im}}$,

where $H_i = \{h\in H \mid i = s(h)\neq t(h)\}$.

Lemma 2.11. *For every $i\in I^{\mathrm{im}}$, we write $\Omega(i)=\{b_{i,1},\dots,b_{i,\omega_i}\}$. For every $C\in\mathrm{B}(\lambda)$, there exists $(x,f)\in C$ such that*

$$\begin{cases}(x,f)\ \textit{stable}\\ \forall i\in I^{\mathrm{im}}, \exists \phi_i\in W_i^*\oplus\left(\bigoplus_{h\in H_i} V^*_{\nu_{t(h)}}\right), \mathbb{C}[\sigma_i(x)]\cdot\Sigma_i(x,f)(\phi_i) = V^*_{\nu_i}\end{cases} \tag{2.12}$$

where $\Sigma_i(x,f) = f_i^* + \sum_{h\in H_i} x_h^*$.

Proof. We proceed by induction on ν, with the first step consisting in the case of $C\in\mathrm{B}(\lambda)\cap\operatorname{Irr}\mathsf{L}_\circ(le_i,\lambda)$ for some $l>0$. If $i\notin I^{\mathrm{im}}$, we have $l\leqslant\lambda_i$ by definition of $\mathrm{B}(\lambda)$ and hence we can find $(x,f)\in C$ such that (2.12) holds, since it is equivalent here to f injective. If $i\in I^{\mathrm{im}}$, we have $\lambda_i>0$ by definition of $\mathrm{B}(\lambda)$, and we can use Lemma 2.9.

Now consider $C\in\mathrm{B}(\lambda)\cap\operatorname{Irr}\mathsf{L}_\circ(\nu,\lambda)_{i,l}$ for some ν and $l>0$, and set $(C_1,C_2)=\mathsf{l}_\circ(\nu,\lambda)_{i,l}(C)$. First assume that $i\notin I^{\mathrm{im}}$. Thanks to the induction hypothesis, we can pick $((x,f),z)\in C_1\times C_2$ such that (x,f) satisfies (2.12). Following the notation used in the proof of Proposition 2.4, we build an element of C satisfying (2.12) by choosing (η,θ) such that $\theta+\sum_{h\in H_i}\eta_h$ is injective with values in a supplementary of $\operatorname{Im}(f_i+\sum_{h\in H_i}x_h)$ in $W_i\oplus\ker(\sum_{h\in H_i}x_{\bar h})$: it is possible since $l+\langle e_i,\lambda-C\nu\rangle\geqslant 0$ by definition of $\mathrm{B}(\lambda)$.

If $i\in I^{\mathrm{im}}$, take $(x,f)\in C_1$ satisfying (2.12) and $z\in C_2$ such that

$$\begin{cases}\operatorname{Spec} x_{\bar b_{i,1}}\cap\operatorname{Spec} z_{\bar b_{i,1}}=\emptyset\\ \exists\psi\in V^*_{le_i}, \mathbb{C}[\sigma_i(z)]\cdot\psi = V^*_{le_i},\end{cases}$$

which is possible thanks to Lemma 2.9. Still following the notation of the proof of Proposition 2.4, we build an element of C mapped to $((x,f),z)$ by considering (η,θ) such that

$$\left(\theta^*+\sum_{h\in H_i}\eta_h^*\right)(\phi_i)=\psi,$$

where $\phi_i\in W_i^*\oplus(\bigoplus_{h\in H_i}V^*_{\nu_{t(h)}})$ satisfies $\mathbb{C}[\sigma_i(x)]\cdot\Sigma_i(x,f)(\phi_i)=\mathfrak{I}^*$ (we use the induction hypothesis), which is possible even if $\mathfrak{I}=\{0\}$ since we have $W_i^*\oplus(\bigoplus_{h\in H_i}V^*_{\nu_{t(h)}})\neq\{0\}$ by definition of $\mathrm{B}(\lambda)$. Put $\eta_{b_{i,j}}=\eta_{\bar b_{i,j}}=0$ for every $j\geqslant 2$, so that

$$\psi_i(\eta,\theta)=0\Leftrightarrow x_{\bar b_{i,1}}\eta_{b_{i,1}}-\eta_{b_{i,1}}z_{\bar b_{i,1}}=\sum_{h\in H_i}\epsilon(h)(x_{\bar h}\eta_h+\eta_{\bar h}z_h).$$

Hence, we can choose $\eta_{b_{i,1}}$ in order to satisfy the right-hand-side equation, since

$$\operatorname{Spec} x_{\bar b_{i,1}}\cap\operatorname{Spec} z_{\bar b_{i,1}}=\emptyset\Rightarrow(\eta_{b_{i,1}}\mapsto x_{\bar b_{i,1}}\eta_{b_{i,1}}-\eta_{b_{i,1}}z_{\bar b_{i,1}})\text{ invertible.}$$

Thanks to Lemma 2.7, $(X,F)\in C$ satisfies

$$\mathbb{C}[\sigma_i(X)]\cdot\Sigma_i(X,F)(\phi_i)=V_{\nu_i}^*.$$

We finally have to check the stability of (X,F) to conclude the proof. Consider $S\subseteq\ker F$ stable by X. We have $S\cap\mathfrak{I}=\{0\}$ by stability of (x,f); thus, $S\simeq S_i$ and we see S as a subspace of $\ker F\cap(\bigcap_{h\in H_i}\ker X_h)$. But then S^* is stable by $\sigma_i(X)$ and contains $\operatorname{Im}F^*+\sum_{h\in H_i}\operatorname{Im}X_h^*$, and thus ϕ_i. Hence, $S^*=V_{\nu_i}$ and $S=\{0\}$. □

Proposition 2.13. *We have* $\mathrm{B}(\lambda)=\operatorname{Irr}\mathsf{L}(\lambda)$.

Proof. Thanks to Lemma 2.11, we have $\mathrm{B}(\lambda)\subseteq\operatorname{Irr}\mathsf{L}(\lambda)$. Consider $Z\in\operatorname{Irr}\mathsf{L}(\nu,\lambda)_{i,l}\backslash\mathrm{B}(\lambda)$ for some $l>0$. We know (cf. [Nak98, Corollary 4.6]) that if $i\in I^{\mathrm{re}}$, we necessarily have $l+\langle e_i,\nu-C\lambda\rangle\geqslant 0$ and, thus, by definition of $\mathrm{B}(\lambda)$,

$$\mathsf{I}_\circ(\nu,\lambda)_{i,l}(Z)\in(\operatorname{Irr}\mathsf{L}(\nu-le_i,\lambda)\backslash\mathrm{B}(\lambda))\times\operatorname{Irr}\Lambda(le_i).$$

If $i\in I^{\mathrm{im}}$, $Z\in\operatorname{Irr}\mathsf{L}(\nu,\lambda)_{i,l}$ necessarily implies $\lambda_i+\sum_{h\in H_i}\nu_{t(h)}>0$, and we get to the same conclusion. By descending induction on ν, we obtain that the only irreducible component of $\mathsf{L}(0,\lambda)$ does not belong to $\mathrm{B}(\lambda)$, which is absurd. □

Corollary 2.14. *Take* $i\in I^{\mathrm{im}}$ *and assume that* $\operatorname{Irr}\mathsf{L}(\nu,\lambda)_{i,l}\subseteq\mathrm{B}(\lambda)$. *We have the following commutative diagram.*

$$\begin{tikzcd}
\operatorname{Irr}\mathsf{L}(\nu,\lambda)_{i,l} \arrow[r, "\mathsf{I}(\nu,\lambda)_{i,l}", "\sim"'] \arrow[d, "\sim"'] & \operatorname{Irr}\mathsf{L}(\nu-le_i,\lambda)_{i,0}\times\operatorname{Irr}\Lambda(le_i) \arrow[d, "\sim"] \\
\operatorname{Irr}\mathfrak{L}(\nu,\lambda)_{i,l} \arrow[r, "\mathsf{I}(\nu,\lambda)_{i,l}", "\sim"'] & \operatorname{Irr}\mathfrak{L}(\nu-le_i,\lambda)_{i,0}\times\operatorname{Irr}\Lambda(le_i)
\end{tikzcd} \tag{2.15}$$

Proof. By definition of stability, the action of G_ν on $\mathsf{L}(\nu,\lambda)$ is free. □

2.3 Tensor product on Irr $\mathfrak{L}$

2.3.1 *Another Lagrangian subvariety.* Embed W in a $(\lambda+\lambda')$-dimensional I-graded vector space, and fix a supplementary subspace W' of W. We still denote by $I(X,Y)$ the set of I-graded isomorphisms between two I-graded spaces X and Y.

For every $\mathbf{v}\in\mathbb{N}^I$, denote by $\mathsf{Z}_\circ(\mathbf{v})\subseteq\mathsf{M}_\circ(\mathbf{v},\lambda+\lambda')$ elements (x,f,g) such that there exists an I-graded subspace V of $V_{\mathbf{v}}$ satisfying:

(i) $x(V)\subseteq V$;
(ii) $f(V)\subseteq W$;
(iii) $g(W\oplus W')\subseteq V$;
(iv) $g(W)=\{0\}$

and denote by $V(x,f,g)$ the larger x-stable subspace of $f^{-1}(W)$ containing $\operatorname{Im}g$. We will then denote by $\widetilde{\mathsf{Z}}_\circ(\mathbf{v})\subset\mathsf{Z}_\circ(\mathbf{v})$ the subvariety of elements (x,f,g) such that

$$(x,f)_{|V\times V}^{|V\times W}\quad\text{and}\quad(x,f)_{|(V_{\mathbf{v}}/V)\times(V_{\mathbf{v}}/V)}^{|(V_{\mathbf{v}}/V)\times(W\oplus W'/W)}\text{ are seminilpotents,}$$

where we have written V instead of $V(x,f,g)$. We get a stratification of $\widetilde{\mathsf{Z}}_\circ(\mathbf{v})$ by setting, for any ν,ν' such that $\nu+\nu'=\mathbf{v}$,

$$\widetilde{\mathsf{Z}}_\circ(\nu,\nu')=\{(x,f,g)\in\widetilde{\mathsf{Z}}_\circ(\nu+\nu') \mid \dim V(x,f,g)=\nu\}.$$

Define the following incidence variety:

$$\check{\mathsf{Z}}_\circ(\nu,\nu')=\left\{(x,f,g,V',\beta)\;\middle|\;\begin{array}{l}(x,f,g)\in\widetilde{\mathsf{Z}}_\circ(\nu,\nu'),\\ V(x,f,g)\oplus V'=V_{\nu+\nu'},\\ \beta\in I(V(x,f,g),V_\nu)\times I(V',V_{\nu'})\end{array}\right\}.$$

By definition of $V(x,f,g)$ (again denoted by V hereunder), we have

$$(x,f,g)\in\mathsf{Z}_\circ(\mathbf{v})\Rightarrow(x,f)_{|(V_{\mathbf{v}}/V)\times(V_{\mathbf{v}}/V)}^{|(V_{\mathbf{v}}/V)\times(W\oplus W'/W)}\text{ stable}$$

and hence the following application is well defined:

$$\mathsf{T}_\circ\;\left|\;\begin{array}{l}\check{\mathsf{Z}}_\circ(\nu,\nu')\to\mathsf{L}_\circ(\nu,\lambda)\times\mathsf{L}(\nu',\lambda')\\ (x,f,g,V',\beta)\mapsto\beta_*\Big((x,f)_{|V\times V}^{|V\times W},(x,f)_{|V'\times V'}^{|V'\times(W\oplus W'/W)}\Big).\end{array}\right.$$

PROPOSITION 2.16. *The map $\mathsf{T}_\circ$ is smooth with connected fibers.*

Proof. Let (x,f) and (x',f') be elements of $\mathsf{L}_\circ(\nu,\lambda)$ and $\mathsf{L}(\nu',\lambda')$ and take I-graded spaces V and V' of dimensions ν and ν'. Define (X,F,G,V',β) in the fiber $\mathsf{T}_\circ^{-1}((x,f),(x',f'))$ by:

(i) $\beta\in I(V,V_\nu)\times I(V',V_{\nu'})$;

(ii) $G=0\oplus\tau$, where

$$\nu\in\bigoplus_{i\in I}\mathrm{Hom}(W'_i,V_i);$$

(iii) $X=\beta^*x\oplus(\beta^*x'+\eta):V\oplus V'\to V\oplus V'$, where

$$\eta\in\bigoplus_{h\in H}\mathrm{Hom}(V'_{s(h)},V_{t(h)});$$

(iv) $F=\beta^*f\oplus(\beta^*f'+\theta):V\oplus V'\to W\oplus W'$, where

$$\theta\in\bigoplus_{i\in I}\mathrm{Hom}(V'_i,W_i)$$

such that $\mu_{\nu+\nu',\lambda+\lambda'}(X,F,G)=0$.

LEMMA 2.17. *This equation is linear in the variables (τ,η,θ), and the associated linear map is surjective.*

Proof. We first identify x,x' and f' with β^*x,β^*x' and β^*f'. Then the linear map $\zeta=(\zeta_i)$ we are interested in is given by

$$\zeta_i(\tau,\eta,\theta)=\tau_if'_i+\sum_{h\in H:s(h)=i}\epsilon(\bar h)(x_{\bar h}\eta_h+\eta_{\bar h}x'_h).$$

Take $L\in\bigoplus_{i\in I}\mathrm{Hom}(V_i,V'_i)$ such that for every (τ,η,θ),

$$\sum_{i\in I}\mathrm{Tr}(\zeta(\tau,\eta,\theta)L_i)=0.$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Then, for every edge h such that $s(h)=i$, $t(h)=j$, we have for every η_h,

$$\mathrm{Tr}(x_{\bar h}\eta_h L_i)-\mathrm{Tr}(\eta_h x'_{\bar h}L_j)=0.$$

But

$$\mathrm{Tr}(\eta_h L_i x_{\bar h})-\mathrm{Tr}(\eta_h x'_{\bar h}L_j)=\mathrm{Tr}(\eta_h L_i x_{\bar h}-\eta_h x'_{\bar h}L_j)=\mathrm{Tr}(\eta_h(L_i x_{\bar h}-x'_{\bar h}L_j)).$$

Hence, $L_i x_{\bar h}=x'_{\bar h}L_j$, and thus $\mathrm{Im}\,L$ is stable by x'. Moreover,

$$\forall i,\forall\tau_i,\quad \mathrm{Tr}(\tau_i f'_i L_i)=0\Rightarrow\forall i, f'_i L_i=0\Rightarrow \mathrm{Im}\,L\subset\ker f'$$

and hence the lemma comes from the stability of (x',f'). □

We have to check that $V=V(X,F,G)$. It is easy to see that $V\subset V(X,F,G)$. Moreover,

$$F^{-1}(W)=\{v+v'\in V\oplus V' \mid f(v)+\theta(v')+f'(v')\in W\}=V\oplus\ker f'$$

and hence, if Y is an X-stable subspace of $F^{-1}(W)$, Y/V is an x'-stable subspace of $\ker f'$. Since (x',f') is stable, we have $Y\subset V$, and thus $V=V(X,F,G)$.

