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BROWNIAN MOTION AND DIMENSION OF PERFECT SETS

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0. Let $X(t)$ denote real-valued Brownian motion on the interval $0 \leqq t \leqq 1$, so normalized that $E(X^2(t)) \equiv t$. We prove some theorems about transforms $X(F)$ of closed sets F: in general, F is not known in advance but depends on X. The main point of comparison among sets is taken to be their Hausdorff dimension, and in this respect the linear process is quite different from the planar. We state and discuss briefly two theorems.

(A) *It is almost sure that, for every closed set F in* $[0, 1]$,

$$\dim X(F) \geqq -1 + 2 \dim F.$$

(B) *For each closed set F in* $(-\infty, \infty)$ *and number* $\alpha < \frac{1}{2} + \frac{1}{2} \dim F$,

$$P\{\dim X^{-1}(F) \geqq \alpha\} > 0.$$

Plainly, statements (A) and (B) are nearly best possible. For the planar process $\dim X(F) = 2 \dim F$ (with the same quantification as in (A)) [**6**]. In the linear case again, $\dim X(F) = \min(1, 2 \dim F)$ holds almost surely for each *fixed* F [**2**], so that the set $X^{-1}(F)$ in (B) is essentially stochastic.

The proof of (A) depends largely on independence of increments, and it is presented apart from the other proofs. The proof of (B), however, depends upon a principle exposed in [**5**], capable of application to a variety of random trigonometrical series. These form a second topic of this paper and some investigations are necessary to establish (B) for an interesting class of series. The objective is to show that a certain random choice of coefficients yields a series that is "in the small" a Brownian motion. However, the proof of (B) for Brownian motion is implicit in the more general case. The main reference for these matters is [**1**].

It will be convenient to use $a \ll b$ as a substitute for $a = O(b)$.

1. We denote by φ a measure function on $(0, \infty)$: φ is positive and increasing, $\varphi(0+) = 0$, $\varphi(2u) \ll \varphi(u)$ for all $u > 0$.

In a pair (φ, ψ) of measure functions, it is always supposed that for small u (say $0 < u < e^{-1}$),

$$\psi(u) \geqq u^{-1} \log^{3/2}(u^{-1}) \varphi(u^2).$$

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Thus if $\varphi(u) = u^\alpha$, with $\frac{1}{2} < \alpha \leqq 1$, we can choose $\psi(u) = u^{2\alpha-1}\log^2(u^{-1})$. This is the link between (A) and the following theorem.

THEOREM 1. *It is almost sure that, for any closed set F, and any pair of measure functions*

$$\varphi - \text{meas } F > 0 \Rightarrow \psi - \text{meas } X(F) > 0.$$

This theorem is in turn a consequence of a simpler statement.

(i) *It is almost sure that if I is an interval of length* 2^{-m} ($m = 1, 2, 3, \ldots$), *then* $X^{-1}(I)$ *meets* $\ll m^{3/2}2^m$ *of the intervals*

$$[k4^{-m}, (k+1)4^{-m}], \qquad 0 \leqq k < 4^m.$$

In view of Lévy's theorem [**7**, p. 547] on the modulus of continuity of almost all paths X, when $X^{-1}(I)$ meets $[k4^{-m}, (k+1)4^{-m}]$, then $X(k4^{-m})$ falls in an interval I' of length $\ll m^{1/2}2^m$, and concentric with I. Thus (i) is reduced to

(ii) *It is almost sure that if* I' *is an interval of length* $m^{1/2}2^{-m}$, *then* $X^{-1}(I')$ *contains* $\ll m^{3/2}2^m$ *of the points* $k4^{-m}$.

