SERIAL DEPENDENCE OF A MARKOV PROCESS

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1. Introduction

Consider a Markov process defined in discrete time $t = 1, 2, 3, \cdots$ on a state space S. The state of the process at time t will be specified by a random variable V_t, taking values in S. This paper presents some results concerning the behaviour of the sequence $V_1, V_2, V_3, \cdots$, considered as a time series. In general, S will be assumed to be a Borel subset of an h-dimensional Euclidean space, where h is finite. The results apply, in particular, to a continuous state space, taking S to be an interval of the real line, or to a discrete process having finitely or enumerably many states. Certain results, which are indicated in what follows, apply also to more general (infinite-dimensional) state spaces.

The Markov process may be partially described by the probability distribution functions (p.d.f.) of V_t, for each value of $t = 1, 2, 3, \cdots$, namely

$$(1) \qquad F_t(x_0) = F_t^{(0)}(x_0) = \Pr(V_t \leqq x_0).$$

Here, if S is h-dimensional, and V_t and x_0 are h-vectors with components $x_{0(i)}$, $V_{t(i)}$ $(i = 1, 2, \cdots, h)$, then $V_t \leqq x_0$ means $V_{t(i)} \leqq x_{0(i)}$ for each i.

More information about the process is given by the joint p.d.f. of $p+1$ consecutive variables V_t, namely

$$(2) \qquad F_t^{(p)}(x_0, x_1, \cdots, x_p) = \Pr(V_t \leqq x_0, V_{t-1} \leqq x_1, \cdots, V_{t-p} \leqq x_p).$$

For various Markov processes — e.g. Markov chains imbedded into queueing systems — the p.d.f. of V_t has been calculated both (a) for the stationary state

$$\lim_{t\to\infty} F_t(x_0),$$

assuming that this exists, and (b) for the transient distribution, i.e. obtaining $F_t^{(0)}(x)$ as a function of t, for given $F_1^{(0)}(x)$. If, however, V_t, or some variable correlated with V_t, forms the input to a second stochastic process, then the behaviour of the second process, to the extent that the sequence $\{V_t\}$ determines it, depends not only on $F_t^{(0)}(x)$, but on $F_t^{(p)}(x_0, \cdots, x_p)$, the joint p.d.f. of $p+1$ consecutive state variables V_t, for some appropriate value of p, possibly infinite. This joint distribution has only rarely been calculated. Benes [1], in discussing the virtual waiting time for a queue with Poisson arrivals, obtained its covariance function (in continuous time); this amounts to a partial specification of (2). There appear to be no similar results for such a process considered in discrete time. The study of the joint distribution (2) is equivalent to analyzing the stochastic sequence $\{V_t\}$ as a time series.

This paper investigates the relation between the joint p.d.f. (2) and the p.d.f. (1). A general expression is given for (2). If (1) is known explicitly, both (a) for the stationary case, and (b) for the transient case, then any moment of the joint stationary distribution (2) is shown to be explicitly determined by a finite recursion; the number of steps depends on the order of the moment, but not on the time t. In particular, the quadratic moment $E(V_tV_{t-i})$ (this is an $h\times h$ matrix if S is h-dimensional), is, apart from a constant multiplicative factor, equal to the expectation of the transient p.d.f. (1) at $t=i$, given that the stochastic process begins at $t=0$ with a particular initial p.d.f.

Two applications are discussed — to waiting times for successive customers in the $M/M/1$ queue, and to a class of queues, or storage systems, with periodic input.

2. Basic results for a Markov process

Let the process be specified by a transition function $G_t(x,u)$, which may or may not depend on t, so that

$$F_{t+1}^{(0)}(x_0)=\int_S G_t(x_0,u)dF_t^{(0)}(u) \qquad t=1,2,\cdots. \tag{3}$$

Construct from the process $\{V_t\}$ a vector stochastic process whose state variable is

$$W_t=(W_t^0, W_t^1,\cdots, W_t^p),$$

where

$$W_t^i=V_{t-i} \qquad (i=0,1,\cdots,p).$$

Then

$$\begin{aligned}&\Pr\{W_{t+1}^0\leqq x_0,\cdots, W_{t+1}^p\leqq x_p\}\\ &=\Pr\{V_{t+1}\leqq x_0, V_t\leqq x_1,\cdots, V_{t-p+1}\leqq x_p\}\\ &=\Pr\{W_{t+1}^0\leqq x_0, W_t^0\leqq x_1,\cdots, W_t^{p-1}\leqq x_p\}.\end{aligned}$$

Now, by (3), $\Pr\{W_{t+1}^0\leqq x_0\}$ depends only on $G_t(x_0,u)$ and on $\Pr\{W_t^0\leqq u\}$. Therefore $\Pr\{W_{t+1}^0\leqq x_0,\cdots, W_{t+1}^p\leqq x_p\}$ depends only on $G_t(x_0,u)$ and on

$$F_t^{(p)}(x_0, \cdots, x_p) = \Pr\{W_t^0 \leqq x_0, \cdots, W_t^p \leqq x_p\}.$$