We have proved that the fiber $\mathsf{T}_\circ^{-1}((x,f),(x',f'))$ is isomorphic to

$$G_{\nu+\nu'}\times\mathbb{C}^{\langle\lambda',\nu\rangle+(\nu',\nu)+\langle\nu',\lambda\rangle-\langle\nu',\nu\rangle}$$

and thus is connected. □

Lemma 2.18. *Consider $(x,f,g)\in\widetilde{\mathsf{Z}}_\circ(\nu,\nu')$ and $V=V(x,f,g)$. Then*

$$(x,f,g)\text{ stable }\Leftrightarrow (x,f)^{|V\times W}_{|V\times V}\text{ stable}$$

and we denote by $\widetilde{\mathsf{Z}}(\nu,\nu')$ the subvariety of stable points of $\widetilde{\mathsf{Z}}_\circ(\nu,\nu')$, and

$$\widetilde{\mathfrak{Z}}(\nu,\nu')=\widetilde{\mathsf{Z}}(\nu,\nu')/\!\!/G_{\nu+\nu'}.$$

Proof. The equivalence is a consequence of the following facts:

– the restriction of a stable point is stable;
– the extension of a stable point by a stable point is stable;
– the point $(x,f)^{|(V_{\nu+\nu'}/V)\times(W\oplus W'/W)}_{|(V_{\nu+\nu'}/V)\times(V_{\nu+\nu'}/V)}$ is stable. □

Theorem 2.5. *We have the following bijection:*

$$\mathrm{Irr}\,\mathfrak{L}(\nu,\lambda)\times\mathrm{Irr}\,\mathfrak{L}(\nu,\lambda')\xrightarrow[\sim]{\otimes}\mathrm{Irr}\,\widetilde{\mathfrak{Z}}(\nu,\nu').$$

Proof. Define $\check{\mathsf{Z}}(\nu,\nu')$ as the variety of stable points of $\check{\mathsf{Z}}_\circ(\nu,\nu')$. We have the following diagram:

$$\begin{tikzcd}
\check{\mathsf{Z}}(\nu,\nu') \arrow[r, "\mathsf{T}"] \arrow[d] & \mathsf{L}(\nu,\lambda)\times\mathsf{L}(\nu',\lambda') \arrow[d] \\
\widetilde{\mathfrak{Z}}(\nu,\nu') \arrow[r, "\mathfrak{T}"] & \mathfrak{L}(\nu,\lambda)\times\mathfrak{L}(\nu',\lambda')
\end{tikzcd}$$

where the rightmost vertical map is just the free quotient by $G_\nu\times G_{\nu'}$. The leftmost map being a principal bundle with fibers isomorphic to

$$G_\nu\times G_{\nu'}\times\mathrm{Grass}^I_{\nu,\nu'}(\nu+\nu')\times G_{\nu+\nu'},$$

we get our bijection thanks to Proposition 2.16 and Lemma 2.18. □

Again, there is an alternative definition for $\widetilde{\mathfrak{Z}}(\nu,\nu')$, given in [Nak01]. Denote by $*$ the $\mathbb{C}^*$-action on $\mathfrak{M}(\mathbf{v},\lambda+\lambda')$ induced by the one-parameter subgroup $\mathbb{C}^* \to \mathrm{GL}(W \oplus W')$, $t \mapsto t\,\mathrm{id}_W \oplus \mathrm{id}_{W'}$. We have

$$\mathfrak{M}(\mathbf{v},\lambda+\lambda')^{\mathbb{C}^*} \simeq \bigsqcup_{\nu+\nu'=\mathbf{v}} \mathfrak{M}(\nu,\lambda) \times \mathfrak{M}(\nu',\lambda')$$

and

$$\widetilde{\mathfrak{Z}}(\nu,\nu') = \Big\{[x,f,g] \in \mathfrak{M}(\mathbf{v},\lambda+\lambda') \mid \lim_{t\to 0} t * [x,f,g] \in \mathfrak{L}(\nu,\lambda) \times \mathfrak{L}(\nu',\lambda')\Big\}.$$

Hence, we also have, as in [Nak01, Proposition 3.15], the following result.

PROPOSITION 2.19. *The subvariety $\widetilde{\mathfrak{Z}}(\nu,\nu') \subset \mathfrak{M}(\nu+\nu',\lambda+\lambda')$ is Lagrangian.*

The results of § 2.2 lead to the following result, completing [Nak01, Proposition 4.3], which deals with the case $\omega_i = 0$.

PROPOSITION 2.20. *Consider i such that $\omega_i > 0$ and $l > 0$. If*

$$\lambda_i + \lambda'_i + \sum_{h\in H_i} \mathbf{v}_{t(h)} > 0,$$

we have a bijection

$$\mathrm{Irr}\,\widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l} \xrightarrow{\sim} \mathrm{Irr}\,\widetilde{\mathfrak{Z}}(\mathbf{v}-le_i)_{i,0} \times \mathrm{Irr}\,\Lambda(le_i).$$

2.3.2 *Comparison of two crystal-type structures.*

Notation 2.21. For every $X \in \mathrm{Irr}\,\widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l}$, we will denote by $\epsilon_i(X) \in \mathrm{Irr}\,\Lambda(le_i)$ the composition of the second projection with the bijection obtained in Proposition 2.20, so that $|\epsilon_i(X)| = l$. Note that if $(X,X') \in \mathrm{Irr}\,\mathfrak{L}(\nu,\lambda) \times \mathrm{Irr}\,\mathfrak{L}(\nu',\lambda')$, the quantity $\epsilon_i(X \otimes X')$ makes sense thanks to Theorem 2.5 and Proposition 2.20.

We will write $\Omega(i) = \{b_{i,j}\}_{1\leqslant j\leqslant \omega_i}$ for i imaginary, or $\Omega(i) = \{b_j\}_{1\leqslant j\leqslant \omega_i}$ if it is not ambiguous.

LEMMA 2.22. *Let i be an imaginary vertex and assume that $\sum_{h\in H_i} n_{t(h)} > 0$. For every $C \in \mathrm{Irr}\,\mathfrak{L}(\nu,\lambda)$, there exist $(x,f) \in C$, $v \in \mathrm{Im}\sum_{h\in H_i} x_{\bar{h}}$ such that*

$$\mathbb{C}[x_{\bar{b}_1}] \cdot v = \mathfrak{I}_i(x,f).$$

Proof. We proceed by induction on ν_i, the first step being trivial. For the inductive step, we can immediately conclude the proof if $C \in \mathrm{Irr}\,\mathfrak{L}(\nu,\lambda)_{i,l}$ for $l > 0$. Otherwise, $C \in \mathrm{Irr}\,\mathfrak{L}(\nu,\lambda)_{i,0}$, but $C \in \mathrm{Irr}\,\mathfrak{L}(\nu,\lambda)_{j,l}$ for some $j \in I$ and $l > 0$. There exists a minimal chain $(j_k,l_k,C_k)_{1\leqslant k\leqslant s}$ of elements of $I \times \mathbb{N}_{>0} \times \mathrm{Irr}\,\mathfrak{L}(-,\lambda)$ such that:

- $(j_1,l_1,C_1) = (j,l,C)$;
- $C_{k+1} = \mathrm{pr}_1 \mathfrak{l}(\nu - l_1 j_1 - \cdots - l_k j_k,\lambda)_{j_k,l_k}(C_k)$, where pr_1 is the first projection;
- $j_s = i$.

We necessarily have j_{s-1} adjacent to i and, by the induction hypothesis, the proposition is satisfied by C_s, and thus by C_{s-1}. But then, thanks to Lemmas 2.7 and 2.9, the proposition is also satisfied by C_{s-2} for a generic choice of $\eta_{\bar{h}}$ (using the notation of the proof of Lemma 2.11, where i is replaced by j_{s-1}). Hence, it is also satisfied by $C = C_1$. □

Proposition 2.23. *Let i be an imaginary vertex and consider $(X, X') \in \operatorname{Irr}\mathfrak{L}(\nu,\lambda)\times\operatorname{Irr}\mathfrak{L}(\nu',\lambda')$. Assume that $|\epsilon_i(X')| < \nu'_i$ or $0 < \lambda'_i$. Then we have*

$$\epsilon_i(X \otimes X') = \epsilon_i(X').$$

Proof. Put $(Y, C) = \mathfrak{l}(n,m)_{i,l}(X)$, where $l = |\epsilon_i(X)|$. Take $((x,f),(x',f')) \in X \times X'$. Consider the equation $\zeta_i = 0$ used in the proof of Lemma 2.17:

$$\tau_i f'_i + \sum_{h\in H: s(h)=i} \epsilon(\bar{h})(x_{\bar{h}}\eta_h + \eta_{\bar{h}}x'_h) = 0.$$

Note that $\eta_{b_j} = \eta_j$, $x_{b_j} = x_j$ and $x_{\bar{b}_j} = \bar{x}_j$ (and the same with x'); take $\eta_{\bar{b}_j} = 0$, so that our equation becomes

$$\begin{aligned}\tau_i f'_i + \sum_{h\in H_i} \eta_{\bar{h}}x'_h &= \sum_{1\leqslant j\leqslant \omega_i}(\bar{x}_j\eta_j - \eta_j\bar{x}'_j)\\ &= \bar{x}_1\eta_1 - \eta_1\bar{x}'_1\end{aligned}$$

if we also set $\eta_j = 0$ for $j \geqslant 2$ (if any). Then we set

$$\begin{aligned}
x' &= f'_i + \bigoplus_{h\in H_i} x'_h : V_{\nu'_i} \to W'_i \oplus \bigoplus_{h\in H_i} V_{\nu'_{t(h)}},\\
\bar{\eta} &= \tau_i + \sum_{h\in H_i}\epsilon(\bar{h})\eta_{\bar{h}} : W'_i \oplus \bigoplus_{h\in H_i} V_{\nu'_{t(h)}} \to V_{\nu_i},\\
\bar{x} &= \sum_{h\in H_i}\epsilon(\bar{h})x_{\bar{h}} : \bigoplus_{h\in H_i} V_{\nu_{t(h)}} \to V_{\nu_i},\\
\eta &= \bigoplus_{h\in H_i}\eta_h : V_{\nu'_i} \to \bigoplus_{h\in H_i} V_{\nu_{t(h)}}
\end{aligned}$$

and our equation finally becomes

$$\bar{\eta}x' + \eta\bar{x} = \bar{x}_1\eta_1 - \eta_1\bar{x}'_1.$$

Consider the open subvariety of $X \times X'$, where:

(i) there exists $\mathbf{v} \in V_{\nu_i}$ such that its image $\bar{\mathbf{v}} \in V_{\nu_i}/\mathfrak{I}_i(x,f)$ satisfies

$$\mathbb{C}[\bar{x}_{1|V_{\nu_i}/\mathfrak{I}_i(x,f)}]\cdot\bar{\mathbf{v}} = V_{\nu_i}/\mathfrak{I}_i(x,f);$$

(ii) $\bar{x}'_1$, $\bar{x}_{1|\mathfrak{I}_i(x,f)}$ and $\bar{x}_{1|\mathbb{C}^{n_i}/\mathfrak{I}_i(x,f)}$ have disjoint spectra;

(iii) there exist v and v' such that $\mathbf{w} = \sum_{h\in H_i} x_{\bar{h}}(v)$ and $\mathbf{w}' = \sum_{h\in H_i} x'_{\bar{h}}(v')$ satisfy

$$\mathbb{C}[\bar{x}_1 \oplus \bar{x}'_1]\cdot\mathbf{w}\oplus\mathbf{w}' = \mathfrak{I}_i(x,f)\oplus\mathfrak{I}_i(x',f'),$$

which is nonempty, thanks to Lemmas 2.9, 2.22 and 2.7. Take:

- $\bar{\eta} = \tau_i$ and $\mathbf{v} \in \operatorname{Im}\tau_i$ if $\lambda'_i > 0$;
- $\bar{\eta}$ such that $\bar{\eta}(v') = \mathbf{v}$ if $\nu'_i > |\epsilon_i(X')|$ (possible since $v' \neq 0$).

From Lemma 2.7, we get (with the notation used in the proof of Proposition 2.16)

$$\mathbb{C}[X_{\bar{b}_1}]\cdot\operatorname{Im}\left(\sum_{h\in H_i} X_{\bar{h}}\right) = V_{\nu_i}\oplus\mathfrak{I}_i(x',f').$$

We have to check that we can choose η such that the equations $\zeta_{t(h)} = 0$ are satisfied for every $h \in H_i$ (if $\lambda'_i > 0$ and $\bar{\eta} = \tau_i$, just take $\eta = 0$). It suffices to set $\eta_h x'_{\bar{h}}(v'_{t(h)}) = -x_h \eta_{\bar{h}}(v'_{t(h)})$ (possible since $\nu'_i > |\epsilon_i(X')|$ and since we may assume that $v'_{t(h)} = 0$ if $x'_{\bar{h}}(v'_{t(h)}) = 0$) and to set η and $\bar{\eta}$ equal to zero on supplementaries of $\mathbb{C}\mathbf{w}'$ and $\mathbb{C}v'$, respectively. We can finally choose η_1 such that $\bar{\eta}x' + \eta\bar{x} = \bar{x}_1\eta_1 - \eta_1\bar{x}'_1$ (possible since $\operatorname{Spec}\bar{x}'_1 \cap \operatorname{Spec}\bar{x}_1 = \emptyset$). Since

$$\operatorname{codim}\mathfrak{I}_i(x,f) \geqslant |\epsilon_i(X')|$$

for every $(x,f) \in X \otimes X'$, the subvariety of $X \otimes X'$ defined by

$$\operatorname{codim}\mathfrak{I}_i(x,f) = |\epsilon_i(X')|$$

is open, and we have shown it is nonempty; hence, the theorem is proved. □

Proposition 2.24. *Assume that $\lambda'_i = 0$, $|\epsilon_i(X')| = \nu'_i$ and $\sum_{h\in H_i} \nu'_{t(h)} > 0$. Then we still have $\epsilon_i(X \otimes X') = \epsilon_i(X')$.*

Proof. Thanks to the previous proof, the result is clear if there exists an imaginary vertex j adjacent to i: the choice of $x_{\bar{b}_{j,1}}$ and $x'_{\bar{b}_{j,1}}$ with disjoint spectra enables to use $\eta_{b_{j,1}}$ for $\zeta_j = 0$ to be satisfied (with the usual notation $\Omega(j) = \{b_{j,1}, \dots, b_{j,\omega_j}\}$).

Assume that every neighbour of i is real. Following the previous proof, assume that $\bar{\eta} = \eta_{\bar{h}}$ is of rank 1 for some $h : i \to j$. We have to check that $\zeta_j = 0$ can be satisfied. It is clear if $f'_j \neq 0$: just choose τ_j such that $\tau_j f'_j = -\epsilon(h)x_h\eta_{\bar{h}}$ and $\eta_p = 0 = \eta_{\bar{p}}$ if $p \in H_j\backslash\{\bar{h}\}$, so that $\zeta_j = 0$ is satisfied. Otherwise, there necessarily exists an edge $q : j \to k \neq i$ such that $x'_q \neq 0$ (if not, $V'_{\nu'_i} \oplus V'_{\nu'_j} \subseteq \ker f'$ would be x'-stable, which is not possible for every vertex j adjacent to i since $\sum_{h\in H_i} \nu'_{t(h)} > 0$). Hence, it is possible to choose $\eta_{\bar{q}}$ so that $\epsilon(\bar{q})\eta_{\bar{q}}x'_q = -\epsilon(h)x_h\eta_{\bar{h}}$ and $\eta_p = 0 = \eta_{\bar{p}}$ if $p \in H_j\backslash\{\bar{h}, q\}$, and thus get $\zeta_j = 0$ satisfied. □

We have proved the following result.

Theorem 2.6. *Let i be an imaginary vertex and consider $(X, X') \in \operatorname{Irr}\mathfrak{L}(\nu,\lambda) \times \operatorname{Irr}\mathfrak{L}(\nu',\lambda')$. We have*

$$\epsilon_i(X \otimes X') = \begin{cases} \epsilon_i(X') & \text{if } \lambda'_i + \sum_{h\in H_i} \nu'_{t(h)} > 0, \\ \epsilon_i(X) & \text{otherwise.} \end{cases}$$

3. Generalized crystals

Let $(-,-)$ denote the symmetric Euler form on $\mathbb{Z}I$: (i,j) is equal to the opposite of the number of edges of Ω between i and j for $i \neq j \in I$, and $(i,i) = 2 - 2\omega_i$. We will still denote by I^{re} (respectively, I^{im}) the set of real (respectively, imaginary) vertices, and by $I^{\mathrm{iso}} \subseteq I^{\mathrm{im}}$ the set of *isotropic* vertices: vertices i such that $(i,i) = 0$, i.e. such that $\omega_i = 1$. We also set $I_\infty = (I^{\mathrm{re}} \times \{1\}) \cup (I^{\mathrm{im}} \times \mathbb{N}_{\geqslant 1})$, and $(\iota, j) = l(i,j)$ if $\iota = (i,l) \in I_\infty$ and $j \in I$.

3.1 A generalized quantum group

We recall some of the definitions and results discussed in [Boz15, § 2].