To prove (ii) we estimate the number of m-tuples $0 \leqq k_1 < \ldots < k_m \leqq 4^m$ for which

$$|X(k_{i+1}4^{-m}) - X(k_i4^{-m})| \leqq m^{1/2}2^{-m}, \qquad 1 \leqq i < m.$$

Each m-tuple thereby defines an event of probability

$$\leqq (C_1 m^{1/2})^{m-1}(k_2 - k_1)^{-1/2} \ldots (k_m - k_{m-1})^{-1/2},$$

where $C_1 = (2/\pi)^{1/2}$. The sum of this probability for all m-tuples is

$$\leqq (C_1 m^{1/2})^{m-1} 4^m (C_2 2^m)^{m-1} \leqq (C_4 2^m m^{1/2})^m, \quad \text{say.}$$

By the Chebyshev inequality, the number of m-tuples is $\ll m^2(C_4 m^{1/2}2^m)^m$ almost surely. If, however, there is an interval I' of length $m^{1/2}2^{-m}$ such that $X(k4^{-m}) \in I'$ for s values of k, then

$$\binom{s}{m} \ll m^2(C_4 m^{1/2}2^m)^m$$

and after some calculation we find $s \leqq C_5 m^{3/2}2^m$, as required.

2. Let $\Lambda = (\lambda_n)_1^\infty$ be a sequence of mutually independent integer-valued random variables, with λ_n uniformly distributed among the values $1, 2, \ldots, n$. Set

$$g(t) = \sum_{n=1}^{\infty} n^{-2}\cos(2\pi\lambda_n t).$$

LEMMA 1. *Almost surely* $g(t)$ *differs from* $\sum_{n=1}^\infty \frac{1}{2}n^{-2}\cos(2\pi nt)$ *by a continuously differentiable function.*

Proof. We compute first the sum of the expectations of the various terms, obtaining

$$\sum_{m=1}^{\infty} m^{-3} \sum_{n=1}^{m} \cos(2\pi nt) = \sum_{n=1}^{\infty} \left(\sum_{m=n}^{\infty} m^{-3}\right)\cos(2\pi nt)$$
$$= \sum_{n=1}^{\infty} \tfrac{1}{2}n^{-2} \cos(2\pi nt) + \sum_{n=1}^{\infty} O(n^{-3})\cos(2\pi nt).$$

Next we observe that the nth term of the derived series has expectation

$$-2\pi n^{-3} \sum_{m=1}^{n} m \sin 2\pi mt \ll n^{-2}(t - t^2)^{-1}$$

on $(0, 1)$. Now, following [**1**], we see that the derived series, after being centred at expectation, converges uniformly, because the general term is $\ll n^{-1}$. The expectations, however, converge uniformly on compact subsets to the derivative of the function found before. This completes the proof.

Set

$$h(s, t) = \sum_{n=1}^{\infty} n^{-2} \cos(2\pi\lambda_n t)\cos(2\pi\lambda_n s)$$
$$= \tfrac{1}{2}g(t + s) + \tfrac{1}{2}g(t - s).$$

LEMMA 2. *Let* $0 < a < b < 1$, $\frac{1}{2} \notin [a, b]$. *Then the inequality*

$$h(s, s) + h(t, t) - 2h(s, t) \gg |t - s|$$

holds uniformly on $[a, b]$ *for* $|t - s|$ *sufficiently small.* (By Schwarz's inequality, $(h(t, t) + h(s, s)) \geqq 2h(s, t)$.)

Proof. The function to be estimated is $g(0) - g(s - t) + \frac{1}{2}g(2s) + \frac{1}{2}g(2t) - g(s + t)$. Referring to the form of g obtained in the last lemma, and noting that $2s$ and $2t$ are not near 0 (modulo 1), we conclude that the sum of the last three terms is $o(|t - s|)$. Finally,

$$g(0) - g(u) \sim \tfrac{1}{2}\pi|u| \quad \text{as } |u| \to 0$$

and this yields the lemma.

LEMMA 3. *Almost surely* $\inf h(t, t) > 0$.

Proof. By definition, $h(t, t)$ is continuous on $0 \leqq t \leqq 1$ and $h(t, t) > 0$ there unless $t = \frac{1}{4}, \frac{3}{4}$. For each of these values of t the equation $\cos(2\pi\lambda_n t) = 0$ has at most $1 + \frac{1}{2}n$ solutions in the range $1 \leqq \lambda_n \leqq n$. The variables λ_n are independent, whence $h(t, t) > 0$ for all t, almost surely.