Thus the sequence $\{W_t\}$ forms a Markov process. For this process, (3) shows that

$$\begin{aligned}\Pr\{V_{t+1} \leqq x_0, V_t \leqq x_1, \cdots, V_{t-p+1} \leqq x_p\} \\ = \int_{S'} G_t(x_0, u_0) dF_t^{(p)}(u_0, u_1, \cdots, u_p),\end{aligned} \tag{4}$$

integrated over the subset S' of the product space S^{p+1} defined by $u_0 \leqq x_1$, $u_1 \leqq x_2, \cdots, u_{p-1} \leqq x_p$, $u_p \leqq \infty$; here $\{V_t : V_t \leqq \infty\}$ denotes the whole space S. Then, performing explicitly the integrations with respect to $u_1, \cdots, u_p$ in (4),

$$F_{t+1}^{(p)}(x_0, \cdots, x_p) = \int_{u_0 \leqq x_1} G_t(x_0, u_0) dF_t^{(p)}(u_0, x_2, x_3, \cdots, x_p, \infty). \tag{5}$$

Since

$$F_t^{(p)}(u_0, x_2, \cdots, x_p, \infty) = F_t^{(p-1)}(u_0, x_2, \cdots, x_p),$$

(5) may be written as

$$F_{t+1}^{(p)}(x_0, \cdots, x_p) = \int_{u_0 \leqq x_1} G_t(x_0, u_0) dF_t^{(p-1)}(u_0, x_2, x_3, \cdots, x_p). \tag{6}$$

Equation (6) specifies $F_{t+1}^{(p)}$ in terms of $F_t^{(p-1)}$; hence, if $F_t^{(0)}(x_0)$ and $G_t(x_0, u_0)$ are given as fuctions of t, then the joint p.d.f. $F_t^{(p)}(x_0, \cdots, x_p)$ is obtained by a p-step recursion.

For the stationary state of the process, the recurrence relations (6), for $p = 1, 2, 3, \cdots$ may be combined into a single integral equation. Define the generating function

$$\psi(x_0, x_1, \cdots; \lambda) = \sum_{p=0}^{\infty} F^{(p)}(x_0, x_1, \cdots)\lambda^p. \tag{7}$$

The suffixes t are dropped, since the stationary state ($t \to \infty$) is considered. The expression (7) is well defined for $|\lambda| < 1$. It is sufficient, in fact, to consider only functions of finitely many arguments; if $x_i = \infty$ for all $i > N$, where N is an arbitrarily large constant integer, then

$$F^{(p)}(x_0, x_1, \cdots, x_N, \cdots) = F^{(N)}(x_0, x_1, \cdots, x_N) \text{ for } p \geqq N.$$

Then from (6)

$$\begin{aligned}\lambda \int_{u \leqq x_1} & G(x_0, u) d\psi(u, x_2, x_3, \cdots; \lambda) \\ &= \lambda \sum_{p=0}^{\infty} \lambda^p \int_{u \leqq x_1} G(x_0, u) dF^{(p)}(u, x_2, x_3, \cdots) \\ &= \lambda \sum_{p=0}^{\infty} \lambda^p F^{(p+1)}(x_0, x_1, x_2, \cdots) \\ &= \psi(x_0, x_1, \cdots; \lambda) - F^{(0)}(x_0) \\ &= \psi(x_0, x_1, \cdots; \lambda) - \psi(x_0, \cdots; 0).\end{aligned} \tag{8}$$

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This integral equation for ψ has a recursive solution given by (6), but there appears to be no general approach to solving (8), other than by a recursion in p. A direct solution may be possible for particular transition functions G.

The joint moment generating function (m.g.f.) of $V_t, V_{t-1}, \cdots, V_{t-p}$ is

$$\boldsymbol{E}\{\exp(-s_0 V_t - \cdots - s_p V_{t-p})\};$$

in which, if each V_t is a h-vector, then so is each s_i, and the expressions $s_i V_{t-i}$ are interpreted as inner products. The joint m.g.f. may be calculated in terms of an unnormalized joint conditional m.g.f., defined by

$$(9)\quad \boldsymbol{Q}_t^{(p)}(x; s_1, s_2, \cdots, s_p) = \boldsymbol{E}\{\exp(-s_1 V_{t-1} - s_2 V_{t-2} - \cdots - s_p V_{t-p}) | V_t \leqq x\}.$$

Then

$$(10)\quad \begin{aligned} &\boldsymbol{Q}_t^{(p)}(x; s_1, s_2, \cdots, s_p) \\ &= \int \cdots \int \exp(-s_1 x_1 - \cdots - s_p x_p) d_{x_1 \cdots x_p} F_t^{(p)}(x, x_1, \cdots, x_p) \end{aligned}$$

Here $x, x_1, \cdots, x_p$ are h-vectors, and each $s_i x_i$ is an inner product. (Terminals of integration are omitted, here and in later equations, when the range of integration is the whole of S.) It is assumed that the stochastic process is such that (9) is finite, for a suitable domain of $(s_1, \cdots, s_p)$, which includes the origin.

The (unconditional) joint m.g.f. is then given by

$$\boldsymbol{E}\{\exp(-s_0 V_t - \cdots - s_p V_{t-p})\} = \int e^{-s_0 x} d_x \boldsymbol{Q}_t^{(p)}(x; s_1, \cdots, s_p).$$

Now, substituting (6) in (10),

$$(11)\quad \begin{aligned} &\boldsymbol{Q}_t^{(p)}(x; s_1, s_2, \cdots, s_p) \\ &= \int \cdots \int e^{-s_1 x_1 - \cdots - s_p x_p} d_{x_1 \cdots x_p} \int_{u \leqq x_1} G_{t-1}(x, u) d_u F_{t-1}^{(p-1)}(u, x_2, \cdots, x_p) \\ &= \int e^{-s_1 x_1} G_{t-1}(x, x_1) d_{x_1} \int \cdots \int e^{-s_2 x_2 - \cdots - s_p x_p} d_{x_2 \cdots x_p} F_{t-1}^{(p-1)}(x_1, x_2, \cdots, x_p) \\ &\qquad\qquad \text{on integrating with respect to } u \text{, and rearranging} \\ &= \int e^{-s_1 x_1} G_{t-1}(x, x_1) d\boldsymbol{Q}_{t-1}^{(p-1)}(x_1; s_2, s_3, \cdots, s_p) \\ &\qquad\qquad \text{on substituting } \boldsymbol{Q}_{t-1}^{(p-1)} \text{ from (10).} \end{aligned}$$

Thus the conditional m.g.f. $\boldsymbol{Q}_t^{(p)}$ is determined, by (11), recursively in terms of $\boldsymbol{Q}_{t-1}^{(p-1)}$, in a similar manner to the recursive determination of the joint p.d.f. $F_t^{(p)}$ by (6).