Definition 3.1. Let F denote the $\mathbb{Q}(v)$-algebra generated by $(E_\iota)_{\iota\in I_\infty}$, naturally $\mathbb{N}I$-graded by $\deg(E_{i,l}) = li$ for $(i,l) \in I_\infty$. We put $\mathsf{F}[A] = \{x \in \mathsf{F} \mid |x| \in A\}$ for any $A \subseteq \mathbb{N}I$, where we denote by $|x|$ the degree of an element x.

For $\nu = \sum \nu_i i \in \mathbb{Z}I$, we set:

- ▷ $\mathrm{ht}(\nu) = \sum \nu_i$ its height;
- ▷ $v_\nu = \prod v_i^{\nu_i}$ if $v_i = v^{(i,i)/2}$.

We endow $\mathsf{F} \otimes \mathsf{F}$ with the following multiplication:

$$(a \otimes b)(c \otimes d) = v^{(|b|,|c|)}(ac) \otimes (bd)$$

and equip F with a comultiplication δ defined by

$$\delta(E_{i,l}) = \sum_{t+t'=l} v_i^{tt'} E_{i,t} \otimes E_{i,t'},$$

where $(i,l) \in I_\infty$ and $E_{i,0} = 1$.

Proposition 3.1. *For any family $(\nu_\iota)_{\iota \in I_\infty}$, we can endow F with a bilinear form $\{-,-\}$ such that:*

- ▷ *$\{x,y\} = 0$ if $|x| \neq |y|$;*
- ▷ *$\{E_\iota, E_\iota\} = \nu_\iota$ for all $\iota \in I_\infty$;*
- ▷ *$\{ab, c\} = \{a \otimes b, \delta(c)\}$ for all $a,b,c \in \mathsf{F}$.*

Notation 3.2. Take $i \in I^{\mathrm{im}}$ and c a composition or a partition. We put $E_{i,\mathsf{c}} = \prod_j E_{i,\mathsf{c}_j}$ and $\nu_{i,\mathsf{c}} = \prod_j \nu_{i,\mathsf{c}_j}$. If i is real, we will often use the index i instead of $i,1$.

Proposition 3.3. *Consider $(\iota, j) \in I_\infty \times I^{\mathrm{re}}$. The element*

$$\sum_{t+t'=-(\iota,j)+1} (-1)^t E_j^{(t)} E_\iota E_j^{(t')} \tag{3.4}$$

belongs to the radical of $\{-,-\}$.

Definition 3.2. We denote by $\tilde{U}^+$ the quotient of F by the ideal spanned by the elements (3.4) and the commutators $[E_{i,l}, E_{i,k}]$ for every isotropic vertex i, so that $\{-,-\}$ is still defined on $\tilde{U}^+$. We denote by U^+ the quotient of $\tilde{U}^+$ by the radical of $\{-,-\}$.

Definition 3.3. Let $\hat{U}$ be the quotient of the algebra generated by $K_i^\pm$, E_ι, F_ι ($i \in I$ and $\iota \in I_\infty$) subject to the following relations:

$$\begin{aligned}
K_i K_j &= K_j K_i,\\
K_i K_i^- &= 1,\\
K_j E_\iota &= v^{(j,\iota)} E_\iota K_j,\\
K_j F_\iota &= v^{-(j,\iota)} F_\iota K_j,\\
\sum_{t+t'=-(\iota,j)+1} (-1)^t E_j^{(t)} E_\iota E_j^{(t')} &= 0 \quad (j \in I^{\mathrm{re}}),\\
\sum_{t+t'=-(\iota,j)+1} (-1)^t F_j^{(t)} F_\iota F_j^{(t')} &= 0 \quad (j \in I^{\mathrm{re}}),\\
[E_{i,l}, E_{j,k}] &= 0 \quad \text{if } (i,j) = 0,\\
[F_{i,l}, F_{j,k}] &= 0 \quad \text{if } (i,j) = 0,
\end{aligned}$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

$$[E_{i,l}, E_{i,k}] = 0 \quad (i \in I^{\mathrm{iso}}),$$
$$[F_{i,l}, F_{i,k}] = 0 \quad (i \in I^{\mathrm{iso}}).$$

We extend the graduation by $|K_i| = 0$ and $|F_\iota| = -|E_\iota|$, and we set $K_\nu = \prod_i K_i^{\nu_i}$ for every $\nu \in \mathbb{Z}I$.

We endow $\hat{U}$ with a comultiplication Δ defined by

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(E_{i,l}) = \sum_{t+t'=l} v_i^{tt'} E_{i,t} K_{t'i} \otimes E_{i,t'},$$
$$\Delta(F_{i,l}) = \sum_{t+t'=l} v_i^{-tt'} F_{i,t} \otimes K_{-ti} F_{i,t'}.$$

We extend $\{-,-\}$ to the subalgebra $\hat{U}^{\geqslant 0} \subseteq \hat{U}$ spanned by $(K_i^{\pm})_{i\in I}$ and $(E_\iota)_{\iota \in I_\infty}$ by setting $\{xK_i, yK_j\} = \{x, y\} v^{(i,j)}$ for $x, y \in \tilde{U}^+$.

We use the Drinfeld double process to define $\tilde{U}$ as the quotient of $\hat{U}$ by the relations

$$\sum \{a_{(1)}, b_{(2)}\} \omega(b_{(1)}) a_{(2)} = \sum \{a_{(2)}, b_{(1)}\} a_{(1)} \omega(b_{(2)}) \tag{3.5}$$

for any $a, b \in \tilde{U}^{\geqslant 0}$, where ω is the unique involutive automorphism of $\hat{U}$ mapping E_ι to F_ι and K_i to K_{-i}, and where we use the Sweedler notation, for example $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

Setting $x^- = \omega(x)$ for $x \in \tilde{U}$, we define $\{-,-\}$ on the subalgebra $\tilde{U}^- \subseteq \tilde{U}$ spanned by $(F_\iota)_{\iota \in I_\infty}$ by setting $\{x, y\} = \{x^-, y^-\}$ for any $x, y \in \tilde{U}^-$. We will denote by U^- (respectively, U) the quotient of $\tilde{U}^-$ (respectively, $\tilde{U}$) by the radical of $\{-,-\}$ restricted to $\tilde{U}^-$ (respectively, restricted to $\tilde{U}^- \times \tilde{U}^+$).

Proposition 3.6. *Assume that*

$$\{E_\iota, E_\iota\} \in 1 + v^{-1}\mathbb{N}[\![v^{-1}]\!].$$

for every $\iota \in I_\infty$. Then we have $\tilde{U}^- \simeq U^-$.

Notation 3.7. We denote by $\mathcal{C}_{i,l}$ the set of compositions c (respectively, partitions) such that $|\mathsf{c}| = l$ if $(i,i) < 0$ (respectively, $(i,i) = 0$), and $\mathcal{C}_i = \bigsqcup_{l \geqslant 0} \mathcal{C}_{i,l}$. If i is real, we just put $\mathcal{C}_{i,l} = \{l\}$.

Denote by $u \mapsto \bar{u}$ the involutive $\mathbb{Q}$-morphism of U stabilizing E_ι, F_ι, and mapping K_i to K_{-i} and v to v^{-1}.

Proposition 3.8. *For any imaginary vertex i and any $l \geqslant 1$, there exists a unique element $a_{i,l} \in U^+[li]$ such that, if we set $b_{i,l} = a_{i,l}^-$, we get:*

(i) $\mathbb{Q}(v)\langle E_{i,l} \mid l \geqslant 1\rangle = \mathbb{Q}(v)\langle a_{i,l} \mid l \geqslant 1\rangle$ *and* $\mathbb{Q}(v)\langle F_{i,l} \mid l \geqslant 1\rangle = \mathbb{Q}(v)\langle b_{i,l} \mid l \geqslant 1\rangle$ *as algebras;*
(ii) $\{a_{i,l}, z\} = \{b_{i,l}, z^-\} = 0$ *for any* $z \in \mathbb{Q}(v)\langle E_{i,k} \mid k < l\rangle$*;*
(iii) $a_{i,l} - E_{i,l} \in \mathbb{Q}(v)\langle E_{i,k} \mid k < l\rangle$ *and* $b_{i,l} - F_{i,l} \in \mathbb{Q}(v)\langle F_{i,k} \mid k < l\rangle$*;*
(iv) $\bar{a}_{i,l} = a_{i,l}$ *and* $\bar{b}_{i,l} = b_{i,l}$*;*
(v) $\delta(a_{i,l}) = a_{i,l} \otimes 1 + 1 \otimes a_{i,l}$ *and* $\delta(b_{i,l}) = b_{i,l} \otimes 1 + 1 \otimes b_{i,l}$.

Notation 3.9. Consider $i \in I^{\mathrm{im}}$ and $\mathsf{c} \in \mathcal{C}_{i,l}$. We set $\tau_{i,l} = \{a_{i,l}, a_{i,l}\}$, $a_{i,\mathsf{c}} = \prod_j a_{i,\mathsf{c}_j}$ and $\tau_{i,\mathsf{c}} = \prod_j \tau_{i,\mathsf{c}_j}$. Notice that $\{a_{i,\mathsf{c}} \mid \mathsf{c} \in \mathcal{C}_{i,l}\}$ is a basis of $U^+[li]$.

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

DEFINITION 3.4. We denote by $\delta_{i,\mathsf{c}}, \delta^{i,\mathsf{c}} : U^+ \to U^+$ the linear maps defined by

$$\delta(x) = \sum_{\mathsf{c} \in \mathcal{C}_{i,l}} \delta_{i,\mathsf{c}}(x) \otimes a_{i,\mathsf{c}} + \text{obd},$$
$$\delta(x) = \sum_{\mathsf{c} \in \mathcal{C}_{i,l}} a_{i,\mathsf{c}} \otimes \delta^{i,\mathsf{c}}(x) + \text{obd},$$

where 'obd' stands for terms of bidegree not in $\mathbb{N}I \times \mathbb{N}i$ in the former equality, $\mathbb{N}i \times \mathbb{N}I$ in the latter one.

3.2 Kashiwara operators

PROPOSITION 3.10. *Let i be an imaginary vertex, $l > 0$, $\mathsf{c} = (\mathsf{c}_1, \ldots, \mathsf{c}_r) \in \mathcal{C}_i$ and $(y, z) \in (U^+)^2$. We have the following identities:*

(i) $\delta^{i,l}(yz) = \delta^{i,l}(y)z + v^{l(i,|y|)} y \delta^{i,l}(z)$;

(ii) $[a_{i,l}, z^-] = \tau_{i,l}\{\delta_{i,l}(z)^- K_{-li} - K_{li}\delta^{i,l}(z)^-\}$;

(iii) $\delta^{i,l}(a_{i,\mathsf{c}}) = \sum_{k:\mathsf{c}_k = l} v_i^{2l\mathsf{c}_{k-1}} a_{i,\mathsf{c}\backslash\mathsf{c}_k}$,

where $\mathsf{c}_0 = 0$ and $\mathsf{c}\backslash\mathsf{c}_k = (\mathsf{c}_1, \ldots, \hat{\mathsf{c}}_k, \ldots, \mathsf{c}_r)$; the notation $\hat{\mathsf{c}}_k$ meaning that c_k is removed from c.

Proof. The first equality comes from the definition of $\delta^{i,l}$, the second from the primitive character of $a_{i,l}$ and the formula (3.5) with $a = a_{i,l}$ and $b = z^-$. The third comes from the definition of $\delta_{i,l}$ and the primitive character of the $a_{i,h}$. □

DEFINITION 3.5. Define $e'_{i,l} : U^- \to U^-$ by $e'_{i,l}(z^-) = \delta^{i,l}(z)^-$ for any $z \in U^+$.

PROPOSITION 3.11. *Set*

$$\mathcal{K}_i = \bigcap_{l>0} \ker e'_{i,l}$$

for any $i \in I^{\text{im}}$. We have the following decomposition:

$$U^- = \bigoplus_{\mathsf{c} \in \mathcal{C}_i} b_{i,\mathsf{c}} \mathcal{K}_i.$$

Proof. Let us first prove the existence. Consider $u \in U^-$, and assume first that u is of the following form: $u = m b_{i,\mathsf{c}} m'$ for some $\mathsf{c} \in \mathcal{C}_i$ and some $m, m' \in \mathcal{K}_i$. We proceed by induction on $|\mathsf{c}|$. If $|\mathsf{c}| = 0$, we have $mm' \in \mathcal{K}_i$ thanks to Proposition 3.10(1). Otherwise, set $[y, z]^\circ = v^{-(|y|,|z|)} yz - zy$ for any $y, z \in U^+$. Thanks to Proposition 3.10(1), and since $\delta^{i,l}(a_{i,k}) = \delta_{l,k}$, we have for any $y \in \bigcap_{l>0} \ker \delta^{i,l}$ and any $k > 0$

$$\begin{aligned}
\delta^{i,l}([y, a_{i,k}]^\circ) &= v^{-k(i,|y|)} \delta^{i,l}(y a_{i,k}) - \delta^{i,l}(a_{i,k} y) \\
&= v^{-k(i,|y|)} v^{l(i,|y|)} y \delta^{i,l}(a_{i,k}) - \delta^{i,l}(a_{i,k}) y \\
&= \delta_{l,k} v^{(l-k)(i,|y|)} y - \delta_{l,k} y \\
&= 0.
\end{aligned}$$

Hence, the following equality:

$$u = v^{\mathsf{c}_1(|m|,i)} [m, b_{i,\mathsf{c}_1}]^\circ b_{i,\mathsf{c}\backslash\mathsf{c}_1} m' + v^{-\mathsf{c}_1(|m|,i)} b_{i,\mathsf{c}_1} m b_{i,\mathsf{c}\backslash\mathsf{c}_1} m'$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

along with the induction hypothesis allow us to conclude the proof, since $|\mathsf{c}\backslash\mathsf{c}_1| < |\mathsf{c}|$ and since $\bigoplus_{\mathsf{c}\in\mathcal{C}_i} b_{i,\mathsf{c}}\mathcal{K}_i$ is stable by left multiplication by b_{i,c_1}.

Then we prove the existence of the decomposition for a general $u \in U^-$, using induction on $-|u|$. If $u \neq 1$, we can write

$$u = \sum_{\iota\in I_\infty} b_\iota u_\iota$$

for some finitely many nonzero $u_\iota \in U^-$. Thanks to our induction hypothesis, we have

$$u = \sum_{\iota\in I_\infty,\mathsf{c}\in\mathcal{C}_i} b_\iota b_{i,\mathsf{c}} z_{\iota,\mathsf{c}}$$

for some finitely many nonzero $z_{\iota,\mathsf{c}} \in \mathcal{K}_i$. Then

$$u = \sum_{l>0,\mathsf{c}\in\mathcal{C}_i} b_{i,(l,\mathsf{c})} z_{(i,l),\mathsf{c}} + \sum_{\substack{\iota\in I_\infty\backslash(\{i\}\times\mathbb{N}_{>0})\\ \mathsf{c}\in\mathcal{C}_i}} b_\iota b_{i,\mathsf{c}} z_{\iota,\mathsf{c}}$$

and we have the result since $b_\iota b_{i,\mathsf{c}} z_{\iota,\mathsf{c}}$ is of the form $m b_{i,\mathsf{c}} m'$ for some $m, m' \in \mathcal{K}_i$. Indeed, it is straightforward from the definitions that $\delta^{i,l}(a_{j,h}) = 0$ for any $l, h > 0$ if $j \neq i$. Note that if $i \notin I^{\mathrm{iso}}$, the composition (l,c) is the composition c', where $\mathsf{c}'_1 = l$ and $\mathsf{c}'_k = \mathsf{c}_{k-1}$ if $k \geqslant 2$, but if $i \in I^{\mathrm{iso}}$, (l,c) stands for the partition $\mathsf{c} \cup l$.