The significance of the function $h(s, t)$ can now be explained. Let $Y = (Y_n)_1^{\infty}$ be a sequence of mutually independent Gaussian variables of type $N(0, 1)$ and

$$H(t) = \sum_{n=1}^{\infty} n^{-1} Y_n \cos(2\pi\lambda_n t), \qquad 0 \leqq t \leqq 1.$$

For each fixed choice of Λ, $H(t)$ is a Gaussian process, the series being almost surely uniformly convergent [**1**]. Then

$$E(H(s)H(t)) = h(s, t).$$

The component of $H(t)$, *orthogonal to* $H(s)$, has variance

$$\begin{aligned} E(H^2(s)) - E^2(H(t)H(s))/E(H^2(t)) &= h(s, s) - h^2(s, t)h^{-1}(t, t) \\ &\gg h(s, s)h(t, t) - h^2(s, t). \end{aligned}$$

In the identity

$$uv - w^2 = (u + v - 2w)w + (u - w)(v - w)$$

we set $u = h(s, s)$, $v = h(t, t)$, and $w = h(s, t)$. Then $(u + v - 2w)w \gg |t - s|$ for small $|t - s|$ on an interval $[a, b]$ as in Lemma 2. Also $(u - w)(v - w) = o(|t - s|)$ from the smoothness property established for g. Thus

$$h(s, s)h(t, t) - h^2(s, t) \gg |t - s|.$$

From the same property of g,

$$\begin{aligned} E((H(s) - H(t))^2) &= h(s, s) + h(t, t) - 2h(s, t) \\ &\gg |t - s|. \end{aligned}$$

Observe that Brownian motion trivially satisfies the conditions established for almost all processes H, provided we omit an interval $0 \leqq t \leqq \epsilon$ from the real line.

4. In this section we establish a useful fact about sets of fractional dimension.

LEMMA 4. *Let F be a compact set in $(-\infty, \infty)$, $0 < \eta < \dim F$. Then there is a positive constant C_1 and an integer N_0 with the following property: for each $N \geqq N_0$ there exist points $\xi_1, \ldots, \xi_N$ in F such that the inequality*

$$|\xi_i - \xi_j| < R, \qquad i \neq j,$$

has at most $C_1 R^\eta N^2$ solutions for $R = 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

Proof. Let $\eta < \eta_1 < \dim F$; following Frostman [**4**, p. 27], there exists a probability measure μ in F such that $\mu(x, x + h) \leqq C_2 h^{\eta_1}$ for all intervals of length h. Let $\xi_1, \ldots, \xi_N$ be mutually independent random variables whose law of distribution is μ.

The expected number of solutions of the inequality $|\xi_i - \xi_j| < R$, $i \neq j$, is $N(N - 1)P\{|\xi_1 - \xi_2| < R\} \leqq 2N(N - 1)C_2 R^{\eta_1}$. Therefore the event, that the inequality have at least $C_1 R^\eta N^2$ solutions, is of probability at most $2C_2 C_1 R^{\eta_1 - \eta}$. We have only to choose C_1 so large that

$$C_1^{-1} C_2 \sum_{j=1}^{\infty} 2^{j(\eta - \eta_1)} < 1$$

to obtain at least one suitable choice of $\xi_1, \ldots, \xi_N$.

Of course the lemma holds for $0 < R \leqq 1$ if C_1 is increased to $2C_1$.

5. Let $\varphi(N) = N^{-\eta^{-1}}$ for $N \geqq 1$. Having chosen by Lemma 4 the points $\xi_1, \ldots, \xi_N$, we define A_N to be the union of the intervals $[\xi_i, \xi_j + \varphi(N)]$, $1 \leqq i \leqq N$. Among distinct intervals, the number of intersections is at most $C_1\varphi^\eta(N) \cdot N^2$ by Lemma 4; by using a value slightly larger than η in Lemma 4 we can make this $o(\varphi^\eta(N)) \cdot N^2 = o(N)$. Thus the Lebesgue measure of A_N is $(1 + o(1))N\varphi(N)$.

We now fix some sequence Λ for which all the "almost sure" conclusions of the previous lemmas actually hold, and also an interval $[a, b]$ not containing $\frac{1}{2}$. The following estimation of probabilities is fundamental.