Equation (6) applies also to a state space S more general than a finite-dimensional Euclidean space. It applies, in fact, to any space with a vector ordering $\leqq$, such that $\{V_t : V_t \leqq x\}$ is always defined, and possessing a measure for which the integral in (3) is defined. Similarly, (7) to (11) remain valid if S is a Hilbert space of sequences $(x_{(1)}, x_{(2)}, \cdots)$, with $x \leqq y$ defined by $x_{(i)} \leqq y_{(i)}$.

3. Moments of the joint distribution

If the Markov process begins with $V_1 = \xi$, a constant, then

$$F_1^{(0)}(x) = H(x-\xi) \equiv \prod_1^h H(x_{(i)}-\xi_{(i)})$$

where $H(\cdot)$ denotes Heaviside's unit function, and S is h-dimensional. Denote the consequent $F_t^{(0)}(x)$, determined using (3) for $t = 2, 3, 4, \cdots$, by $\phi_t(x, \xi)$; thus

$$\phi_t(x, \xi) = \Pr(V_t \leqq x | V_1 = \xi).$$

If, instead, the initial distribution $F_1^{(0)}(x)$ is any arbitrary p.d.f., then

$$F_t^{(0)}(x) = \int \phi_t(x, \xi) dF_1^{(0)}(\xi). \tag{12}$$

Consider now the stationary state. Let $\phi(x)$ denote the stationary p.d.f.

$$\phi(x) = \lim_{t\to\infty} F_t^{(0)}(x).$$

For $1 \leqq k \leqq p$, define the function $C(k; x)$, whose values are h-vectors, by

$$C(k; x) = \left[-\frac{\partial}{\partial s_k} \mathbf{Q}^{(p)}(x; s_1, s_2, \cdots, s_p) \right]_{s_1=\cdots=s_p=0}. \tag{13}$$

Then, substituting for $\mathbf{Q}^{(p)}$ from (10),

$$C(k; x) = \int \cdots \int x_k d_{x_1\cdots x_p} F^{(p)}(x, x_1, \cdots, x_p). \tag{14}$$

In this integral, and in various subsequent expressions, x_k is an h-vector; likewise $\partial \mathbf{Q}/\partial s_k$ is a h-vector. From (14), $C(k; x)$ is independent of p, provided $p \geqq k$.

Now for $k \geqq 2$, substituting in (14) for $F^{(p)}$, and using (6), gives

$$\begin{aligned} C(k; x) &= \int \cdots \int x_k d_{x_1\cdots x_p} \int_{u\leqq x_1} G(x, u) d_u F^{(p-1)}(u, x_2, \cdots, x_p) \\ &= G(x, u) d_u \int \cdots \int x_k d_{x_2\cdots x_p} F^{(p-1)}(u, x_2, \cdots, x_p) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{since } 2 \leqq k \leqq p \\ &= \int G(x, x_1) dC(k-1; x_1) \qquad\qquad \text{again using (14).} \end{aligned} \tag{15}$$

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Also, for $k = 1$, (14) and (6) give

$$C(1;x) = \int x_1 dF^{(1)}(x, x_1)$$

(16)
$$= \int x_1 d \int_{u \leq x_1} G(x, u) dF^{(0)}(u)$$
$$= \int x_1 G(x, x_1) d\phi(x_1),$$

since $F^{(0)}(x_1) = \phi(x_1)$ for the stationary state.

In (3), if k is written for $t+1$, and x_1 for u, then

(17)
$$F_k^{(0)}(x) = \int_S G(x, x_1) dF_{k-1}^{(0)}(x_1);$$

and this implies (12), i.e.

$$F_k^{(0)}(x) = \int \phi_k(x, \xi) dF_1^{(0)}(\xi),$$

where ϕ_k does not depend on $F_1^{(0)}$. Comparing (17) with (15), it is seen that the recurrence relation which connects $C(k;x)$ with $C(k-1;x)$ is identical with that connecting $F_k^{(0)}(x)$ with $F_{k-1}^{(0)}(x)$. So from (12),

(18)
$$C(k;x) = \int \phi_k(x, \xi) dC(1;\xi) \quad \text{for} \quad k = 2, 3, \cdots.$$

It does not follow that $C(k;x)$ is a p.d.f. However, the substitution $F_{k-1}^{(0)}(x) = H(x_1 - \xi)$ into (17) shows that $G(x, x_1)$ is a p.d.f. of x for each fixed x_1. So $C(1;x)$, which, by (16), is obtained from $G(x, x_1)$ by integration over x_1, can differ from a p.d.f. only by a constant multiplier. Therefore $C(1;x)/C(1;\infty)$ is a p.d.f.