To prove the unicity of the decomposition, consider a minimal nontrivial relation of dependence

$$0 = \sum_{\mathsf{c}\in\mathcal{C}_i} a_{i,\mathsf{c}} z_{\mathsf{c}},$$

where $z_{\mathsf{c}} \in \mathcal{K}_i^-$. We have to consider separately the cases $i \in I^{\mathrm{iso}}$ and $i \notin I^{\mathrm{iso}}$. First, consider $i \notin I^{\mathrm{iso}}$. Consider r maximal such that there exists $\mathsf{c} = (\mathsf{c}_1, \dots, \mathsf{c}_r)$ such that $z_{\mathsf{c}} \neq 0$. Using Proposition 3.10(1) and applying repeatedly Proposition 3.10(3), we see that for any $\mathsf{c}' \in \mathfrak{S}_r\mathsf{c}$ (with the convention $(\sigma\mathsf{c})_k = \mathsf{c}_{\sigma(k)}$),

$$\begin{aligned} 0 &= \delta^{i,\mathsf{c}'}\left(\sum_{\mathsf{c}''\in\mathcal{C}_i} a_{i,\mathsf{c}''} z_{\mathsf{c}''}\right) \\ &= \sum_{\mathsf{c}''\in\mathcal{C}_i} \delta^{i,\mathsf{c}'}(a_{i,\mathsf{c}''}) z_{\mathsf{c}''} \\ &= \sum_{\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}} \delta^{i,\mathsf{c}'}(a_{i,\mathsf{c}''}) z_{\mathsf{c}''} \\ &= \sum_{\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}} P_{\mathsf{c}',\mathsf{c}''}(v_i) z_{\mathsf{c}''}, \end{aligned}$$

where $P_{\mathsf{c}',\mathsf{c}''}(v) \in \mathbb{Z}[v]$. The third equality is true by maximality of r. Since $(z_{\mathsf{c}''})_{\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}} \neq 0$, we have to prove that

$$\Delta(v) = \det(P_{\mathsf{c}',\mathsf{c}''}(v))_{\mathsf{c}',\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}} \neq 0 \in \mathbb{Z}[v]$$

to end our proof in the case $(i,i) < 0$ (since then we have $v_i \neq 1$). But, for any $\mathsf{c}' = (\mathsf{c}'_1, \dots, \mathsf{c}'_r) \in \mathfrak{S}_r\mathsf{c}$, one has, using Proposition 3.10(3),

$$P_{\mathsf{c}',\mathsf{c}''}(0) \neq 0 \Leftrightarrow \mathsf{c}'' = (\mathsf{c}'_r, \dots, \mathsf{c}'_1).$$

Hence, $\Delta(0) \neq 0$, and in particular $\Delta \neq 0$.

We finally have to prove the uniqueness in the case $(i,i)=0$. Write a relation of dependence of minimal degree:

$$0=\sum_{\lambda\in\mathcal{C}_i} a_{i,\lambda}z_\lambda,$$

where $z_\lambda\in\mathcal{K}_i^-$. For any λ and $l>0$, set $m_l(\lambda)=|\{s:\lambda_s=l\}|$, and denote by $\lambda\backslash l$ the partition obtained by removing one of the $\lambda_s=l$ when $m_l(\lambda)\geqslant 1$. Hence, $m_l(\lambda\backslash l)=m_l(\lambda)-1$. We have, thanks to Proposition 3.10(1,3),

$$\begin{aligned}\delta^{i,l}\Bigg(\sum_{\lambda\in\mathcal{C}_i} a_{i,\lambda}z_\lambda\Bigg)&=\sum_{\lambda\in\mathcal{C}_i} m_l(\lambda)a_{i,\lambda\backslash l}z_\lambda\\&=\sum_{\mu\in\mathcal{C}_i} a_{i,\mu},\{(m_l(\mu)+1)z_{\mu\cup l}\}\end{aligned}$$

which contradicts the minimality of the first relation. Note that the proof is easier in this case because we are dealing with partitions and hence the quantity $\mu\cup l$ is 'uniquely defined'. □

The following definition generalizes the Kashiwara operators (see e.g. [KS97, Lemma 2.3.1]).

Definition 3.6. If i is imaginary and $z=\sum_{\mathsf{c}\in\mathcal{C}_i} b_{i,\mathsf{c}}z_{\mathsf{c}}\in U^-$, set

$$\tilde{e}_{i,l}(z)=\begin{cases}\displaystyle\sum_{\mathsf{c}:\mathsf{c}_1=l} b_{i,\mathsf{c}\backslash\mathsf{c}_1}z_{\mathsf{c}} & \text{if } i\notin I^{\text{iso}},\\ \displaystyle\sum_{\lambda\in\mathcal{C}_i}\sqrt{\frac{m_l(\lambda)}{l}}\,b_{i,\lambda\backslash l}z_\lambda & \text{if } i\in I^{\text{iso}},\end{cases}$$

$$\tilde{f}_{i,l}(z)=\begin{cases}\displaystyle\sum_{\mathsf{c}\in\mathcal{C}_i} b_{i,(l,\mathsf{c})}z_{\mathsf{c}} & \text{if } i\notin I^{\text{iso}},\\ \displaystyle\sum_{\lambda\in\mathcal{C}_i}\sqrt{\frac{l}{m_l(\lambda)+1}}\,b_{i,\lambda\cup l}z_\lambda & \text{if } i\in I^{\text{iso}},\end{cases}$$

where $m_l(\lambda)=|\{s:\lambda_s=l\}|$.

Remark 3.12. Note that the fact that $v_i=1$ for isotropic vertices makes this case somehow degenerate. The use of partitions instead of compositions, or the presence of square roots in the above definition, are consequences of this particularity. The importance of these square roots will become clear in the proofs of Lemmas 3.33 and 3.34. Note that we need to consider an extension of $\mathbb{Q}$ for these square roots to be defined.

3.3 Definition of generalized crystals

Denote by P the lattice $\mathbb{Z}^I$, still endowed with the pairing $\langle -,-\rangle$ defined by $\langle e_i,e_j\rangle=\delta_{i,j}$, where $e_i=(\delta_{i,j})_{j\in I}$ for every $i\in I$. We will also denote α_i instead of Ce_i, where $C=((i,j))_{i,j\in I}$ still denotes the Cartan matrix associated to Q.

Definition 3.7. We call Q-crystal a set $\mathcal{B}$ together with maps

$$\begin{aligned}\text{wt}&:\mathcal{B}\to P,\\ \epsilon_i&:\mathcal{B}\to\mathcal{C}_i\sqcup\{-\infty\},\\ \phi_i&:\mathcal{B}\to\mathbb{N}\sqcup\{\pm\infty\},\end{aligned}$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

$$\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \quad i \in I^{\mathrm{re}},$$
$$\tilde{e}_{i,l}, \tilde{f}_{i,l} : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \quad i \in I^{\mathrm{im}}, l > 0$$

such that for every $b, b' \in \mathcal{B}$,

(A1) $\langle e_i, \mathrm{wt}(b)\rangle \geqslant 0$ if $i \in I^{\mathrm{im}}$;

(A2) $\mathrm{wt}(\tilde{e}_{i,l}b) = \mathrm{wt}(b) + l\alpha_i$ if $\tilde{e}_{i,l}b \neq 0$;

(A3) $\mathrm{wt}(\tilde{f}_{i,l}b) = \mathrm{wt}(b) - l\alpha_i$ if $\tilde{f}_{i,l}b \neq 0$;

(A4) $\tilde{f}_{i,l}b = b' \Leftrightarrow b = \tilde{e}_{i,l}b'$;

(A5) if $\tilde{e}_{i,l}b \neq 0$, $\epsilon_i(\tilde{e}_{i,l}b) = \begin{cases} \epsilon_i(b) - l & \text{if } i \in I^{\mathrm{re}}, \\ \epsilon_i(b)\backslash l & \text{if } i \in I^{\mathrm{im}}\backslash I^{\mathrm{iso}} \text{ and } l = \epsilon_i(b)_1, \\ \epsilon_i(b)\backslash l & \text{if } i \in I^{\mathrm{iso}}; \end{cases}$

(A6) if $\tilde{f}_{i,l}b \neq 0$, $\epsilon_i(\tilde{f}_{i,l}b) = \begin{cases} \epsilon_i(b) + l & \text{if } i \in I^{\mathrm{re}}, \\ (l, \epsilon_i(b)) & \text{if } i \in I^{\mathrm{im}}; \end{cases}$

(A7) $\phi_i(b) = \begin{cases} \epsilon_i(b) + \langle e_i, \mathrm{wt}(b)\rangle & \text{if } i \in I^{\mathrm{re}}, \\ +\infty & \text{if } i \in I^{\mathrm{im}} \text{ and } \langle e_i, \mathrm{wt}(b)\rangle > 0, \\ 0 & \text{otherwise,} \end{cases}$

where, for $i \in I^{\mathrm{re}}$, we write $\tilde{e}_{i,1}, \tilde{f}_{i,1}$ instead of $\tilde{e}_i, \tilde{f}_i$ and $\tilde{e}_{i,l}, \tilde{f}_{i,l}$ instead of $\tilde{e}^l_{i,1}, \tilde{f}^l_{i,1}$. Also, as earlier, $(l, \epsilon_i(b))$ stands for the partition $\epsilon_i(b) \cup l$ if $i \in I^{\mathrm{iso}}$.

Remark 3.13. – We will use the following notation: $\mathrm{wt}_i = \langle e_i, \mathrm{wt}\rangle$.

- Note that this definition of ϕ_i already appears in [JKK05]. Also note that since we will only be interested in normal crystals (see Definition 3.9), we require $|\epsilon_i|$ and ϕ_i to be nonnegative, except if $\epsilon_i = -\infty$, in which case we set $|\epsilon_i| = -\infty$.
- Set $\tilde{e}_{i,\mathsf{c}} = \tilde{e}_{i,\mathsf{c}_1} \ldots \tilde{e}_{i,\mathsf{c}_r}$ and $\tilde{f}_{i,\mathsf{c}} = \tilde{f}_{i,\mathsf{c}_1} \ldots \tilde{f}_{i,\mathsf{c}_r}$ for every $\mathsf{c} = (\mathsf{c}_1, \ldots, \mathsf{c}_r)$. Set $\bar{\mathsf{c}} = (\mathsf{c}_r, \ldots, \mathsf{c}_1)$ if $\omega_i \geqslant 2$, $\bar{\mathsf{c}} = \mathsf{c}$ if $\omega_i \leqslant 1$. We have

$$\tilde{f}_{i,\mathsf{c}}b = b' \Leftrightarrow b = \tilde{e}_{i,\bar{\mathsf{c}}}b'.$$

Example 3.14. For every vertex i, we define a crystal $\mathcal{B}_i$ by endowing $\mathcal{C}_i$ with the following maps:

$$\begin{aligned} \mathrm{wt}(\mathsf{c}) &= -|\mathsf{c}|\alpha_i, \\ \epsilon_i(\mathsf{c}) &= \mathsf{c}, \\ \epsilon_j(\mathsf{c}) &= -\infty \quad \text{if } j \neq i, \\ \tilde{e}_{i,l}(\mathsf{c}) &= \begin{cases} \mathsf{c} - l & \text{if } i \in I^{\mathrm{re}} \text{ and } \mathsf{c} \geqslant l, \\ \mathsf{c}\backslash l & \text{if } i \in I^{\mathrm{im}}\backslash I^{\mathrm{iso}} \text{ and } l = \epsilon_i(b)_1, \\ \mathsf{c}\backslash l & \text{if } i \in I^{\mathrm{iso}}, \\ 0 & \text{otherwise,} \end{cases} \\ \tilde{f}_{i,l}(\mathsf{c}) &= \begin{cases} \mathsf{c} + l & \text{if } i \in I^{\mathrm{re}}, \\ (l, \mathsf{c}) & \text{if } i \in I^{\mathrm{im}}. \end{cases} \end{aligned}$$

We will denote by $(0)_i$ the trivial element of $\mathcal{C}_i$.

Definition 3.8. A morphism of crystals $\mathcal{B}_1 \to \mathcal{B}_2$ is a map $\mathcal{B}_1 \sqcup \{0\} \to \mathcal{B}_2 \sqcup \{0\}$ mapping 0 to 0, preserving the weight, ϵ_i, and commuting with the respective actions of the $\tilde{e}_\iota, \tilde{f}_\iota$ on $\mathcal{B}_1$ and $\mathcal{B}_2$.

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Definition 3.9. A crystal $\mathcal{B}$ is said to be *normal* if for every $b \in \mathcal{B}$ and $i \in I$, we have

$$\epsilon_i(b) = \overline{\max\{\bar{\mathsf{c}} \mid \tilde{e}_{i,\mathsf{c}}(b) \neq 0\}},$$
$$\phi_i(b) = \max\{|\mathsf{c}| \mid \tilde{f}_{i,\mathsf{c}}(b) \neq 0\}.$$

Definition 3.10. The *tensor product* $\mathcal{B} \otimes \mathcal{B}' = \{b \otimes b' \mid b \in \mathcal{B}, b' \in \mathcal{B}'\}$ of two crystals is defined by:

(i) $\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$;

(ii) if $i \in I^{\mathrm{re}}$, $\epsilon_i(b \otimes b') = \max\{\epsilon_i(b'), \epsilon_i(b) - \mathrm{wt}_i(b')\}$;

(iii) if $i \in I^{\mathrm{im}}$, $\epsilon_i(b \otimes b') = \begin{cases} \epsilon_i(b') & \text{if } \phi_i(b) \geqslant |\epsilon_i(b')|, \\ \epsilon_i(b) & \text{if } \phi_i(b) < |\epsilon_i(b')|; \end{cases}$

(iv) if $i \in I^{\mathrm{re}}$, $\phi_i(b \otimes b') = \max\{\phi_i(b') + \mathrm{wt}_i(b), \phi_i(b)\}$;

(v) if $i \in I^{\mathrm{im}}$, $\phi_i(b \otimes b') = \begin{cases} \phi_i(b') & \text{if } \phi_i(b) \geqslant |\epsilon_i(b')|, \\ \phi_i(b) & \text{if } \phi_i(b) < |\epsilon_i(b')|; \end{cases}$

(vi) for every $\iota = (i, l) \in I_\infty$, $\tilde{e}_\iota(b \otimes b') = \begin{cases} b \otimes \tilde{e}_\iota(b') & \text{if } \phi_i(b') \geqslant |\epsilon_i(b)|, \\ \tilde{e}_\iota(b) \otimes b' & \text{if } \phi_i(b') < |\epsilon_i(b)|; \end{cases}$

(vii) for every $\iota = (i, l) \in I_\infty$, $\tilde{f}_\iota(b \otimes b') = \begin{cases} b \otimes \tilde{f}_\iota(b') & \text{if } \phi_i(b') > |\epsilon_i(b)|, \\ \tilde{f}_\iota(b) \otimes b' & \text{if } \phi_i(b') \leqslant |\epsilon_i(b)|. \end{cases}$

Remark 3.15. Note that when i is imaginary, the condition $\phi_i(b') > |\epsilon_i(b)|$ is equivalent to

$$\phi_i(b') = +\infty \quad \text{or} \quad [\phi_i(b') = 0 \text{ and } \epsilon_i(b) = -\infty],$$

and $\phi_i(b') = |\epsilon_i(b)|$ is equivalent to $\phi_i(b') = 0 = |\epsilon_i(b)|$.

Proposition 3.16. *$\mathcal{B} \otimes \mathcal{B}'$ is a crystal if $\mathcal{B}'$ is normal.*

Proof. Note that the result is already known if $I^{\mathrm{im}} = \emptyset$ and hence we just have to check the axioms of Definition 3.7 that concern imaginary vertices. Axioms (A1), (A2), (A3) and (A7) are clearly satisfied.