LEMMA 5. *For $|t - s|$ sufficiently small and N sufficiently large, we have the conditional probability*

$$P^\Lambda\{H(s) \in A_N, H(t) \in A_N\} \ll N^2\varphi^2(N) \log(|t - s|^{-1})|t - s|^{-1/2+\eta/2}.$$

Proof. For each s and t, $H(s)$ and $H(t)$ can be represented by means of *independent* Gaussian variables Z_1 and Z_2:

$$H(s) = A_1Z_1, \qquad H(t) = A_2Z_1 + BZ_2;$$

here A_1, A_2, and B are functions of Λ with

$$A_1 \gg 1, \qquad |A_1 - A_2| \ll |t - s|^{1/2}$$
$$|t - s|^{1/2} \ll B \ll |t - s|^{1/2}.$$

The probability in question is at most

$$\sum_{i=1}^{n} \sum_{j=1}^{n} P^\Lambda\{\xi_i \leqq H(s) \leqq \xi_i + \varphi(N), \xi_j \leqq H(t) \leqq \xi_j + \varphi(N)\}.$$

We observe that $P^\Lambda\{H(s) \in A_N\} \ll N\varphi(N)$ so that the lemma is obvious except in the case

$$1 \geqq N\varphi(N) \log(|t - s|^{-1})|t - s|^{-1/2+\eta/2},$$

an inequality that is now assumed to be true. We then divide the N^2 pairs (ξ_i, ξ_j) into three classes and estimate the total probability for each.

(a) $|\xi_i - \xi_j| \leqq \log(|t - s|^{-1})|t - s|^{1/2}$, $i \neq j$. The number of these pairs is $\ll N^2 \log^\eta(|t - s|^{-1})|t - s|^{\eta/2}$. Recalling that $H(t) - H(s)$ has a component *independent* of $H(t)$, of variance $\gg |t - s|$, we find a total contribution

$$\ll N^2 \log^\eta(|t - s|^{-1})|t - s|^{\eta/2} \cdot \varphi^2(N)|t - s|^{-1/2},$$

as required.

(b) $|\xi_i - \xi_j| \geqq \log(|t - s|^{-1})|t - s|^{1/2}$. Now we use the inequality assumed before and find that for large N and small $|t - s|$, $\varphi(N) = o(|\xi_i - \xi_j|)$. Thus the intervals $[\xi_i, \xi_i + \varphi(N)]$, $[\xi_j, \xi_j + \varphi(N)]$ have a mutual distance $\gg |\xi_i - \xi_j|$.

The event whose probability is to be estimated is

$$|A_1Z_1 - \xi_i| \leqq \varphi(N), \qquad |A_2Z_1 + BZ_2 - \xi_j| \leqq \varphi(N).$$

By the upper bound on $A_1 - A_2$ and the lower bound on A_1, we have $|A_1 - A_2| \ |Z_1| \ll |t - s|^{1/2}$ and therefore $|BZ_2| \gg \log(|t - s|^{-1})|t - s|^{1/2}$, $|Z_2| \gg \log(|t - s|^{-1})$.

This requires that (Z_1, Z_2) belong to a planar set of Lebesgue measure $\ll \varphi^2(N)|t - s|^{-1/2}$, and also that $|Z_2| \gg \log(|t - s|^{-1})$. The probability of this event is $\ll |t - s|^{-1/2}\varphi^2(N) \cdot \exp[\delta \log^2(|t - s|^{-1})]$ for some $\delta > 0$. Plainly, this is $o(\varphi^2(N))$ for $|t - s|$ sufficiently small. Since the number of pairs is N^2, the total probability under (b) is $o(N^2\varphi^2(N))$.

(c) $i = j$. The total here has magnitude $\ll N\varphi^2(N)|t - s|^{-1/2}$ and this is within the bounds required by the inequality assumed above.

THEOREM 2. *Let $F_1 \subseteq [0, 1]$ and $F \subseteq (-\infty, \infty)$ be compact sets and let*

$$0 < \beta < \dim F_1 + \tfrac{1}{2} \dim F - \tfrac{1}{2}.$$

Then $P\{\dim[H^{-1}(F) \cap F_1] \geqq \beta\} > 0$.