Let

$$C^*(k;x) = C(k;x)/C(k;\infty) \qquad k = 1, 2, 3, \cdots.$$

By setting $x = \infty$ in (18) and in (16), it follows that, for $k > 1$,

$$C(k;\infty) = C(1;\infty) = \boldsymbol{E}(V_t).$$

So equations (15) and (18) remain valid with $C^*(k;x)$ substituted for $C(k;x)$. Now in (17), if $F_{k-1}^{(0)}(x)$ is any p.d.f., then $F_k^{(0)}(x)$ is also a p.d.f. Consequently, from (15), if $C^*(k-1;x)$ is a p.d.f., then so is $C^*(k;x)$. But $C^*(1;x)$ was shown above to be a p.d.f. From this result and (15), therefore, the functions $C^*(k;x)$ are the transient probability distribution functions $F_k^{(0)}(x)$ of a Markov process, with the same transition function $G(x, u)$ as the given process, and commencing with the particular p.d.f.

$$F_1^{(0)}(x) = C^*(1;x),$$

calculable from (16).

Considering still the stationary state, let $V_{t,\alpha}^m$ denote component α

of the h-vector V_t, raised to the power m. For given integers $i, j, \cdots, k$; $\alpha, \beta, \cdots, \gamma$; $m, n, \cdots, q$, define the product moment

$$M^r(i^m, j^n, \cdots, k^q) = \boldsymbol{E}(V^m_{t-i,\alpha} V^n_{t-j,\beta} \cdots V^q_{t-k,\gamma}).$$

The indices $\alpha, \beta, \cdots, \gamma$ will be considered fixed throughout the calculation, and are, for brevity, omitted from the symbol $M^r(\cdot)$. The superscript $r = m+n+\cdots+q$ will be omitted whenever no ambiguity can result. Then in particular, from (14), the moments of order two are given by

$$M(0, k) = \int x dC(k; x). \tag{19}$$

(The values of (19), for all possible α, β, form an $h \times h$ matrix.) It follows from (19) that $M(0, k)/C(k, \infty) = \boldsymbol{E}(V_t V_{t-k})/\boldsymbol{E}(V_t)$ equals the expectation of the Markov random variable whose p.d.f. is $C^*(k; x)$.

Now from (18),

$$\begin{aligned} \lim_{k\to\infty} C(k; x)/C(k; \infty) &= \int \lim_{k\to\infty} \phi_k(x; \xi) dC(1; \xi)/C(1; \infty) \\ &= \phi(x); \end{aligned} \tag{20}$$

by Lebesgue's theorem on limits of integrals, assuming that the stationary p.d.f. $\phi(x)$ is unique. Therefore

$$\lim_{k\to\infty} \{M(0, k) - [\boldsymbol{E}(V_t)]^2\} = 0; \tag{21}$$

i.e. the covariance of V_t with V_{t-k} tends to zero as $k \to \infty$, as would be expected.

Equation (19) may be compared with Benes' expression [1] for the covariance function $R(t)$ for virtual waiting time for a queue with Poisson arrivals;

$$R(t) = \int_{0-}^{\infty} w \cdot E\{W(t)|W(0) = w\} dA(w) - [E(W(0))]^2 \tag{22}$$

where $A(w)$ is the stationary distribution of virtual waiting time $W(t)$, and t is continuous time. This comparison shows that a result similar to (19) holds for some Markov processes in continuous time.

To determine moments of higher order than two, let

$$1 < i < j < \cdots < \kappa < \lambda \leqq p; \text{ and } r = g+m+\cdots+\mu+\nu.$$

Then from (8), assuming the stationary state, define the "conditional product moment"

$$\begin{aligned} &K^r(1^g, i^m, j^n, \cdots, \kappa^\mu, \lambda^\nu; x) \\ &\quad = \left[(-1)^r \frac{\partial^r}{\partial s_1^g \cdots \partial s_\lambda^\nu} Q^{(p)}(x; s_1, \cdots, s_p)\right]_{s_1=\cdots=s_p=0}. \end{aligned} \tag{23}$$

Substituting for $Q^{(p)}$ from (11), and performing the differentiation with respect to s_1,

$$K^r(1^g, i^m, j^n, \cdots, \kappa^\mu, \lambda^\nu; x)$$

$$(24)\qquad = \int \xi^g G(x, \xi) d\left[(-1)^{r-g} \frac{\partial^{r-g}}{\partial s_i^m \cdots \partial s_\lambda^\nu} Q^{(p-1)}(\xi; s, \cdots, s_p)\right]_{s_2=\cdots=s_p=0}$$

$$= \int \xi^g G(x, \xi) dK^{r-g}((i-1)^m, \cdots, (\lambda-1)^\nu; \xi),$$

using (23) again.

A similar calculation, using (11), shows that

$$(25)\qquad K^r(i^m, j^n, \cdots, \lambda^\nu; x) = \int G(x, \xi) dK^r((i-1)^m, (j-1)^n, \cdots, (\lambda-1)^\nu; \xi).$$

This equation is of the same form as (15), if $m, n, \cdots, \nu, r$ are held constant, and $(i, j, \cdots, \lambda)$ is considered as a multiple index, in place of k in (15). The argument leading from (15) to (18) shows, therefore, that

$$(26)\quad K^r(i^m, j^n, \cdots, \lambda^\nu; x) = \int \phi_i(x, \xi) dK^r(1^m, (j-i+1)^n, \cdots, (\lambda-i+1)^\nu; \xi).$$