To prove that (A4) is satisfied, we first consider b and b' such that $\tilde{e}_{i,l}(b \otimes b') \neq 0$. If $\phi_i(b') \geqslant |\epsilon_i(b)|$, we have $\tilde{e}_{i,l}(b \otimes b') = b \otimes \tilde{e}_{i,l}(b')$. The crystal $\mathcal{B}'$ being normal, $\phi_i(\tilde{e}_{i,l}(b')) > 0$ since $\tilde{f}_{i,l}\tilde{e}_{i,l}(b') = b' \neq 0$. But i is imaginary, so by definition $\phi_i(\tilde{e}_{i,l}(b')) \in \{0, +\infty\}$, and we get $\phi_i(\tilde{e}_{i,l}(b')) = +\infty$. Also by definition, $|\epsilon_i(b)| < +\infty$ and hence we get

$$\tilde{f}_{i,l}(b \otimes \tilde{e}_{i,l}b') = b \otimes \tilde{f}_{i,l}\tilde{e}_{i,l}b' = b \otimes b'.$$

If $\phi_i(b') < |\epsilon_i(b)|$, we have $\tilde{e}_{i,l}(b \otimes b') = \tilde{e}_{i,l}(b) \otimes b'$, where $|\epsilon_i(\tilde{e}_{i,l}(b))| = |\epsilon_i(b)| - l$ is necessarily nonnegative by definition of ϵ_i. Also, $\phi_i(b')$ cannot be equal to $+\infty$ and hence is 0, and

$$\tilde{f}_{i,l}(\tilde{e}_{i,l}(b) \otimes b') = \tilde{f}_{i,l}(\tilde{e}_{i,l}(b)) \otimes b' = b \otimes b'.$$

Assume now that $\tilde{f}_{i,l}(b \otimes b') \neq 0$. If $\phi_i(b') > |\epsilon_i(b)|$, we get $\tilde{f}_{i,l}(b \otimes b') = b \otimes \tilde{f}_{i,l}b'$. If $\epsilon_i(b) = -\infty$, we have $\phi_i(\tilde{f}_{i,l}b') > \epsilon_i(b)$. Otherwise, we necessarily have $\phi_i(b') = +\infty$. But

$$\mathrm{wt}_i(\tilde{f}_{i,l}(b')) = \mathrm{wt}_i(b') - l\langle e_i, \alpha_i\rangle \geqslant \mathrm{wt}_i(b')$$

since $\langle e_i, \alpha_i\rangle \leqslant 0$ for every $i \in I^{\mathrm{im}}$. Hence, $\phi_i(\tilde{f}_{i,l}(b')) = +\infty$, and

$$\tilde{e}_{i,l}(b \otimes \tilde{f}_{i,l}b') = b \otimes \tilde{e}_{i,l}\tilde{f}_{i,l}b' = b \otimes b'.$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

If $\phi_i(b') \leqslant |\epsilon_i(b)|$, then $\tilde{f}_{i,l}(b \otimes b') = \tilde{f}_{i,l}(b) \otimes b'$, where

$$|\epsilon_i(\tilde{f}_{i,l}(b))| = |\epsilon_i(b)| + l > |\epsilon_i(b)| \geqslant \phi_i(b')$$

and hence

$$\tilde{e}_{i,l}(\tilde{f}_{i,l}(b) \otimes b') = \tilde{e}_{i,l}(\tilde{f}_{i,l}(b)) \otimes b' = b \otimes b'.$$

From the definitions and the proof of (A4) above, it is easy to check that

$$\epsilon_i(\tilde{e}_{i,l}(b \otimes b')) = \begin{cases} \epsilon_i(\tilde{e}_{i,l}b') & \text{if } \epsilon_i(b \otimes b') = \epsilon_i(b'), \\ \epsilon_i(\tilde{e}_{i,l}b) & \text{if } \epsilon_i(b \otimes b') = \epsilon_i(b) \end{cases}$$

and hence (A5) is satisfied by $\mathcal{B} \otimes \mathcal{B}'$ since it is by $\mathcal{B}$ and $\mathcal{B}'$. For the same reasons, (A6) is satisfied, except if $\phi_i(b') = |\epsilon_i(b)|$, which can only happen if both are equal to 0 (we still consider $i \in I^{\mathrm{im}}$). But then $\tilde{e}_{i,l}(b') = 0$, so there is nothing to prove. Otherwise, we would have $\tilde{f}_{i,l}\tilde{e}_{i,l}b' = b' \neq 0$ and hence $\phi_i(\tilde{e}_{i,l}b') = +\infty$ by normality. But then

$$\mathrm{wt}_i(b') = \mathrm{wt}_i(\tilde{e}_{i,l}(b')) - l\langle e_i, \alpha_i\rangle \geqslant \mathrm{wt}_i(\tilde{e}_{i,l}(b')) > 0$$

would imply $\phi_i(b') = +\infty$, which contradicts the assumption. □

3.4 The crystal $\mathcal{B}(\infty)$

3.4.1 *Algebraic definition.* Let $\mathcal{A} \subset \mathbb{Q}(v^{-1})$ be the subring consisting of rational functions without pole at $v^{-1} = 0$, and $\mathcal{L}(\infty)$ be the sub-$\mathcal{A}$-module of U^- generated by the elements $\tilde{f}_{\iota_1} \dots \tilde{f}_{\iota_s}.1$, where $\iota_k \in I_\infty$ and the operators $\tilde{f}_\iota$ are those defined in Definition 3.6 together with the original ones for $\iota = i \in I^{\mathrm{re}}$. Define the following set:

$$\mathcal{B}(\infty) = \{\tilde{f}_{\iota_1} \dots \tilde{f}_{\iota_s} \cdot 1 \mid \iota_k \in I_\infty\} \subset \frac{\mathcal{L}(\infty)}{v^{-1}\mathcal{L}(\infty)}.$$

The following theorem will be proved in § 3.6.

Theorem 3.11. *The Kashiwara operators are still defined on $\mathcal{B}(\infty)$, which is a crystal once equipped with the following maps:*

$$\mathrm{wt}(b) = \sum_{i \in I} \nu_i \alpha_i \text{ if } |b| = \nu \in -\mathbb{N}I,$$
$$\epsilon_i(b) = \overline{\max\{\bar{\mathsf{c}} \mid \tilde{e}_{i,\mathsf{c}}(b) \neq 0\}}.$$

We have the following characterization, analogous to [KS97, Proposition 3.2.3].

Proposition 3.17. *Let $\mathcal{B}$ be a crystal, and $b_0 \in \mathcal{B}$ with weight 0, such that:*

(i) $\mathrm{wt}(\mathcal{B}) \subset -\sum_{i \in I} \mathbb{N}\alpha_i$;
(ii) *the only element of $\mathcal{B}$ with weight 0 is b_0;*
(iii) $\epsilon_i(b_0) = 0$ *for every* $i \in I$;
(iv) *there exists an embedding $\Psi_i : \mathcal{B} \to \mathcal{B}_i \otimes \mathcal{B}$ for every $i \in I$;*
(v) *for every $b \neq b_0$, there exists $i \in I$ such that $\Psi_i(b) = \mathsf{c} \otimes b'$ for some $b' \in \mathcal{B}$ and $\mathsf{c} \in \mathcal{C}_i \backslash \{(0)_i\}$;*
(vi) *for every i, the crystal $\mathcal{B}'_i = \pi_i \Psi_i(\mathcal{B})$ is normal, where π_i is the second projection $\mathcal{B}_i \otimes \mathcal{B} \to \mathcal{B}$.*

Then $\mathcal{B} \simeq \mathcal{B}(\infty)$.

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Remark 3.18. The crystal structure we consider on $\mathcal{B}'_i$ is the following: if $b' \in \mathcal{B}'_i$, we set $\tilde{e}_\iota(b') = 0$ (respectively, $\tilde{f}_\iota(b') = 0$) if, with respect to the structure of $\mathcal{B}$, $\tilde{e}_\iota(b') \in \mathcal{B}\backslash\mathcal{B}'_i$ (respectively, $\tilde{f}_\iota(b') \in \mathcal{B}\backslash\mathcal{B}'_i$).

Proof. First note that we necessarily have $\Psi_i(b_0) = (0)_i \otimes b_0$, thanks to (1). Let us show that for any $b \in \mathcal{B}\backslash\{b_0\}$ there exists $\iota \in I_\infty$ such that $\tilde{e}_\iota(b) \neq 0$. Consider $i \in I$ such that $\Psi_i(b) = \mathsf{c} \otimes b'$ for some $b' \in \mathcal{B}$ and nontrivial $\mathsf{c} \in \mathcal{C}_i$, and assume that i is imaginary since the result is already known from [KS97, Proposition 3.2.3] when i is real. If $b' = b_0$, since $\phi_i(b_0) = 0$, we have $\tilde{e}_{i,\mathsf{c}_1}(b) = \mathsf{c}\backslash\mathsf{c}_1 \otimes b_0 \neq 0$. Otherwise, by induction on the weight, we can assume that there exists $\iota \in I_\infty$ such that $\tilde{e}_\iota(b') \neq 0$. If $\iota = (j, 1)$ for some real vertex j, we get $\tilde{e}_\iota(b) \neq 0$. If $\iota = (j, l)$ for some imaginary vertex j, we have to prove that $\phi_j(b') = +\infty$ to get to the same result. But we have $b' = \tilde{f}_{j,l}\tilde{e}_{j,l}b' \neq 0$ and hence $\phi_j(\tilde{e}_{j,l}(b')) \neq 0$ by normality of $\mathcal{B}'_j$. Since $(j, j) \leqslant 0$, we have

$$\mathrm{wt}_j(b') = \mathrm{wt}_j(\tilde{e}_{j,l}(b')) - l\langle e_j, \alpha_j\rangle \geqslant \mathrm{wt}_j(\tilde{e}_{j,l}(b')) > 0;$$

hence, $\phi_j(b') = +\infty$, and

$$\Psi_i(\tilde{e}_{j,l}(b)) = \tilde{e}_{j,l}(\mathsf{c} \otimes b') = \mathsf{c} \otimes \tilde{e}_{j,l}(b') \neq 0,$$

which proves that $\tilde{e}_{j,l}(b) \neq 0$.

Hence, any element can be written $b = \tilde{f}_{\iota_1} \dots \tilde{f}_{\iota_1}(b_0)$ for some $\iota_k \in I_\infty$. The end of the proof is analogous to the one given in [KS97]; one just has to replace I by I_∞ (which is countably infinite). □

3.4.2 *Geometric realization.*

Notation 3.19. From Proposition 1.11, we have the following bijections:

$$\mathrm{Irr}\,\Lambda(\nu)_{i,l} \xrightarrow[\sim]{\mathfrak{k}_{i,l}} \mathrm{Irr}\,\Lambda(\nu - le_i)_{i,0} \times \mathcal{C}_{i,l},$$

where $\nu \in P^+ = \mathbb{N}^I$, $i \in I$, $l > 0$. Set, for $\mathsf{c} \in \mathcal{C}_{i,l}$,

$$\mathrm{Irr}\,\Lambda_{i,l} = \bigsqcup_{\nu \in P^+} \mathrm{Irr}\,\Lambda(\nu)_{i,l},$$
$$\mathrm{Irr}\,\Lambda(\nu)_{i,\mathsf{c}} = \mathfrak{k}_{i,l}^{-1}(\mathrm{Irr}\,\Lambda(\nu - le_i)_{i,0} \times \{\mathsf{c}\}),$$
$$\mathrm{Irr}\,\Lambda_{i,\mathsf{c}} = \bigsqcup_{\nu \in P^+} \mathrm{Irr}\,\Lambda(\nu)_{i,\mathsf{c}},$$
$$\mathrm{Irr}\,\Lambda = \bigsqcup_{\nu \in P^+} \mathrm{Irr}\,\Lambda(\nu)$$

and denote by $\tilde{e}_{i,\mathsf{c}}$ and $\tilde{f}_{i,\mathsf{c}}$ the inverse bijections

$$\tilde{e}_{i,\mathsf{c}} : \mathrm{Irr}\,\Lambda_{i,\mathsf{c}} \rightleftarrows \mathrm{Irr}\,\Lambda_{i,0} : \tilde{f}_{i,\mathsf{c}}$$

induced by $\mathfrak{k}_{i,l}$. Then, for every $l > 0$, we define

$$\tilde{e}_{i,l} = \bigsqcup_{\mathsf{c} \in \mathcal{C}_i} \tilde{f}_{i,\mathsf{c}\backslash l}\tilde{e}_{i,\mathsf{c}} : \mathrm{Irr}\,\Lambda \to \mathrm{Irr}\,\Lambda \sqcup \{0\},$$
$$\tilde{f}_{i,l} = \tilde{f}_{i,(l)} \sqcup \left(\bigsqcup_{\mathsf{c} \in \mathcal{C}_i} \tilde{f}_{i,(l,\mathsf{c})}\tilde{e}_{i,\mathsf{c}}\right) : \mathrm{Irr}\,\Lambda \to \mathrm{Irr}\,\Lambda \sqcup \{0\},$$

where $\tilde{f}_{i,\mathsf{c}\backslash l} = 0$ if $\omega_i \geqslant 2$ and $l \neq \mathsf{c}_1$, or if $\omega_i = 1$ and $m_l(\mathsf{c}) = 0$.

It is obvious from the definitions that we have the following result.

Proposition 3.20. *The set* $\operatorname{Irr}\Lambda$ *is a crystal with respect to* $\mathrm{wt} : Z \in \operatorname{Irr}\Lambda(\nu) \mapsto -C\nu$, ϵ_i *the composition of* $\bigsqcup_{l>0} \mathfrak{k}_{i,l}$ *and the second projection, and* $\tilde{e}_{i,l}, \tilde{f}_{i,l}$ *the maps defined above.*

The duality $\Lambda \to \Lambda$, $x \mapsto x^*$ induces a bijection $* : \operatorname{Irr}\Lambda \to \operatorname{Irr}\Lambda$, $Z \mapsto Z^*$ preserving the grading. Following [KS97], we note that

$$\epsilon_i^* = *\epsilon_i *,$$
$$\tilde{e}_{i,l}^* = *\tilde{e}_{i,l}*,$$
$$\tilde{f}_{i,l}^* = *\tilde{f}_{i,l} * .$$

Note that $\epsilon_i^*(Z)$ is the dimension of the largest subspace of $\bigcap_{h\in H_i} \ker x_h$ stable by $(x_h)_{h\in H_i}$, for a generic element $x \in Z$. We will denote $\tilde{e}_i^{*\max}(Z)$ instead of $\tilde{e}_{i,\mathsf{c}}(Z)$ when $\mathsf{c} = \epsilon_i^*(Z)$. We have the following result, corresponding to [KS97, 5.3.1] when i is real.

Proposition 3.21. *Consider* $Z \in \operatorname{Irr}\Lambda(\nu)$ *such that* $\epsilon_i^*(Z) = \mathsf{c} \neq 0$ *for some imaginary vertex* i, *and set* $\bar{Z} = \tilde{e}_{i,\mathsf{c}}^*(Z)$. *Assume that* $\mathrm{wt}_i(\bar{Z}) > 0$. *We have:*

(i) $\epsilon_i(Z) = \epsilon_i(\bar{Z})$;

(ii) $\begin{cases} \epsilon_i^*(\tilde{e}_\iota(Z)) = \epsilon_i^*(Z) \\ \tilde{e}_i^{*\max}(\tilde{e}_\iota(Z)) = \tilde{e}_\iota(\bar{Z}) \end{cases}$ *for every* $\iota \in I_\infty$.

Proof. The proof is actually simpler than in the real case. Indeed, in the proof of Proposition 1.11, consider $y \in \bar{Z}^*$ and $z \in \mathsf{c}^*$ (we abusively identify $\operatorname{Irr}\Lambda(le)$ with $\mathcal{C}_{i,l}$). We want

$$0 = \sum_{h\in H_i} y_{\bar{h}}\eta_h + \sum_{h\in\Omega(i)} [(y_{\bar{h}}\eta_h - \eta_h z_{\bar{h}}) + (y_h\eta_{\bar{h}} - \eta_{\bar{h}} z_h)].$$

Note that

$$0 < \mathrm{wt}_i(\bar{Z}) = \sum_{h\in H_i} \nu_{t(h)} - (i,i)(\nu_i - |\mathsf{c}|) \Leftrightarrow 0 < \sum_{h\in H_i} \nu_{t(h)}$$

since $(i,i) \leqslant 0$, and since it is impossible to have $\sum_{h\in H_i} \nu_{t(h)} = 0$ and $\nu_i - |\mathsf{c}| > 0$. Hence, there exists $h_0 \in H_i$ such that $\nu_{h_0} > 0$. We have $\operatorname{Spec}(z_{\bar{h}_1}) \cap \operatorname{Spec}(y_{\bar{h}_1}) = \emptyset$ for a generic choice of y, z, where $h_1 \in \Omega(i)$. But then the map

$$\eta_{h_1} \mapsto y_{\bar{h}_1}\eta_{h_1} - \eta_{h_1} z_{\bar{h}_1}$$

is invertible, and we can generically choose η_{h_0} so that

$$\dim \mathbb{C}\langle z_h^* \mid h \in H_i\rangle \cdot \operatorname{Im}\eta_{h_0}^* = |\mathsf{c}|.$$

This proves that $\epsilon_i^*(Z^*) = \epsilon_i^*(\bar{Z}^*)$ and hence (1). The second statement directly follows from the proof of (1). □

Theorem 3.12. *We have* $\operatorname{Irr}\Lambda \simeq \mathcal{B}(\infty)$.