Proof. Let $\eta > 0$ and $\delta > 0$ numbers such that

$$\eta < \dim F, \qquad \delta < \dim F_1, \qquad \beta < \delta + \tfrac{1}{2}\eta - \tfrac{1}{2}.$$

(This η is used in the previous estimations.) Let σ be a probability measure in F_1 such that

$$\sigma(x, x + h) \leqq C_2h^\delta \quad \text{for all intervals of length } h > 0.$$

We can suppose also that $0, 1, \frac{1}{2} \notin F_1$. For large integers N let σ_N be the random measure in F_1 defined as follows:

$$\begin{aligned} \sigma_N(T) &= 0 && \text{if } T \cap H^{-1}(A_N) = 0, \\ \sigma_N(T) &= N^{-1}\varphi^{-1}(N)\sigma(T) && \text{if } H(T) \subseteq A_N. \end{aligned}$$

Then $E(||\sigma_N||) \gg 1$, by Fubini's Theorem and our estimate of the Lebesgue measure of A_N. Further,

$$\begin{aligned} E(||\sigma_N||^2) &\ll E(\textstyle\iint |t - s|^{-\beta}\sigma_N(ds)\sigma_N(dt)) \\ &= N^{-2}\varphi^{-2}(N)\textstyle\iint P\{H(s) \in A_N, H(t) \in A_N\}|t - s|^{-\beta}\,\sigma(ds)\sigma(dt) \\ &\qquad \ll \textstyle\iint |t - s|^{-1/2+\eta/2}\log(|t - s|^{-1})|t - s|^{-\beta}\,\sigma(ds)\sigma(dt) < \infty \end{aligned}$$

since $-\beta - \frac{1}{2} + \frac{1}{2}\eta > -\delta$.

Arguing as in **[5]**, we find that with positive probability, $F_1 \cap H^{-1}(F)$ carries a positive measure whose potential in dimension β is finite, and so has dimension $\geqq \beta$ **[4]**.

The final theorem shows that if F_1 is suitably restricted, the number β is in fact best possible. We suppose for this purpose that F_1 meets $\ll 2^{m\lambda}$ intervals $[k2^{-m}, (k + 1)2^{-m}]$, $k = 0, \pm 1, \pm 2, \ldots$, $(0 < \lambda \leqq 1)$.

THEOREM 3. *Let F be a compact set of dimension $<\delta$. Then*

$$P\{\dim[H^{-1}(F) \cap F_1] \leqq -\tfrac{1}{2} + \tfrac{1}{2}\delta + \lambda\} = 1$$

and the set in braces is almost surely void if $\lambda + 2\delta < 1$.

Since there are compact sets F_1 for which $\lambda = \dim F_1$, e.g. the ternary Cantor set, Theorem 3 does provide a partial converse to Theorem 2. We only sketch the proof, referring to [**5**] for the details.

For each $c < \frac{1}{2}$, H is almost surely of class Lip^c[1]. Consider a covering of F by intervals $I_1, \ldots, I_N$ with lengths $|I_1|, \ldots, |I_N|$. For each I_k we can choose, by the definition of λ, a covering of F_1 by intervals J_l of length $|I_k|^2$, in number $\ll |I_k|^{-2\lambda}$, and with centres a_l. Then $H(J_l) \cap I_k = \emptyset$ unless $H(a_l)$ has distance $\ll |I_k|^{2c}$ from I_k. This defines an event of probability $\ll |I_k|^{2c}$. For each interval I_k, the contribution to the "expected Hausdorff measure", in dimension $-\frac{1}{2} + \frac{1}{2}\delta + \lambda$, of $H^{-1}(F) \cap F_1$ is thus

$$\ll |I_k|^{-2\lambda} \cdot |I_k|^{2c} \cdot |I_k|^{-1+\delta+2\lambda}.$$

Choosing $c < \frac{1}{2}$ so that $\dim F < \delta - 1 + 2c$, we obtain the inequality on dimension. In case $-1 + \delta + 2\lambda < 0$, we suppress the last factor above and choose c so that $2c - 2\lambda > \delta$.

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