P 86. Let π be a projectivity on a line in the real projective plane. Show that if a single point P has period $n > 1$ under π, then π is periodic of period n, and every non-invariant point has period n.

John Wilker, University of Toronto

P 87. Let $0 < a_1 < a_2 < \ldots$ be an infinite sequence of integers and let $d_n = [a_1, \ldots, a_n]$ be the least common multiple of $a_1, \ldots, a_n$. Prove that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} d_n^{-\varepsilon}$$

converges.

P. Erdős, McMaster University

P 88. Let G be a graph with n vertices and more than $k(n-k) + \binom{k}{2}$ edges. Prove that G has a subgraph G_1 each vertex of which has valence $> k$.

P. Erdős, McMaster University

P 89. Exercise 2, p. 132, in Distance Geometry by L. M. Blumenthal reads:

Prove that for $p = 3, 4, 5, 6$, the minimum of the maximum angle determined by planar subsets of p points is $(p-2)\pi/p$. Show that this formula fails for $p = 7$. (The problem of determining the desired minimax is unsolved for $p > 6$.)

Find the minimax for $p = 7$.

Robert A. Melter, University of Massachusetts

P 90. Let $\log_s x$ be the log function iterated s times, and let m be the smallest positive integer such that $\log_4 m > 1$. Then show that the sum

$$\sum_{k=m}^{\infty} \frac{1}{k(\log k)(\log_2 k)(\log_3 k)(\log_4 k)^2}$$

is approximately 1, - correct to more than one million decimal places!

John D. Dixon, California Institute of Technology

SOLUTIONS

P 75. In a certain isolated community the marriage contract is for one year only. So great is the satisfaction with this arrangement that each January 1st the entire population, consisting of an equal number of men and women, gathers together and marries (in pairs) for the coming year. It may happen that a couple marry who have been married to each other in the past, there being no stigma attached to this. A "marriage graph" may be defined as a bipartite graph whose two vertex sets correspond to the men and women and in which two vertices are joined by an edge if and only if the corresponding people have been married to each other at least once.

What are necessary and sufficient conditions for a bipartite graph G to be a "marriage graph" of some such community for a period of n years during which the population remains fixed?

J. W. Moon, University College, London

Solution by the proposer.

Let the bipartite graph G be such a "marriage graph"; then the degree $\lambda(x)$ of each vertex x can be at most n, the number of years involved. For any subset M of one of the vertex sets of G denote by $\bar{M}$ the complement of M in that vertex set, by M' the set of vertices joined by an edge to at least one element of M, and by $c(M', \bar{M})$ the number of edges joining vertices of M' and $\bar{M}$. Since there are $n|M|$ marriages between members of M' and M and at least $c(M', \bar{M})$ between members of M' and $\bar{M}$ it follows that

$$(1) \qquad c(M', \bar{M}) + n|M| \leq n|M'| ,$$

the total number of marriages involving members of M'.

Now let G be a bipartite graph no vertex of which has degree greater than n and which satisfies (1) for all subsets M. We will presently show that this implies that it is possible to assign a positive integer to each edge in G so that the sum of the integers assigned to the edges incident to any vertex equals n. When this is done add new edges joining vertices already joined until the number of edges joining any two vertices equals the integer assigned to the edge originally joining them. The resulting bipartite graph is regular of degree n and hence its edges may be coloured with n colours in such a way that no two edges with the same colour have a vertex in common. This will suffice to show that G is a "marriage graph", since the edges of the i'th colour can be interpreted as representing the marriages of the i'th year, for $i = 1, 2, \ldots, n$.

It remains to prove the above assertion regarding the positive integers. Let X and Y be the two vertex sets of G. From (1) it follows that X and Y have the same number of elements. Form a transport network from G by adding a source z which is connected with vertex x by an edge of capacity $n - \lambda(x)$ for each x in X, and a sink z' which is connected with vertex y by an edge of capacity $n - \lambda(y)$ for each y in Y. Let each of the edges originally in G have capacity n. From the saturation theorem for network flows it follows that there will exist a flow from z to z' which will saturate the ingoing and outgoing edges if for each subset M of Y the maximum amount $F(M)$ of material that can flow into M is not less than the demand $d(M)$ of M. It is not difficult to see that

$$d(M) = \Sigma(n - \lambda(y)) = n|M| - \Sigma\lambda(y)$$

and

$$F(M) = \Sigma(n-\lambda(x)) = n|M'| - c(M', \bar{M}) - \Sigma\lambda(y) ,$$

where the sums are over all x in M' and y in M. The fact that G satisfies (1) implies that $F(M) \geq d(M)$ for all such M and hence the required flow exists. This flow defines non-negative integers which may be assigned to the edges in G so that the sum of the integers assigned to the edges incident to any vertex equals n minus the degree of the vertex. Adding one to each of these integers gives the positive integers with the properties originally described and completes the solution of the problem.