Proof. Set $\Psi_i(Z) = \epsilon_i^*(Z) \otimes \tilde{e}_i^{*\max}(Z)$, which is clearly injective. By Proposition 3.21 and the definition of our generalized crystals, Ψ_i is a morphism. Note that we have $\Psi(\tilde{e}_{i,l}(Z)) = \tilde{e}_{i,l}(\epsilon_i^*(Z)) \otimes \bar{Z}$ if $\mathrm{wt}_i(\bar{Z}) = 0$. By Remark 1.10 (or its dual analog), the condition (5) of Proposition 3.17 is satisfied. The condition (6) is satisfied because it is clear that $\tilde{f}_{i,l}(\bar{Z}) \notin \mathcal{B}_i'$ if $\phi_i(\bar{Z}) = 0$. Hence, we get the result. □

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

3.4.3 *Semicanonical basis.* The following proposition is proved in [Boz15].

Proposition 3.22. *There exists a surjective morphism $\Phi : U^+_{v=1} \to \mathcal{M}_\circ$ defined by*

$$\begin{cases} E_{i,a} \mapsto 1_{i,l} & \text{if } i \in I^{\mathrm{im}}, \\ E_i \mapsto 1_i & \text{if } i \in I^{\mathrm{re}}. \end{cases}$$

Thanks to Theorem 3.12, we now have the following result.

Theorem 3.13. *The morphism Φ is an isomorphism: $U^+_{v=1} \xrightarrow{\sim} \mathcal{M}_\circ$.*

Proof. The family $(f_Z)_{Z \in \operatorname{Irr}\Lambda}$ is clearly free, so we have

$$|\operatorname{Irr}\Lambda(\nu)| \leqslant \dim \mathcal{M}_\circ(\nu) \leqslant \dim U^+_{v=1}[\nu],$$

the latter inequality being true thanks to Proposition 3.22. From Theorem 3.12, we have $|\operatorname{Irr}\Lambda(\nu)| = \dim U^+_{v=1}[\nu]$; hence, $(f_Z)_{Z \in \operatorname{Irr}\Lambda}$ is a basis of $\mathcal{M}_\circ$, and Φ is an isomorphism. □

Definition 3.14. The semicanonical basis of $U^+_{v=1}$ is the pullback of $(f_Z)_{Z \in \operatorname{Irr}\Lambda}$.

3.5 The crystals $\mathcal{B}(\lambda)$

3.5.1 *Algebraic definition.* We will use the *fundamental weights* $(\Lambda_i)_{i \in I}$ defined by $(i, \Lambda_j) = \delta_{i,j}$ for every $i, j \in I$. Note that the isomorphism $P \xrightarrow{\sim} \sum \mathbb{Z}\Lambda_i$, $e_i \mapsto \Lambda_i$ maps α_i to i. We use this isomorphism to identify $\sum \mathbb{Z}\Lambda_i$ with P and $\sum \mathbb{N}\Lambda_i$ with P^+. We call *dominant* the elements $\lambda \in P^+$, which are the ones satisfying $(i, \lambda) \geqslant 0$ for every $i \in I$.

Definition 3.15. We denote by $\mathcal{O}$ the category of U-modules satisfying:

(i) $M = \bigoplus_{\mu \in P} M_\mu$, where $M_\mu = \{m \in M \mid \forall i, K_i m = v^{(\mu,i)} m\}$;
(ii) for any $m \in M$, there exists $p \geqslant 0$ such that $xm = 0$ as soon as $x \in U^+[\nu]$ and $\operatorname{ht}(\nu) \geqslant p$.

For any $\lambda \in P$, we define a *Verma module*

$$M(\lambda) = \frac{U}{\sum_{\iota \in I_\infty} U E_\iota + \sum_{i \in I} U\big(K_i - v^{(i,\lambda)}\big)} \in \mathcal{O}$$

and the following simple quotient:

$$\pi_\lambda : U \twoheadrightarrow V(\lambda) = \frac{M(\lambda)}{M(\lambda)^-} \in \mathcal{O},$$

where $M(\lambda)^-$ is the sum of all strict submodules of $M(\lambda)$. We will denote by $v_\lambda \in V(\lambda)_\lambda$ the image of $1 \in U$.

Remark 3.23. Note that thanks to Proposition 3.10(2), we have a triangular decomposition, and thus $M(\lambda) = U^- v_\lambda$.

We have the following proposition, generalizing the case $i \in I^{\mathrm{re}}$.

Proposition 3.24. *Assume that $(i, \lambda) \geqslant 0$ for some imaginary vertex i. Then we have the following decomposition:*

$$V(\lambda) = \bigoplus_{\mathsf{c} \in \mathcal{C}_i} b_{i,\mathsf{c}} \mathcal{K}_i,$$

where $\mathcal{K}_i = \bigcap_{l>0} \ker E_{i,l}$.

Proof. Let us first prove the existence. Consider $v \in V(\lambda)$, and assume first that v is of the following form: $v = ub_{i,\mathsf{c}}z$ for some $\mathsf{c} \in \mathcal{C}_i$, $u \in U^-$ satisfying $[a_{i,l}, u] = 0$ for every l, and $z \in \mathcal{K}_i$. We proceed by induction on $|\mathsf{c}|$. First note that if $(i, |u|) = 0$, since i is imaginary, one necessarily has

$$\mathrm{supp}|u| \subseteq \{j \in I \mid (i, j) = 0\}.$$

Hence, $[u, b_{i,l}] = 0$ for any l (whether i is isotropic or not) and we get the result by induction. Otherwise, $(i, |u|) > 0$, and we set

$$\begin{aligned} l &= \mathsf{c}_1, \\ [u, b_{i,l}]^\circ &= ub_{i,l} - R(v)b_{i,l}u \quad \text{for some } R \in \mathbb{Q}(v), \\ z' &= b_{i,\mathsf{c}\backslash \mathsf{c}_1} z \in V(\lambda)_\mu. \end{aligned}$$

For any $k > 0$, we have

$$\begin{aligned} [a_{i,k}, [u, b_{i,l}]^\circ]z' &= \delta_{l,k}\tau_{i,l}\{u(K_{-li} - K_{li}) - R(v)(K_{-li} - K_{li})u\}z' \quad [\text{cf. Definition 3.10(2)}] \\ &= \delta_{l,k}\tau_{i,l}u\{(v^{-l(i,\mu)} - v^{l(i,\mu)}) - R(v)(v^{-l(i,|u|+\mu)} - v^{l(i,|u|+\mu)})\}z' \\ &= 0 \end{aligned}$$

if

$$R(v) = \frac{v^{-l(i,\mu)} - v^{l(i,\mu)}}{v^{-l(i,|u|+\mu)} - v^{l(i,|u|+\mu)}},$$

which is possible since

$$(i, |u| + \mu) = (i, \lambda) + (i, |u|) + (i, \mu - \lambda) > (i, \lambda) + (i, \mu - \lambda) \geqslant 0.$$

We have used that since i is imaginary, we have

$$\mu - \lambda \in -\mathbb{N}I \Rightarrow (i, \mu - \lambda) \geqslant 0.$$

Hence, the following equality:

$$v = [u, b_{i,l}]^\circ b_{i,\mathsf{c}\backslash \mathsf{c}_1} z + R(v)b_{i,l}ub_{i,\mathsf{c}\backslash \mathsf{c}_1} z$$

along with the induction hypothesis allow us to conclude the proof since $|\mathsf{c}\backslash \mathsf{c}_1| < |\mathsf{c}|$ and since $\bigoplus_{\mathsf{c}\in\mathcal{C}_i} b_{i,\mathsf{c}}\mathcal{K}_i$ is stable by left multiplication by $b_{i,l}$.

Then we prove the existence of the decomposition for a general $v \in V(\lambda)_\mu$, using induction on $\sum(\lambda_i - \mu_i)$. If $v \neq v_\lambda$, thanks to Remark 3.23, we can write

$$v = \sum_{\iota \in I_\infty} b_\iota v_\iota$$

for some finitely many nonzero $v_\iota \in V(\lambda)$. Thanks to our induction hypothesis, we have

$$v = \sum_{\iota\in I_\infty, \mathsf{c}\in\mathcal{C}_i} b_\iota b_{i,\mathsf{c}} z_{\iota,\mathsf{c}}$$

for some finitely many nonzero $z_{\iota,\mathsf{c}} \in \mathcal{K}_i$. Then

$$v = \sum_{l>0, \mathsf{c}\in\mathcal{C}_i} b_{i,(l,\mathsf{c})} z_{(i,l),\mathsf{c}} + \sum_{\substack{\iota\in I_\infty\backslash(\{i\}\times\mathbb{N}_{>0}) \\ \mathsf{c}\in\mathcal{C}_i}} b_\iota b_{i,\mathsf{c}} z_{\iota,\mathsf{c}}$$

and we have the result since $b_\iota b_{i,\mathsf{c}} z_{\iota,\mathsf{c}}$ is of the form $ub_{i,\mathsf{c}}z$ already treated. Indeed, thanks to Proposition 3.10(2), $[a_{i,l}, b_{j,k}] = 0$ for any $l, k > 0$ if $j \neq i$.

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

To prove the unicity of the decomposition, consider a minimal nontrivial relation of dependence:

$$0=\sum_{\mathsf{c}\in\mathcal{C}_i} b_{i,\mathsf{c}} z_{\mathsf{c}},$$

where $z_{\mathsf{c}}\in V(\lambda)_{\mu+|\mathsf{c}|i}\cap\mathcal{K}_i$. We have to consider separately the cases $i\in I^{\mathrm{iso}}$ and $i\notin I^{\mathrm{iso}}$. First, consider $i\notin I^{\mathrm{iso}}$. Consider r maximal such that there exists $\mathsf{c}=(\mathsf{c}_1,\ldots,\mathsf{c}_r)$ such that $z_{\mathsf{c}}\neq 0$. Set, for any $k\in[\![1,r]\!]$,

$$\begin{aligned}\mathsf{c}_{<k}&=(\mathsf{c}_1,\ldots,\mathsf{c}_{k-1}),\\ \mathsf{c}_{>k}&=(\mathsf{c}_{k+1},\ldots,\mathsf{c}_r)\end{aligned}$$

with the convention $\mathsf{c}_{<1}=\emptyset=\mathsf{c}_{>r}$. Then, if $l>0$, we get the following from Proposition 3.10(2), where by convention $b_{i,\emptyset}=1$:

$$\begin{aligned}[][a_{i,l},b_{i,\mathsf{c}}]&=\tau_{i,l}\sum_{k:\mathsf{c}_k=l} b_{i,\mathsf{c}_{<k}}(K_{-li}-K_{li})b_{i,\mathsf{c}_{>k}}\\ &=\tau_{i,l}\sum_{k:\mathsf{c}_k=l} b_{i,\mathsf{c}\setminus\mathsf{c}_k}(v_i^{2l|\mathsf{c}_{>k}|}K_{-li}-v_i^{-2l|\mathsf{c}_{>k}|}K_{li}).\end{aligned}$$

Then, since $z_{\mathsf{c}}\in\mathcal{K}_i$,

$$\begin{aligned}a_{i,l}b_{i,\mathsf{c}}z_{\mathsf{c}}&=\tau_{i,l}\sum_{k:\mathsf{c}_k=l} b_{i,\mathsf{c}\setminus\mathsf{c}_k}(v_i^{2l|\mathsf{c}_{>k}|}K_{-li}-v_i^{-2l|\mathsf{c}_{>k}|}K_{li})z_{\mathsf{c}}\\ &=\tau_{i,l}\sum_{k:\mathsf{c}_k=l} b_{i,\mathsf{c}\setminus\mathsf{c}_k}(v^{-l(i,\mu+|\mathsf{c}_{\leqslant k}|i)}-v^{l(i,\mu+|\mathsf{c}_{\leqslant k}|i)})z_{\mathsf{c}},\end{aligned}$$

where $\mathsf{c}_{\leqslant k}=(\mathsf{c}_{<k},\mathsf{c}_k)$. We see that for any $\mathsf{c}'\in\mathfrak{S}_r\mathsf{c}$ (with the convention $(\sigma\mathsf{c})_k=\mathsf{c}_{\sigma(k)}$), since r is maximal, we have

$$0=a_{i,\mathsf{c}'}\sum_{\mathsf{c}''\in\mathcal{C}_i} b_{i,\mathsf{c}''}z_{\mathsf{c}''}=a_{i,\mathsf{c}'}\sum_{\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}} b_{i,\mathsf{c}''}z_{\mathsf{c}''}=\tau_{i,\mathsf{c}}\sum_{\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}} P_{\mathsf{c}',\mathsf{c}''}(v)z_{\mathsf{c}''},$$

where $P_{\mathsf{c}',\mathsf{c}''}(v)\in\mathbb{Z}[v,v^{-1}]$. Since $(z_{\mathsf{c}''})_{\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}}\neq 0$, we have to prove that

$$\Delta(v)=\det(P_{\mathsf{c}',\mathsf{c}''}(v))_{\mathsf{c}',\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}}\neq 0\in\mathbb{Z}[v,v^{-1}]$$

to end our proof in the case $(i,i)<0$. Note that $\lambda-(\mu+|\mathsf{c}|i)\in\mathbb{N}I$; hence, since i is imaginary,

$$(i,\mu+|\mathsf{c}|i)=(i,\lambda)+(i,\mu+|\mathsf{c}|i-\lambda)\geqslant 0.$$

Then, for any $\mathsf{c}'\in\mathfrak{S}_r\mathsf{c}$, one has

$$\max_{\mathsf{c}''\in\mathfrak{S}_r\mathsf{c}}\{\deg(P_{\mathsf{c}',\mathsf{c}''})\}=\deg(P_{\mathsf{c}',\bar{\mathsf{c}}'})=\sum_{1\leqslant k\leqslant r}\mathsf{c}_k(i,\mu+\mathsf{c}_k i)=m,$$

which is only reached for $\mathsf{c}''=\bar{\mathsf{c}}'$. However, this is not true if $m=0$, which can only happen if our initial relation of dependence is of the form $b_{i,l}z_l=0$, with $(i,\mu+li)=0$. But, if $(i,\mu+li)=0$, the module generated by $b_{i,l}z_l$ is a nontrivial strict submodule of $V(\lambda)$ since, for every $k>0$ and $j\neq i$,

$$\begin{aligned}a_{i,k}b_{i,l}z_l&=0\\ a_{j,k}b_{i,l}z_l&=b_{i,l}a_{j,k}z_l.\end{aligned}$$

Hence, the relation of dependence $b_{i,l}z_l=0$ is actually trivial by definition of $V(\lambda)$.

Otherwise, the application $\mathfrak{S}_r\mathsf{c} \to \mathfrak{S}_r\mathsf{c}$, $\mathsf{c}' \mapsto \bar{\mathsf{c}}'$ being a permutation, the degree of Δ is $|\mathfrak{S}_r\mathsf{c}|m > 0$, and in particular $\Delta \neq 0$.

We finally have to prove the uniqueness in the case $(i,i) = 0$. Write a relation of dependence of minimal degree:

$$0 = \sum_{\nu \in \mathcal{C}_i} b_{i,\nu} z_\nu,$$

where $z_\nu \in V(\lambda)_{\mu+|\nu|i} \cap \mathcal{K}_i$. For any $l > 0$, we have, thanks to Proposition 3.10(2),

$$\begin{aligned} a_{i,l} \sum_{\nu \in \mathcal{C}_i} b_{i,\nu} z_\nu &= \sum_{\nu \in \mathcal{C}_i} m_l(\nu) \tau_{i,l}(v^{-l(i,\mu)} - v^{l(i,\mu)}) b_{i,\nu \setminus l} z_\nu \\ &= \tau_{i,l}(v^{-l(i,\mu)} - v^{l(i,\mu)}) \sum_{\nu' \in \mathcal{C}_i} b_{i,\nu'} \{(m_l(\nu')+1) z_{\nu' \cup l}\}, \end{aligned}$$

which contradicts the minimality of the first relation. Note that we can assume that $l(i,\mu) \neq 0$: otherwise, we would again have an initial trivial relation of dependence (more precisely, for every ν we would have $b_{i,\nu} z_\nu = 0 \in V(\lambda)$). $\square$

This proposition allows us to define Kashiwara operators $\tilde{e}_\iota$, $\tilde{f}_\iota$ on each $V(\lambda)$, exactly as in Definition 3.6.