Equations (24) and (26) thus enable $K^r(1^g, i^m, \cdots, \lambda^\nu; x)$ to be expressed in terms of similar functions with smaller values of $i, j, \cdots, \lambda$. This may be done systematically as follows. Define the linear operators Ω_g and Λ_i by

$$\Omega_g f(x) = \int \xi^g G(x, \xi) df(\xi) \qquad (g = 1, 2, \cdots)$$

$$\Lambda_i f(x) = \int \phi_i(x, \xi) df(\xi) \qquad (i = 2, 3, \cdots).$$

Then

$$K^r(1^g, i^m, j^n, \cdots, \lambda^\nu; x) = \Omega_g K^{r-g}((i-1)^m, \cdots, (\lambda-1)^\nu; x) \text{ by (24)}$$

$$= \Omega_g \Omega_m K^{r-g-m}((j-1)^n, \cdots, (\lambda-i)^\nu; x)$$

if $i-1 = 1$, using (24), again,

or

$$= \Omega_g \Lambda_{i-1} \Omega_m K^{r-g-m}((j-i)^n, \cdots, (\lambda-i)^\nu; x)$$

if $i > 2$, using (26), then (24).

These last two expressions coincide if Λ_1 is defined as the identity operator.

Then successive application of these reduction formulae results in

$$(27)\quad K^r(1^g, i^m, j^n, \cdots, \kappa^\mu, \lambda^\nu; x) = \Omega_g \Lambda_{i-1} \Omega_m \Lambda_{j-i} \cdots \Omega_\mu \Lambda_{\lambda-\kappa} K^\nu(1^\nu; x);$$

where, from (23),

$$K^{\nu}(1^{\nu};x) = \left[(-1)^{\nu}\frac{\partial^{\nu}}{\partial s_1^{\nu}} Q^{(p)}(x; s_1, \cdots, s_p)\right]_{s_1=\cdots=s_p=0}$$

$$= \int \xi^{\nu} G(x, \xi) d[Q^{(p-1)}(\xi; s_2, \cdots, s_p)]_{s_2=\cdots=s_p=0}$$

(28) substituting for $Q^{(p)}$ from (11)

$$= \int \xi^{\nu} G(x, \xi) dF^{(0)}(\xi) \text{ by (10)}$$

$$= \int \xi^{\nu} G(x, \xi) d\phi(\xi)$$

for the stationary state.

Therefore any product moment of the Markov process may be determined as

$$M^{r}(0^{g}, i^{m}, j^{n}, \cdots, \kappa^{\mu}, \lambda^{\nu}) = E(V_t^{g} V_{t-i}^{m} \cdots V_{t-\lambda}^{\nu})$$

with the previous convention regarding $\alpha, \beta, \cdots, \gamma$,

(29) $$= K^{r}(1^{g}, (i+1)^{m}, (j+1)^{n}, \cdots, (\lambda+1)^{\nu}; \infty)$$

assuming the stationary state

$$= \Omega_g \Lambda_i \Omega_m \Lambda_{j-i} \cdots \Omega_{\mu} \Lambda_{\lambda-\kappa} \int \xi^{\nu} d\phi(\xi) \text{ from (27) and (28).}$$

If $\phi_i(x, \xi)$ is known explicitly, then the number of steps involved in applying (29) depends on r, the order of the moment, but not on the "lags" $i, j, \cdots, \kappa, \lambda$.

Some special cases arise when the "exponents" $g, m, \cdots, \nu$ all equal 1.

For example, if $r = 3$, then for $0 < j < k$, (29) reduces to

(30) $$M(0, j, k) = \int\int\int \lambda\eta d_{\lambda}\phi_j(\lambda, \xi) d_{\xi} G(\xi, \eta) d_{\eta} C(k-j; \eta)$$

where $C(k-j;\eta)$ is obtainable from (18). If $j = 1$, then

$$M(0, 1, k) = \int\int \xi\eta d_{\xi} G(\xi, \eta) d_{\eta} C(k-1; \eta).$$

If $j = k$, then

$$M(0, k^2) \equiv M(0, k, k) = \int\int\int \lambda\eta^2 d_{\lambda}\phi_j(\lambda, \xi) d_{\xi} G(\xi, \eta) d_{\eta}\phi(\eta).$$

The last expression may be regarded as a special case of (30), if $\eta d\phi(\eta)$ is formally substituted for the undefined quantity $dC(0;\eta)$.

4. Waiting time in the $M/M/1$ queue

The waiting times V_t $(t = 1, 2, \cdots)$ of successive customers in an $M/M/1$ queue, with traffic intensity ρ, form a Markov process with transition function

$$\begin{aligned} G(x, u) &= 0 && \text{for} \quad x < 0 \\ &= J(x-u) && \text{for} \quad x \geqq 0 \end{aligned}$$

where

$$\begin{aligned} J(\xi) &= 1-[\rho/(1+\rho)]e^{-\xi} && \text{for} \quad \xi \geqq 0 \\ &= [1/(1+\rho)]e^{\rho\xi} && \text{for} \quad \xi < 0. \end{aligned} \tag{31}$$

The stationary distribution is

$$\begin{aligned} \phi(x) &= 1-\rho e^{-(1-\rho)x} && \text{for } x \geqq 0 \\ &= 0 && \text{for } x < 0. \end{aligned} \tag{32}$$

Some results for this process are as follows.