P 76. If H is a normal subgroup of a group G then, in particular,

(1) H commutes with every subgroup K of G, i.e. $HK = KH$;

(2) H is subnormal in G, i.e. there exists a normal series from G to H.

Thus these two properties are each generalisations of the property of being normal. Show that for any finite group G, any subgroup H which has property (1) also has property (2).

J. D. Dixon, California Inst. of Technology

Solution by Mrs. E. Rowlinson, McGill University.

We prove (2) under the weakened assumptions

(i) $[G:H] < \infty$, and

(ii) $H^g H = H H^g$ for all $g \in G$.

Since $[G:H] < \infty$ we can form a normal series $G = N_1 \rhd \ldots \rhd N_k$ where $N = N_k \supseteq H$ and N contains no proper normal subgroup containing H ; we will prove that $N = H$, thus showing that H is subnormal in G.

Let the complete set of conjugate subgroups of H in N be H, $H^{n_2}, \ldots, H^{n_r}$. Since $H^n H = H H^n$ for all $n \in N$,

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$H^n H^{n'} = H^{n'} H^n$ for all n, $n' \in N$. The complex $H_o = H H^{n_2} \dots H^{n_r}$ is therefore a group; moreover $H_o \trianglelefteq N$. But $H \subseteq H_o$, and so $H_o = N$. Let $H_1 = H H^{n_2} \dots H^{n_{r-1}}$; this is also a group, and $H_1 H_1^{n_r} = H_o = N$. Thus, since $n_r \in N$, there are elements h_1 and h_2 in H_1 such that $h_1 n_r^{-1} h_2 n_r = n_r$. Therefore $n_r \in H_1$, $H_1 = H_1^{n_r}$, and $H_1 = H_o = N$. Similarly, let $H_2 = H H^{n_2} \dots H^{n_{r-2}}$; $H_2 H_2^{n_{r-1}} = H_1 = N$, thus $n_{r-1} \in H_2$, and so $H_2 = H_1 = H_o = N$. Continuing we obtain $H = H_{r-1} = \dots = H_o = N$, as required.

Also solved by T. Hawkes, C. G. Thomas, H. Simon, and the proposer. Professor Thomas pointed out that the original problem follows from results of Ore [Duke Math. J., 5(1939), 431-460], and that a generalization of these ideas is given by Kegel [Math. Zeit., 78(1962), 205-221].

P 77. Prove that $n > 3$ lines in the projective plane, no three concurrent, determine at least n triangles.

Leo Moser, University of Alberta, Edmonton

Solution by B. Grünbaum, University of Jerusalem.

Using the notion of convexity, we prove the stronger statement: In any general configuration of $n > 3$ lines in the projective plane, every line is (edge) incident to at least 3 triangles.

Without loss of generality assume $n > 4$. Take any of the lines as the line at infinity of a Euclidean plane, and consider the $(n-1)(n-2)/2$ intersections of the remaining lines. Let C be the convex hull of this point-set. Then C has at least 3 vertices. Obviously, to each vertex of C corresponds exactly one triangle in the configuration of lines (formed by the two lines through the vertex and the line at infinity).

There need be no more than n triangles; this may be shown by different examples. The following shows also that one may find, for each n, n lines in general position such that each of the regions determined by them has at most 5 sides.

We construct, by induction, a family of n such lines with the additional property (which is probably automatically satisfied, but let us require it anyway) that a triangle is edge incident to a quadrangle. (The case $n = 4$ is trivial.) If n such lines are given, take a point in the common edge of the triangle and the quadrangle, and through it a line sufficiently close to the carrier-line of the edge. The triangle then yields a triangle and a quadrangle, the quadrangle a triangle and a pentagon, and our line cuts off a quadrangle from each of the remaining (n-2) regions through which it passes.