DEFINITION 3.16. If i is imaginary and $v = \sum_{\mathsf{c} \in \mathcal{C}_i} b_{i,\mathsf{c}} z_{\mathsf{c}} \in V(\lambda)$, set

$$\tilde{e}_{i,l}(v) = \begin{cases} \displaystyle\sum_{\mathsf{c}:\mathsf{c}_1 = l} b_{i,\mathsf{c} \setminus \mathsf{c}_1} z_{\mathsf{c}} & \text{if } i \notin I^{\text{iso}}, \\ \displaystyle\sum_{\nu \in \mathcal{C}_i} \sqrt{\frac{m_l(\nu)}{l}} b_{i,\nu \setminus l} z_\nu & \text{if } i \in I^{\text{iso}}, \end{cases}$$

$$\tilde{f}_{i,l}(v) = \begin{cases} \displaystyle\sum_{\mathsf{c} \in \mathcal{C}_i} b_{i,(l,\mathsf{c})} z_{\mathsf{c}} & \text{if } i \notin I^{\text{iso}}, \\ \displaystyle\sum_{\nu \in \mathcal{C}_i} \sqrt{\frac{l}{m_l(\nu)+1}} b_{i,\nu \cup l} z_\nu & \text{if } i \in I^{\text{iso}}. \end{cases}$$

The following result will be proved in § 3.6.

THEOREM 3.17. *Assume that λ is dominant. The Kashiwara operators, along with the maps*

$$\begin{aligned} \mathrm{wt}(m) &= \mu \text{ if } m \in V(\lambda)_\mu, \\ \epsilon_i(m) &= \overline{\max\{\bar{\mathsf{c}} \mid \tilde{e}_{i,\mathsf{c}}(m) \neq 0\}}, \end{aligned}$$

induce a structure of crystal on

$$\mathcal{B}(\lambda) = \{\tilde{f}_{\iota_1} \ldots \tilde{f}_{\iota_s} v_\lambda \mid \iota_k \in I_\infty\} \subset \frac{\mathcal{L}(\lambda)}{v^{-1}\mathcal{L}(\lambda)},$$

where

$$\mathcal{L}(\lambda) = \sum_{\iota_1, \ldots, \iota_s} \mathcal{A} \tilde{f}_{\iota_1} \ldots \tilde{f}_{\iota_s} v_\lambda.$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Remark 3.25. These crystals are normal: consider $m \in V(\lambda)_\mu$ and i imaginary (again, the case of real vertices is already known). We have already seen that we necessarily have $(i,\mu) \geqslant 0$ since λ is dominant. If $(i,\mu) = 0$ for some imaginary vertex i, then $a_\iota b_{i,l} m = b_{i,l} a_\iota m$ for any $\iota \in I_\infty$ (use Proposition 3.10(2) if $\iota = (i,l)$). Hence, for any $l > 0$, the submodule of $V(\lambda)$ spanned by $b_{i,l} m$ is a strict submodule, and we get $\tilde{f}_{i,l} m = 0$.

Otherwise, $(i,\mu) > 0$ and, for every $\mu' \in -\mathbb{N}I$, since $(i,i) \leqslant 0$, we get

$$(i, \mu + \mu') = (i,\mu) + (i,\mu') \geqslant (i,\mu) > 0;$$

hence, $\max\{|\mathsf{c}| \mid b_{i,\mathsf{c}} m \neq 0\} = +\infty$.

3.5.2 *Geometric realization.*

Notation 3.26. Consider λ dominant. We have the following bijections:

$$\operatorname{Irr} \mathfrak{L}(\nu,\lambda)_{i,l} \xrightarrow[\sim]{\mathfrak{l}_{i,l}} \operatorname{Irr} \mathfrak{L}(\nu - l e_i, \lambda)_{i,0} \times \mathcal{C}_{i,l};$$

each time the left-hand side is nonempty (cf. Proposition 2.13). Set, for $\mathsf{c} \in \mathcal{C}_{i,l}$,

$$\begin{aligned}
\operatorname{Irr} \mathfrak{L}(\lambda)_{i,l} &= \bigsqcup_{\nu \in P^+} \operatorname{Irr} \mathfrak{L}(\nu,\lambda)_{i,l},\\
\operatorname{Irr} \mathfrak{L}(\nu,\lambda)_{i,\mathsf{c}} &= \mathfrak{l}_{i,l}^{-1}(\operatorname{Irr} \mathfrak{L}(\nu - l e_i,\lambda)_{i,0} \times \{\mathsf{c}\}),\\
\operatorname{Irr} \mathfrak{L}(\lambda)_{i,\mathsf{c}} &= \bigsqcup_{\nu \in P^+} \operatorname{Irr} \mathfrak{L}(\nu,\lambda)_{i,\mathsf{c}},\\
\operatorname{Irr} \mathfrak{L}(\lambda) &= \bigsqcup_{\nu \in P^+} \operatorname{Irr} \mathfrak{L}(\nu,\lambda)
\end{aligned}$$

and denote by $\tilde{e}_{i,\mathsf{c}}$ and $\tilde{f}_{i,\mathsf{c}}$ the inverse bijections

$$\tilde{e}_{i,\mathsf{c}} : \operatorname{Irr} \mathfrak{L}(\lambda)_{i,\mathsf{c}} \rightleftarrows \operatorname{Irr} \mathfrak{L}(\lambda)_{i,0} : \tilde{f}_{i,\mathsf{c}}$$

induced by $\mathfrak{l}_{i,l}$. Then, for every $l > 0$, we define

$$\begin{aligned}
\tilde{e}_{i,l} &= \bigsqcup_{\mathsf{c} \in \mathcal{C}_i} \tilde{f}_{i,\mathsf{c} \setminus \mathsf{c}_1} \tilde{e}_{i,\mathsf{c}} : \operatorname{Irr} \mathfrak{L}(\lambda) \to \operatorname{Irr} \mathfrak{L}(\lambda) \sqcup \{0\},\\
\tilde{f}_{i,l} &= \tilde{f}_{i,(l)} \sqcup \left(\bigsqcup_{\mathsf{c} \in \mathcal{C}_i} \tilde{f}_{i,(l,\mathsf{c})} \tilde{e}_{i,\mathsf{c}} \right) : \operatorname{Irr} \mathfrak{L}(\lambda) \to \operatorname{Irr} \mathfrak{L}(\lambda) \sqcup \{0\}
\end{aligned}$$

with the same conventions as in Notation 3.19.

The following result is a direct consequence of Proposition 2.13.

Proposition 3.27. *The set* $\operatorname{Irr} \mathfrak{L}(\lambda)$ *is a crystal with respect to*

$$\mathrm{wt} : b \in \operatorname{Irr} \mathfrak{L}(\nu,\lambda) \mapsto \lambda - C\nu,$$

ϵ_i *the composition of* $\bigsqcup_{l>0} \mathfrak{l}_{i,l}$ *and the second projection, and* $\tilde{e}_{i,l}, \tilde{f}_{i,l}$ *the maps defined above.*

Remark 3.28. Thanks to Proposition 2.13 and the classical case, we have, for every $i \in I$,

$$\phi_i(b) = \max\{|\mathsf{c}| \in \mathbb{N} \mid \tilde{f}_{i,\mathsf{c}}(b) \neq 0\}.$$

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Indeed, for $b \in \operatorname{Irr} \mathfrak{L}(\nu, \lambda)$, it is impossible to have $\nu_i > 0$ and $\lambda_i + \sum_{h \in H_i} \nu_{t(h)} = 0$; hence,

$$\lambda_i + \sum_{h \in H_i} \nu_{t(h)} > 0 \Leftrightarrow \langle e_i, \lambda - C\nu \rangle > 0,$$

and $\operatorname{Irr} \mathfrak{L}(\lambda)$ is normal.

In an analogous way, one can equip $\operatorname{Irr} \widetilde{\mathfrak{Z}}$ with a structure of crystal, thanks to Proposition 2.20, and get the following result.

THEOREM 3.18. *The crystal structure on* $\operatorname{Irr} \widetilde{\mathfrak{Z}}$ *coincides with that of the tensor product* $\operatorname{Irr} \mathfrak{L}(\lambda) \otimes \operatorname{Irr} \mathfrak{L}(\lambda')$.

Proof. This is a direct consequence of Theorem 2.6 along with the proofs of Propositions 2.23 and 2.24. □

We will see in § 3.6 how the previous theorem leads to the following result.

THEOREM 3.19. *If* λ *is dominant, we have the following isomorphism of crystals:* $\mathcal{B}(\lambda) \simeq \operatorname{Irr} \mathfrak{L}(\lambda)$.

3.6 Grand-loop argument

To prove Theorems 3.11, 3.17 and 3.19, one has to generalize Kashiwara's grand-loop argument to our framework (see [Kas91]).

Instead of giving the whole grand-loop argument, we give a few lemmas that yield its generalization.

Notation 3.29. When working with a $\mathcal{A}$-lattice $\mathcal{L}$, we will write $m \equiv m'$ instead of $m = m' + v^{-1}\mathcal{L}$ for any $m, m' \in \mathcal{L}$.

The following result is about the tensor product.

LEMMA 3.30. *Consider two dominant weights* λ *and* λ', *and* $(m, m') \in \mathcal{L}(\lambda) \times \mathcal{L}(\lambda')_{\mu'}$. *Then, for every imaginary vertex* i *and* $l > 0$, *we have*

$$b_{i,l}(m \otimes m') \equiv \begin{cases} m \otimes b_{i,l} m' & \textit{if } \operatorname{wt}_i(m') > 0, \\ b_{i,l} m \otimes m' & \textit{otherwise.} \end{cases}$$

Remark 3.31. Note that since $\epsilon_i(m) \neq -\infty$ in this situation, this is exactly Definition 3.10(7).

Proof. We have already seen that when $i \in I^{\mathrm{im}}$, since $\mu' - \lambda' \in -\mathbb{N}I$, we have

$$(i, \mu') = (i, \lambda') + (i, \mu' - \lambda') \geqslant 0.$$

We have also already seen that thanks to Proposition 3.10(2), if $(i, \mu') = 0$, then $a_\iota b_{i,l} m' = b_{i,l} a_\iota m'$ for every $\iota \in I_\infty$. Hence, $b_{i,l} m' = 0$ since the module spanned by $b_{i,l} m'$ is a strict submodule of $V(\lambda')$. Hence,

$$\begin{aligned} b_{i,l}(m \otimes m') &= b_{i,l} m \otimes K_{-li} m' + m \otimes b_{i,l} m' \\ &= v^{-l(i,\mu')} b_{i,l} m \otimes m' + m \otimes b_{i,l} m' \\ &\equiv \begin{cases} m \otimes b_{i,l} m' & \text{if } \operatorname{wt}_i(m') > 0 \\ b_{i,l} m \otimes m' & \text{otherwise.} \end{cases} \end{aligned}$$

□

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Lemma 3.32. *Consider $i \in I^{\mathrm{im}}$ and $l > 0$. We have $\tau_{i,l} \equiv 1/l$ if $i \in I^{\mathrm{iso}}$, $\tau_{i,l} \equiv 1$ otherwise.*

Proof. First note that for any $i \in I^{\mathrm{im}}$ and $l > 0$, $\{E_{i,l}, E_{i,l}\} \equiv 1$ is required by Proposition 3.6. Assume moreover that

$$\{E_{i,l}, E_{i,l}\} = \prod_{1 \leqslant k \leqslant l} \frac{1}{1 - v^{-k}},$$

which is consistent with [Boz15, Theorem 1]. Then, when $i \in I^{\mathrm{iso}}$, we have an isomorphism from the ring of symmetric functions $\Lambda = \mathbb{Z}[x_k, k \geqslant 1]$ to $\mathbb{Z}[E_{i,l}, l \geqslant 1]$ mapping the elementary symmetric functions e_l to $v^{-l/2} E_{i,l}$ and such that the pushforward of $\{-,-\}$ is the Hall–Littlewood scalar product (still denoted by $\{-,-\}$). Asking for $a_{i,l}$ to be primitive and to satisfy $E_{i,l} - a_{i,l} \in \mathbb{Q}(v)[E_{i,k}, k < l]$ means that the power sum symmetric functions p_l are mapped to $v^{-l/2}(-1)^{l-1} l a_{i,l}$. Since the Hall–Littlewood scalar product satisfies $\{p_l, p_l\} = (l v^{-l}/(1 - v^{-l}))$, we get

$$\tau_{i,l} = \{a_{i,l}, a_{i,l}\} = \frac{v^l}{l^2} \frac{l v^{-l}}{1 - v^{-l}} = \frac{1/l}{1 - v^{-l}} \equiv 1/l,$$

as expected.

If $i \notin I^{\mathrm{iso}}$, let us prove that $a_{i,l} \equiv E_{i,l}$ by induction on l. Write

$$a_{i,l} - E_{i,l} = \sum_{\mathsf{c} \in \mathcal{C}_{i,l} \setminus \{(l)\}} \alpha_{\mathsf{c}} a_{i,\mathsf{c}}$$

for some $\alpha_{\mathsf{c}} \in \mathbb{Q}(v)$. By Proposition 3.8, for every $\mathsf{c}' \in \mathcal{C}_{i,l} \setminus \{(l)\}$, we have

$$\begin{aligned}
\sum_{\mathsf{c} \in \mathcal{C}_{i,l} \setminus \{(l)\}} \alpha_{\mathsf{c}} \{a_{i,\mathsf{c}}, a_{i,\mathsf{c}'}\} &= -\{E_{i,l}, a_{i,\mathsf{c}'}\} \\
&= -\{\delta(E_{i,l}), a_{i,\mathsf{c}'_1} \otimes a_{i,\mathsf{c}' \setminus \mathsf{c}'_1}\} \\
&= -v_i^{\mathsf{c}'_1 |\mathsf{c}' \setminus \mathsf{c}'_1|} \{E_{i,\mathsf{c}'_1}, a_{i,\mathsf{c}'_1}\} \{E_{i,|\mathsf{c}' \setminus \mathsf{c}'_1|}, a_{i,\mathsf{c}' \setminus \mathsf{c}'_1}\} \\
&= -v_i^{\sum \mathsf{c}'_k \mathsf{c}'_{k+1}} \prod \{E_{i,\mathsf{c}'_k}, a_{i,\mathsf{c}'_k}\} \\
&= -v_i^{\sum \mathsf{c}'_k \mathsf{c}'_{k+1}} \prod \tau_{i,\mathsf{c}'_k} \\
&\equiv 0
\end{aligned}$$

by the induction hypothesis and since $(i,i) < 0$. We have also used that $\tau_{i,k} = \{E_{i,k}, a_{i,k}\}$ since $\{a_{i,k}, a_{i,k} - E_{i,k}\} = 0$. Also, note that

$$\det(\{a_{i,\mathsf{c}}, a_{i,\mathsf{c}'}\})_{\mathsf{c},\mathsf{c}' \in \mathcal{C}_{i,l} \setminus \{(l)\}} \equiv 1$$

since

$$\begin{aligned}
\{a_{i,\mathsf{c}}, a_{i,\mathsf{c}'}\} &= \left\{ a_{i,\mathsf{c}_1} \otimes a_{i,\mathsf{c} \setminus \mathsf{c}_1}, \prod (a_{i,\mathsf{c}'_k} \otimes 1 + 1 \otimes a_{i,\mathsf{c}'_k}) \right\} \\
&= \sum_{k : \mathsf{c}'_k = \mathsf{c}_1} v_i^{\mathsf{c}'_{k-1} \mathsf{c}'_k} \tau_{i,\mathsf{c}_1} \{a_{i,\mathsf{c} \setminus \mathsf{c}_1}, a_{i,\mathsf{c}' \setminus \mathsf{c}'_k}\} \\
&\equiv \delta_{\mathsf{c}_1, \mathsf{c}'_1} \{a_{i,\mathsf{c} \setminus \mathsf{c}_1}, a_{i,\mathsf{c}' \setminus \mathsf{c}'_k}\} \\
&\equiv \cdots \equiv \delta_{\mathsf{c},\mathsf{c}'}.
\end{aligned}$$

Hence, we get $\alpha_{\mathsf{c}} \equiv 0$, which implies that $\tau_{i,l} = \{a_{i,l}, E_{i,l}\} \equiv \{E_{i,l}, E_{i,l}\} \equiv 1$. □

The following lemma deals with the behaviour of the generalized Kashiwara operators regarding our Hopf bilinear form $\{-,-\}$.