From (6), setting $F^{(0)}(x) = \phi(x)$ for the stationary state, are obtained:

$$\begin{aligned} F^{(1)}(x, y) &= 0 && \text{for } x < 0 \text{ or for } y < 0 \\ &= 1-\rho e^{-(1-\rho)x} - \frac{\rho(1-\rho)}{1+\rho} e^{\rho x - y} && \text{for } 0 \leqq x \leqq y \\ &= 1-\rho e^{-(1-\rho)y} - \frac{\rho(1-\rho)}{1+\rho} e^{-x+\rho y} && \text{for } 0 \leqq y < x. \end{aligned} \tag{33}$$

$$\begin{aligned} F^{(2)}(x, y, z) = 1 &- \frac{\rho(1-\rho)}{1+\rho} e^{\rho x - y} - \rho e^{-(1-\rho)x} \\ &+ \frac{\rho(1-\rho)}{(1+\rho)^2}\left[\frac{1+2\rho}{1+\rho} - \rho(y-x)\right] e^{\rho x - z} - \frac{\rho^2(1-\rho)(1+2\rho)}{1+\rho} e^{-x-z} \\ &\qquad \text{for } x < y < z. \end{aligned} \tag{34}$$

Expressions of similar form are obtainable for the other orders of x, y, z.

This calculation may be continued to $F^{(p)}$ for $p > 2$, but the expressions become complicated. The moments of the joint distributions are, however, more readily obtained using equations (16), (18), and (19) for bivariate moments, and equation (29) for moments of higher order. For the $M/M/1$ queue, elementary but somewhat lengthy calculations show that

(35) $\phi_1(x, \xi) = H(x-\xi)$ where $H(\cdot)$ is Heaviside's unit function

$$\phi_2(x, \xi) = G(x, \xi)$$

$$\phi_3(x, \xi) = 1-\frac{\rho}{(1+\rho)^3}e^{-x-\rho\xi}+\left[1-\frac{1+3\rho}{(1+\rho)^3}-\frac{\rho^2(x-\xi)}{(1+\rho)^2}\right]e^{-x+\xi} \quad \text{for } 0<\xi<x$$

$$= -\frac{\rho}{(1+\rho)^3}e^{-x-\rho\xi}+\left[\frac{1+3\rho}{(1+\rho)^3}+\frac{\rho(\xi-x)}{(1+\rho)^2}\right]e^{\rho x-\rho\xi} \quad \text{for } 0<x\leqq\xi.$$

(36)
$$C(1; x) = \frac{\rho}{1-\rho}-\left[\frac{\rho}{1-\rho}-(1-\rho)\right]e^{-(1-\rho)x}-\frac{1-\rho}{1+\rho}e^{-x}-\rho xe^{-(1-\rho)x}$$

$$C(2; x) = \frac{\rho}{1-\rho}-\frac{(1-\rho)(2+5\rho+2\rho^2)}{(1+\rho)^3}e^{-x}-\left[\frac{\rho}{1-\rho}-2(1-\rho)\right]e^{-(1-\rho)x}$$
$$-\rho xe^{-(1-\rho)x}-\frac{\rho(1-\rho)}{(1+\rho)}xe^{-x}$$

$$C(3; x) = \frac{\rho}{1-\rho}-\left[\frac{3(1-\rho)}{1+\rho}+\frac{2\rho(1-\rho)(1+3\rho+\rho^2)}{(1+\rho)^5}\right]e^{-x}$$
$$-\frac{\rho^2(1-\rho)}{2(1+\rho)^3}x^2e^{-x}-\left[\frac{\rho}{1-\rho}-3(1-\rho)\right]e^{-(1-\rho)x}$$
$$-\rho xe^{-(1-\rho)x}-\frac{2\rho(1-\rho)(1+3\rho+\rho^2)}{(1+\rho)^4}xe^{-x}.$$

Now, for the $M/M/1$ queue, (19) gives

$$M(0, k) = \int_{0-}^{\infty} x dC(k; x)$$
$$= \lim_{b\to\infty}\int_{0-}^{b} x dC(k; x).$$

Therefore, integrating by parts

$$M(0, k) = \lim_{b\to\infty}\left[bC(k; b)-\int_0^b C(k; x)dx\right]$$

(37)
$$= \lim_{b\to\infty}\left\{b[C(k; b)-C(k; \infty)]+\int_0^b [C(k; \infty)-C(k; x)]dx\right\}$$
$$= \int_0^{\infty}[C(k, \infty)-C(k; x)]dx.$$

The correlation coefficient of lag k for the time series $\{V_t\}$ is then

$$(38) \qquad R_k = \frac{M(0,k)-[E(V)]^2}{\operatorname{var}(V)} = \frac{(1-\rho)^2M(0,k)-\rho^2}{2\rho-\rho^2}.$$

Hence, using (36) and (37), after some reduction,

$$(39) \qquad \begin{aligned} R_1 &= \{[(1-\rho)^3/(1+\rho)]-2(1-\rho)^2+1\}/\{2\rho-\rho^2\} \\ &= 5\rho/2+O(\rho^2) && \text{for } \rho \ll 1 \\ &= 1-(1-\rho)^2+O((1-\rho)^3) && \text{for } 0<1-\rho\ll 1. \end{aligned}$$

Similarly

$$(40) \qquad \begin{aligned} R_2 &= 7\rho^2+O(\rho^3) && \text{for } \rho \ll 1 \\ &= 1-2(1-\rho)^2+O((1-\rho)^3) && \text{for } 0<1-\rho\ll 1 \end{aligned}$$

$$(41) \qquad \begin{aligned} R_3 &= 21\rho^3+O(\rho^4) \quad \text{for } \rho \ll 1 \\ &= 1-3(1-\rho)^2-O((1-\rho)^3) \quad \text{for } 0<1-\rho\ll 1. \end{aligned}$$

For a near-saturated queue (ρ close to 1) these results illustrate the slow decay of R_k from 1 as k increases. They also put in question the usefulness of a Monte Carlo investigation of a queue in this range, as is sometimes undertaken to estimate properties of the stationary distribution, unless the sample size is extremely large. It is conjectured (but remains unproved) that

$$(42) \qquad R_k = 1-k(1-\rho)^2+O((1-\rho)^3) \quad \text{for } k=4,5,\cdots.$$

5. Queue with periodic input

In Lindley's queue model [2], successive waiting times w_r are given by $w_1 = 0$ and

$$(43) \qquad \begin{aligned} w_{r+1} &= w_r+u_r && \text{if} \quad w_r+u_r > 0 \\ &= 0 && \text{if} \quad w_r+u_r \leqq 0; \end{aligned}$$

where

$$\begin{aligned} u_r = {} & \text{service interval of } r\text{th customer} \\ & - \text{time interval between } r\text{th and } (r+1)\text{th arrival;} \end{aligned}$$

and the u_r are assumed to be independently and identically distributed.

Suppose now that the u_r remain independent, but that their distributions

$$T_r(x) = P_r(u_r \leqq x)$$

are no longer identical. From (43), if

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$$F_r(x) = P_r(w_r \leqq x),$$

then

(44) $$F_{r+1}(x) = \int_{u_r \leqq x} T_r(u_r) dF_r(x-u_r),$$

so that

$$F_2(x) = \int_{u_1 \leqq x} dT_1(u_1) = P_r(u_1 \leqq x),$$
$$F_3(x) = \int_{u_2 \leqq x} \int_{u_1 \leqq x-u_2} dT_1(u_1) dT_2(u_2)$$
$$= \Pr(u_1+u_2 \leqq x, u_2 \leqq x);$$

and generally, by induction,

(45) $$F_{r+1}(x) = \Pr(u_1 + \cdots + u_r \leqq x, u_2 + \cdots + u_r \leqq x, \cdots, u_r \leqq x).$$

If, in particular, the u_r are identically distributed, then (45) is unaltered by renumbering the u_r, so as to replace u_s by u_{r+1-s} for $1 \leqq s \leqq r$. Thus

(46) $$F_{r+1}(x) = \Pr(u_1 + \cdots + u_r \leqq x, u_1 + \cdots + u_{r-1} \leqq x, \cdots, u_1 \leqq x).$$

From (46), Lindley [2] deduces the existence of a stationary waiting time distribution when, and only when, $E(u) < 0$.

Lindley's argument can, however, be extended to certain cases when the u_r are not identically distributed, and in particular to a queue with "periodic input". Suppose that the u_r are independent, but that, for a fixed integer "period" $m > 1$,

(47) $$\Pr(u_{jm+i} \leqq x) = J_i(x) \qquad \begin{array}{l} i = 0, 1, 2, \cdots, m-1; \\ j = 1, 2, 3, \cdots; \end{array}$$

where the distributions $J_i(x)$, for $i = 0, 1, \cdots, m-1$, are not identical. This represents, e.g., the case where every mth customer has the same service-time distribution, but the distributions for consecutive customers differ. Or if, instead, w_r denotes the storage level for a reservoir, and u_r denotes (inflow — required outflow) during the rth period of time, then (47) means that the distribution of (inflow — required outflow) is a periodic function of discrete time r, representing perhaps a seasonal effect. For this model, neither Lindley's theorem, nor the results of Loynes [3], who assumes that the $\{u_r\}$ form a strictly stationary sequence, are applicable.

From (44) and (47)

$$F_{jm+i+1}(x) = \int_{-\infty}^{\infty} G_i(x, u) dF_{jm+i}(u)$$

where

$$G_i(x, u) = 0 \qquad \text{for } x < 0$$
$$= J_i(x-u) \quad \text{for } x \geqq 0.$$

Since this is a special case of the Markov process (3), equation (6) for the joint distribution applies. Applying (6) repeatedly,

$$\begin{aligned}\Pr(w_{jm+m+1} \leqq x_{m+1}, w_{jm+m} \leqq x_m, \cdots, w_{jm+2} \leqq x_2) \\ = \int_{0-}^{\infty} L(x_2, x_3, \cdots, x_{m+1}; \xi) d\psi_j(\xi)\end{aligned} \tag{48}$$

where

$$\psi_j(\xi) = \Pr(w_{jm+1} \leqq \xi);$$

and

$$\begin{aligned}L(x_2, x_3, \cdots, x_{m+1}; \xi) \\ = \int_{0-}^{x_m} G_m(x_{m+1}, v_m) d \int_{0-}^{x_{m-1}} G_{m-1}(v_m, v_{m-1}) d \int_{0-}^{x_{m-2}} \cdots d \int_{0-}^{x_2} G_1(v_2, \xi)\end{aligned} \tag{49}$$

and $G_m(x, m) \equiv G_0(x, u)$. (Denote also $J_m(x) \equiv J_0(x)$.)

Given (47), the process cannot possess a stationary state in the ordinary sense. However, from (48), if $\psi_j(\xi)$ tends to a stationary state $\psi(\xi)$ as $j \to \infty$, then the joint distribution (48) of m consecutive waiting times tends to a joint stationary distribution, since it depends on j only through $\psi_j(\xi)$.