Lemma 3.33. *For any $u, v \in U^-$ and $(i,l) \in I_\infty$, $\{\tilde{f}_{i,l}u, v\} \equiv \{u, \tilde{e}_{i,l}v\}$.*

Proof. We can assume that $v = b_{i,\mathsf{c}}z$ for some $z \in \mathcal{K}_i$. If $i \in I^{\mathrm{im}}$ is not isotropic,

$$\begin{aligned}
\{\tilde{f}_{i,l}u, v\} &= \{b_{i,l}u, v\} \\
&= \{b_{i,l} \otimes u, \delta(b_{i,\mathsf{c}}z)\} \\
&= \tau_{i,l}\{u, \delta^{i,l}(b_{i,\mathsf{c}}z)\} \\
&= \tau_{i,l}\{u, \delta^{i,l}(b_{i,\mathsf{c}})z\} \quad \text{[cf. Definition 3.10(1)]} \\
&\equiv \{u, \delta^{i,l}(b_{i,\mathsf{c}})z\} \quad \text{[cf. Lemma 3.32]} \\
&= \sum_{k:\mathsf{c}_k = l} v_i^{l\mathsf{c}_{k-1}}\{u, b_{i,\mathsf{c}\setminus\mathsf{c}_k}z\} \quad \text{[cf. Definition 3.10(3)]} \\
&\equiv \{u, b_{i,\mathsf{c}\setminus\mathsf{c}_1}z\} \\
&= \{u, \tilde{e}_{i,l}v\}.
\end{aligned}$$

This computation also proves the case $i \in I^{\mathrm{re}}$, $l = 1$, which is already known. If i is isotropic, $v = b_{i,\nu}z$ and $u = b_{i,\nu'}z'$,

$$\begin{aligned}
\{\tilde{f}_{i,l}u, v\} &= \sqrt{\frac{l}{m_l(\nu')+1}}\{b_{i,l}u, v\} \\
&= \sqrt{\frac{l}{m_l(\nu')+1}}\{b_{i,l} \otimes u, \delta(b_{i,\nu}z)\} \\
&= \sqrt{\frac{l}{m_l(\nu')+1}}\tau_{i,l}\{u, \delta^{i,l}(b_{i,\nu}z)\} \\
&= \sqrt{\frac{l}{m_l(\nu')+1}}\tau_{i,l}\{u, \delta^{i,l}(b_{i,\nu})z\} \\
&= \sqrt{\frac{l}{m_l(\nu')+1}}\tau_{i,l}m_l(\nu)\{u, b_{i,\nu\setminus l}z\}.
\end{aligned}$$

We see by induction that $\{\tilde{f}_{i,l}u, v\} = \{u, \tilde{e}_{i,l}v\} = 0$ if $\nu \neq \nu' \cup l$. Otherwise, we get, thanks to Lemma 3.32,

$$\{\tilde{f}_{i,l}u, v\} \equiv \sqrt{\frac{m_l(\nu)}{l}}\{u, b_{i,\nu\setminus l}z\} = \{u, \tilde{e}_{i,l}v\}. \qquad \square$$

In order to get an analogous result regarding the lattices $\mathcal{L}(\lambda)$, first note that there exists for each $\lambda \in P^+$ a unique symmetric bilinear form $(-,-)$ on $V(\lambda)$ satisfying

$$\begin{aligned}
&(K_i u, u') = (u, K_i u'), \\
&(b_{i,l}u, u') = -(u, K_{-li}a_{i,l}u') \quad \text{if } i \in I^{\mathrm{im}}, \\
&(b_i u, u') = \frac{v}{v^{-1}-v}(u, K_{-i}a_i u') \quad \text{if } i \in I^{\mathrm{re}}, \\
&(v_\lambda, v_\lambda) = 1
\end{aligned}$$

for every $u, u' \in V(\lambda)$ and $(i,l) \in I_\infty$. Then we have the following result.

https://doi.org/10.1112/S0010437X1600751X Published online by Cambridge University Press

Lemma 3.34. *For every $u, v \in \mathcal{L}(\lambda)$ and $(i,l) \in I_\infty$, $(\tilde{f}_{i,l}u, v) \equiv (u, \tilde{e}_{i,l}v)$.*

Proof. Assume that $v = b_{i,\mathsf{c}}z$ for some nontrivial c and some $z \in \mathcal{K}_i \cap V(\lambda)_\mu$. Note that we have already seen that $(i,\mu) \geqslant 0$, and that $b_{i,l}z = 0$ if $(i,\mu) = 0$. Hence, we assume that $(i,\mu) > 0$, otherwise there is nothing to prove. If $\omega_i \geqslant 2$, we have

$$\begin{aligned}
(\tilde{f}_{i,l}u, v) &= (b_{i,l}u, v) \\
&= -(u, K_{-li}a_{i,l}v) \\
&= -(u, K_{-li}a_{i,l}b_{i,\mathsf{c}}z) \\
&= -\Big(u, K_{-li}\tau_{i,l}\sum_{k:\mathsf{c}_k=l}(v_i^{-2l|\mathsf{c}_{<k}|}K_{-li} - v_i^{2l|\mathsf{c}_{<k}|}K_{li})b_{i,\mathsf{c}\setminus\mathsf{c}_k}z\Big) \quad \text{[cf. proof of Proposition 3.24]} \\
&\equiv -\Big(u, \sum_{k:\mathsf{c}_k=l}(v_i^{-2l|\mathsf{c}_{<k}|}K_{-2li} - v_i^{2l|\mathsf{c}_{<k}|})b_{i,\mathsf{c}\setminus\mathsf{c}_k}z\Big) \quad \text{[cf. Lemma 3.32]} \\
&= -\Big(u, \sum_{k:\mathsf{c}_k=l}(v_i^{-2l|\mathsf{c}_{<k}|}v^{-2l(i,\mu-|\mathsf{c}\setminus\mathsf{c}_k|i)} - v_i^{2l|\mathsf{c}_{<k}|})b_{i,\mathsf{c}\setminus\mathsf{c}_k}z\Big) \\
&= -\Big(u, \sum_{k:\mathsf{c}_k=l}(v_i^{2l|\mathsf{c}_{<k}|+4l|\mathsf{c}_{>k}|}v^{-2l(i,\mu)} - v_i^{2l|\mathsf{c}_{<k}|})b_{i,\mathsf{c}\setminus\mathsf{c}_k}z\Big) \\
&\equiv (u, b_{i,\mathsf{c}\setminus\mathsf{c}_1}z) \\
&= (u, \tilde{e}_{i,l}v).
\end{aligned}$$

Thanks to the proof of Proposition 3.24 and Lemma 3.32, the same can be proved if $\omega_i = 1$. To that end, consider $v = b_{i,\nu}z$ and $u = b_{i,\nu'}z$ for some partitions ν, ν' and elements $z, z' \in \mathcal{K}_i$, assuming again that $z \in V(\lambda)_\mu$. We have

$$\begin{aligned}
(\tilde{f}_{i,l}u, v) &= \sqrt{\frac{l}{m_l(\nu')+1}}(b_{i,l}u, v) \\
&= -\sqrt{\frac{l}{m_l(\nu')+1}}(u, K_{-li}a_{i,l}v) \\
&= -\sqrt{\frac{l}{m_l(\nu')+1}}(u, K_{-li}a_{i,l}b_{i,\nu}z) \\
&= -\sqrt{\frac{l}{m_l(\nu')+1}}(u, K_{-li}\tau_{i,l}m_l(\nu)(K_{-li} - K_{li})b_{i,\nu\setminus l}z) \quad \text{[cf. proof of Proposition 3.24]} \\
&\equiv -\sqrt{\frac{l}{m_l(\nu')+1}}\Big(u, \frac{m_l(\nu)}{l}(K_{-2li} - 1)b_{i,\nu\setminus l}z\Big) \quad \text{[cf. Lemma 3.32]} \\
&= -\sqrt{\frac{l}{m_l(\nu')+1}}\Big(u, \frac{m_l(\nu)}{l}(v^{-2l(i,\mu)} - 1)b_{i,\nu\setminus l}z\Big) \\
&\equiv \sqrt{\frac{l}{m_l(\nu')+1}}\frac{m_l(\nu)}{l}(u, b_{i,\nu\setminus l}z).
\end{aligned}$$

Iterating this computation, we see that $(\tilde{f}_{i,l}u, v) = (u, \tilde{e}_{i,l}v) = 0$ if $\nu' \cup l \neq \nu$. Otherwise, we get

$$(\tilde{f}_{i,l}u, v) \equiv \sqrt{\frac{m_l(\nu)}{l}}(u, b_{i,\nu\setminus l}z) = (u, \tilde{e}_{i,l}v).$$

The case $i \in I^{\mathrm{re}}$ is already known, but we reproduce the proof adapted to our conventions. The following can be proved by induction:

$$[E_i, F_i^{(n)}] = \tau_i(v^{-n+1}K_{-i} - v^{n-1}K_i)F_i^{(n-1)}.$$

Then note that if $u = f_i^m u_0$ and $u' = f_i^n u'_0$, where $u_0, u'_0 \in \mathcal{K}_i$, it is easy to prove that

$$(\tilde{f}_i u, u') = (F_i^{(m+1)}u_0, F_i^{(n)}u'_0) = 0$$

if $m+1 \neq n$. If $n = m+1$ and $u' \in V(\lambda)_\mu$, we get

$$\begin{aligned}
(\tilde{f}_i u, u') &= \frac{1}{[m+1]}(F_i u, F_i^{(n)}u'_0)\\
&= \frac{1}{[n]}\left(u, \frac{v}{v^{-1}-v}K_{-i}E_iF_i^{(n)}u'_0\right)\\
&= \frac{1}{[n]}\left(u, \frac{v}{v^{-1}-v}K_{-i}\tau_i(v^{-n+1}K_{-i} - v^{n-1}K_i)F_i^{(n-1)}u'_0\right)\\
&\equiv \frac{1}{[n]}\left(u, \frac{v}{v^{-1}-v}K_{-i}(v^{-n+1}K_{-i} - v^{n-1}K_i)F_i^{(n-1)}u'_0\right)\\
&= \frac{1}{[n]}\left(u, \frac{v}{v^{-1}-v}(v^{-n+1}v^{-2(i,\mu+i)} - v^{n-1})F_i^{(n-1)}u'_0\right)\\
&= \left(u, \frac{v^{-n+2}v^{-2(i,\mu+i)} - v^n}{v^{-n}-v^n}F_i^{(n-1)}u'_0\right)\\
&= \left(u, \frac{1 - v^{-2n+2}v^{-2(i,\mu+i)}}{1-v^{-n}}F_i^{(n-1)}u'_0\right)\\
&\equiv (u, F_i^{(n-1)}u'_0)\\
&= (u, \tilde{e}_i u').
\end{aligned}$$

We have assumed that $(i, \mu+i) > 0$, since otherwise we would have $u' = 0$. $\square$

The previous lemmas make it possible to reproduce step by step the original Kashiwara's grand-loop argument (see [Kas91, § 4]).

We also want to prove Theorem 3.19, using the same kind of argument as in [Nak01, Corollary 4.7]. To that end, the characterization of the crystals $\mathcal{B}(\lambda)$ given by Joseph in [Jos95, § 6.4.21] has to be generalized. We first need two definitions.

Definition 3.20. A crystal $\mathcal{B}$ is said to be of *highest weight* λ if:

(i) there exists $b_\lambda \in \mathcal{B}$ with weight λ such that $\tilde{e}_\iota b_\lambda = 0$ for every $\iota \in I_\infty$;
(ii) any element of $\mathcal{B}$ can be written $\tilde{f}_{\iota_1} \dots \tilde{f}_{\iota_r} b_\lambda$ for some $\iota_k \in I_\infty$.

Definition 3.21. Consider a family $\{\mathcal{B}_\lambda \mid \lambda \in P^+\}$ of highest weight normal crystals $\mathcal{B}_\lambda$ of highest weight λ, with elements $b_\lambda \in \mathcal{B}_\lambda$ satisfying the properties of the above definition. It is called *closed* if the subcrystal of $\mathcal{B}_\lambda \otimes \mathcal{B}_\mu$ generated by $b_\lambda \otimes b_\mu$ (i.e. obtained after successive applications of the operators $\tilde{e}_\iota$ and $\tilde{f}_\iota$) is isomorphic to $\mathcal{B}_{\lambda+\mu}$.

Our previous lemmas, along with the definitions and properties given in the previous sections, make it possible to generalize the proof of [Jos95, § 6.4.21] and we get the following result.

Proposition 3.35. *The only closed family of highest weight normal crystals is* $\{\mathcal{B}(\lambda) \mid \lambda \in P^+\}$.

Then it is easy to see that $\{\operatorname{Irr}\mathcal{L}(\lambda) \mid \lambda \in P^+\}$ is a closed family of highest weight normal crystals: thanks to Proposition 2.20, Remark 2.3 adapted to $\widetilde{\mathfrak{Z}}$ and Theorem 2.6, the arguments given in [Nak01] can be reproduced and we get Theorem 3.19. Alternatively (but similarly), the original proof given by Saito in [Sai02] can also be generalized to our framework.

REFERENCES

Boz15 T. Bozec, *Quivers with loops and perverse sheaves*, Math. Ann. **362** (2015), 773–797; MR 3368082.

HLR13 T. Hausel, E. Letellier and F. Rodriguez-Villegas, *Arithmetic harmonic analysis on character and quiver varieties II*, Adv. Math. **234** (2013), 85–128; MR 3003926.

HR08 T. Hausel and F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties*, Invent. Math. **174** (2008), 555–624; with an appendix by Nicholas M. Katz; MR 2453601 (2010b:14094).

JKK05 K. Jeong, S.-J. Kang and M. Kashiwara, *Crystal bases for quantum generalized Kac–Moody algebras*, Proc. Lond. Math. Soc. (3) **90** (2005), 395–438; MR 2142133 (2006e:17020).

Jos95 A. Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 29 (Springer, Berlin, 1995); MR 1315966 (96d:17015).

KKS09 S.-J. Kang, M. Kashiwara and O. Schiffmann, *Geometric construction of crystal bases for quantum generalized Kac–Moody algebras*, Adv. Math. **222** (2009), 996–1015; MR 2553376 (2010h:17016).

Kas91 M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516; MR 1115118 (93b:17045).

KS97 M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), 9–36; MR 1458969 (99e:17025).

KS94 M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292 (Springer, Berlin, 1994), with a chapter in French by Christian Houzel, corrected reprint of the 1990 original; MR 1299726 (95g:58222).

Li Y. Li, *Canonical bases of Cartan–Borcherds type, II*, Preprint.

Lus91 G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421; MR 1088333 (91m:17018).

Lus00 G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), 129–139; MR 1758244 (2001e:17033).

MO12 D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, Preprint (2012), arXiv:1211.1287.

Nak94 H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras*, Duke Math. J. **76** (1994), 365–416.

Nak98 H. Nakajima, *Quiver varieties and Kac–Moody algebras*, Duke Math. J. **91** (1998), 515–560; MR 1604167 (99b:17033).

Nak01 H. Nakajima, *Quiver varieties and tensor products*, Invent. Math. **146** (2001), 399–449; MR 1865400 (2003e:17023).

Nak09 H. Nakajima, *Quiver varieties and branching*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 003, 37 pp.; MR 2470410 (2010f:17034).

Sai02 Y. Saito, *Crystal bases and quiver varieties*, Math. Ann. **324** (2002), 675–688; MR 1942245 (2004a:17023).

Tristan Bozec tbozec@mit.edu
Massachusetts Institute of Technology, 77 Massachusetts Avenue,
Cambridge, MA 02139, USA