The following proof shows that if

$$b = \sum_{0}^{m-1} b_i < 0, \tag{50}$$

where b_i is the expectation of the distribution $J_i(x)$, then $\psi_j(\xi)$ tends to a stationary distribution as $j \to \infty$. The probability (45), with $r = jm$, depends on the distribution of the u_s, but not on their ordering. It is therefore unaltered by the relabelling

$$u_s = v_{jm+1-s} \qquad (s = 1, 2, \cdots, jm), \tag{51}$$

provided that the same distributions are retained, i.e. that v_{km+i} has the distribution $J_{m+1-i}(x)$ for $i = 0, 1, \cdots, m-1$; $k = 0, 1, 2, \cdots$. Then

$$\begin{aligned}\psi_j(x) = F_{jm+1}(x) &= \Pr\{v_1 + \cdots + v_{jm} \leqq x, v_1 + \cdots + v_{jm-1} \leqq x, \cdots, v_1 \leqq x\} \\ &= \Pr\{E_j\}\end{aligned}$$

where E_j denotes the joint event

$$\{v_1 \leqq x, v_1 + v_2 \leqq x, \cdots, v_1 + \cdots + v_{jm} \leqq x\}.$$

For $k = 0, 1, 2, \cdots$ consider $(v_{km+1}, v_{km+2}, \cdots, v_{km+m})$ as a point in an m-space S_k, in which a probability measure is defined by the product of the distributions $J_{m+1-i}(x)$, for $i = 1, 2, \cdots, m$. Then in the product space $S_0 \times S_1 \times S_2 \times \cdots$, the sequence of events $\{E_j : j = 1, 2, 3, \cdots\}$ is contracting, and tends as $j \to \infty$ to the limit event

$$E = \{Z_s \leqq x : s = 1, 2, 3, \cdots\}$$

where $Z_s = v_1 + v_2 + \cdots + v_s$.

Now the probability measure is the same in each subspace S_k; so, by a property of probability measures,

$$\lim_{j\to\infty} \psi_j(x) = \lim_{j\to\infty} \Pr\{E_j\} = \Pr\{E\} \tag{52}$$

exists. Denote this limit by $\psi(x)$.

If $\psi(x)$ is an "honest" probability distribution, i.e. if $\psi(x) \to 1$ as $x \to \infty$, then $\psi_j(x)$ tends to a stationary state, and consequently, so does the joint distribution (48). This is established, given (50), by the following argument. For each integer i in $\{1, 2, \cdots, m\}$, Z_{km+i} equals the sum of m partial sums R_{ks} $(s = 1, 2, \cdots, m)$, where R_{ks} is the sum of those v_r in Z_{km+i} which are distributed as $J_s(x)$. The number of terms in R_{ks} is k (if $s > i$) or $k+1$ (if $s \leqq i$). Since the v_r are independent, the strong law of large numbers shows that $R_{ks}/k \to b_s$, with probability 1, as $k \to \infty$. Therefore, with probability 1,

$$\sum_{s=1}^{m} R_{ks}/(k) \to \sum_{s=1}^{m} b_s = b \quad \text{as } k \to \infty, \tag{53}$$

where b is negative, by hypothesis (50). Choose any δ in $0 < \delta < 1$. Then since b is negative, there is an integer k_0 such that, for all $k > k_0$ and all i in $\{1, 2, \cdots, m\}$,

$$\Pr\{v_1 + \cdots + v_{km+i} \leqq 0\} > 1 - \tfrac{1}{2}\delta. \tag{54}$$

Now, considering the joint distribution of $v_1, v_2, \cdots, v_t$, where $t = k_0 m$, there exists a positive x for which

$$\Pr\{v_1 \leqq x, v_1+v_2 \leqq x, \cdots, v_1 + \cdots + v_t \leqq x\} > 1 - \tfrac{1}{2}\delta. \tag{55}$$

So, combining (53) and (54),

$$\Pr\{v_1 + \cdots + v_r \leqq x \text{ for all } r \geqq 1\} > 1 - \delta, \text{ for } x = x(\delta).$$

Since also $\psi(x) \leqq 1$, and δ is arbitrary, this shows that

$$\lim_{x\to\infty} \psi(x) = 1. \tag{56}$$

In terms of the queue model, (50) does not preclude some b_i from being positive; i.e. the queue may be "over-saturated" for some values of i, but provided that it is sufficiently "under-saturated" for other values of i so that (50) holds, then the stationary joint distribution of m consecutive waiting times w_r still exists.

It can also be shown that $\psi(x)$, and therefore also the joint stationary distribution, does not depend on the waiting time of the first customer. If $w_1 = y > 0$, instead of $w_1 = 0$ as previously assumed, then a calculation similar to the proof of (45) shows that

$$F_{r+1}(x|y) \equiv \Pr\{w_{r+1} \leqq x | w_1 = y\}$$
$$(57) \qquad = \Pr\{u_1 + \cdots + u_r \leqq x-y, u_2 + \cdots + u_r \leqq x, \cdots, u_r \leqq x\}.$$

If $r = jm$, and the variables are relabelled as in (51), then

$$F_{jm+1}(x|y) = \Pr\{v_1 \leqq x, v_1 + v_2 \leqq x, \cdots, v_1 + \cdots + v_{jm-1} \leqq x, v_1 + \cdots + v_{jm} \leqq x-y\}$$

therefore

$$F_{jm+1}(x|0) \geqq F_{jm+1}(x|y) \geqq F_{jm+1}(x|0) - \Pr\{v_1 + \cdots + v_{jm} > x-y\}.$$

Since, for $b < 0$, the last term on the right tends to zero as $j \to \infty$, by (53),

$$(58) \qquad \lim_{j\to\infty} F_{jm+1}(x|y) = \lim_{j\to\infty} F_{jm+1}(x|0) = \psi(x),$$

which proves the stated result.

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