A REMARK ON RELATIVE HOMOLOGY AND COHOMOLOGY GROUPS OF A GROUP

To ZYOITI SUETUNA on his 60th Birthday

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Let G be a group and H a subgroup of G. With a left G-module M, relative cohomology groups $H^n(G, H, M)$ of G on M, relative to H, have been defined by Adamson [1] and may be expressed as $\mathrm{Ext}^n_{(G,H)}(Z, M)$ in the notation of relative homological algebra of Hochschild [2], where Z denotes the G-module of rational integers (acted by G trivially). Regarding M as a right G-module, $\mathrm{Tor}_n^{(G,H)}(M, Z)$ are similarly relative homology groups $H_n(G, H, M)$. In case H is of finite index in G, Hochschild [2] defines further negative-dimensional relative homology and cohomology groups. He then remarks that these complete relative homology and cohomology structures are separate (contrary to the absolute case $H=1$). Indeed he exhibits an example of G, H, M (with H even normal in G) such that $H^n(G, H, M)=0$ for every $n=0, \pm 1, \pm 2, \ldots$ and $H_n(G, H, M)$ is a group of order 2 for every $n=0, \pm 1, \pm 2, \ldots$. This, however, does not exclude the possibility that negative-dimensional relative homology groups $H_{-n}(G, H, M)$ are in close relationship with positive-dimensional relative cohomology groups on some G-module N other than M. In fact, in case H is a normal subgroup of G, we have $H_{-n}(G, H, M) \approx H_{-n}(G/H, M_H) \approx H^{n-1}(G/H, M_H)$ (where M_H denotes as usual the residue-module of M with respect to the submodule generated by the elements of form $u-hu$ ($u \in M$, $h \in H$) and this is isomorphic to $H^{n-1}(G/H, N^H) \approx H^{n-1}(G, H, N)$ if M_H is G-isomorphic to N^H (where N^H is the submodule of N consisting of all elements of N left invariant by H); this holds not only for $n>0$ but for all $n=0, \pm 1, \pm 2, \ldots$. Now we want to show that a similar phenomenon prevails also in case of a non-normal subgroup H.

Thus, let H be a subgroup of finite index in a group G and K_0 be the largest normal subgroup of G contained in H, i.e. the intersection of all conjugates of H in G. For G-modules M and N, we consider the following condition:

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(A) There exists a G-homomorphism κ_0 of M into N such that κ_0 induces a (G-) isomorphism of M_{K_0} onto N^{K_0}, and, moreover, we have $H^{-1}(K/K_0, M_{K_0}) = H^0(K/K_0, M_{K_0}) = 0$ for every subgroup K of H which is an intersection of conjugates of H in G.

This condition may also be formulated as follows:

(A') There exists a G-homomorphism κ_0 of M into N such that κ_0 induces a (G-) isomorphism of M_{K_0} onto N^{K_0} and for every subgroup K of H which is an intersection of conjugates of H in G the homomorphism $\kappa(K)$ of M into N defined by

$$\kappa(K)u = \sum_{K \ni \rho \bmod K_0} \rho\kappa_0 u \qquad (u \in M), \tag{1}$$

ρ running over a representative system of cosets of K_0 in K, induces an isomorphism of M_K onto N^K.

Indeed, the endomorphism $v \to \sum_{K \ni \rho \bmod K_0} \rho v$ ($v \in N^{K_0}$) of N^{K_0} induces a monomorphism (resp. an epimorphism) of $(N^{K_0})_{(K/K_0)}$ to $(N^{K_0})^{(K/K_0)} = N^K$ if and only if $H^0(K/K_0, N^{K_0}) = 0$ (resp. $H^{-1}(K/K_0, N^{K_0}) = 0$). Combining this consideration with G-isomorphism of M_{K_0} and N^{K_0} induced by κ_0, we see the equivalence of the conditions (A), (A') readily.

We want also to note that if (A) (or (A')) is the case and if K is an intersection of non-void set of conjugates of H in G, which is not necessarily a subgroup of H, the homomorphism of M into N defined by the same formula as (1) induces an isomorphism of M_K onto N^K, as follows readily from the G-isomorphism property of κ_0 by an easy conjugation consideration. With this generalized significance of $\kappa(K)$, we observe also that if K, L are two intersections of non-void sets of conjugates of H in G and if $L \supset K$ then

$$\kappa(L) = \sum_{L \ni \rho \text{ r.} \bmod K} \rho\kappa(K), \tag{2}$$

ρ running over a representative system of right cosets of K in L. We have also

$$\kappa(\sigma K \sigma^{-1}) = \sigma\kappa(K)\sigma^{-1}. \tag{3}$$

Now, with the condition (A) (or (A')) we assert

THEOREM.[1] *Let H be a subgroup of finite index in a group G. For G-*

[1] The theorem will be applied in a subsequent paper to a study of fundamental exact sequences in homology and cohomology of finite groups. Cf. remark at the end.

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modules M, N, suppose that the condition (A) (or, equivalently, (A′)) is satisfied. Then

$$H_n(G, H, M) \approx H^{-n-1}(G, H, N)$$

for all $n \gtreqless 0$. *More precisaly, if* $\sum_{n \gtreqless 0} X_n$ *is the standard complete complex G relative to H, the complexes* $\sum M \otimes_G X_n$, $\sum \mathrm{Hom}_G(X_{-n-1}, N)$ *with respective differentiations* ∂, δ *are isomorphic by an isomorphism mapping* $M \otimes_G X_n$ *onto* $\mathrm{Hom}_G(X_{-n-1}, N)$.

Proof. Relative homology and cohomology groups $H_n(G, H; M)$, $H^n(G, H; M)$, with H of finite index in G, are defined most conveniently by means of the standard complete complex of (G, H), i.e. the exact sequence

$$\cdots \longrightarrow x_1 \xrightarrow{\partial_1} x_0 \xrightarrow{\partial_0} X_{-1} \xrightarrow{\partial_{-1}} X_{-2} \xrightarrow{\partial_{-2}} \cdots \tag{4}$$

in which each X_n with $n \geqq 0$ is the G-module having the totality of $(n+1)$-tuples $(\sigma_0 H, \sigma_1 H, \ldots, \sigma_n H)$ of right H-cosets in G as Z-basis and the map $\partial_n : X_n \to X_{n-1}$, $n > 0$, is defined by (linearity and)

$$\partial_n(\sigma_0 H, \ldots, \sigma_n H) = \sum_{i=0}^{n} (-1)^i (\sigma_0 H, \ldots, \sigma_{i-1} H, \sigma_{i+1} H, \ldots, \sigma_n H)$$

while X_{-n}, $n > 0$, is the G-module $\mathrm{Hom}_Z(X_{n-1}, Z)$ dual to X_{n-1} and the map $\partial_{-n} : X_{-n} \to X_{-n-1}$, $n \geqq 0$, is dual of ∂_{n+1}, and further, the map $\partial_0 : X_0 \to X_1$ is the combination of the coefficient sum homomorphism $X_0 \to Z$ and its dual $Z \to X_{-1}$; all the modules X_n, $n \gtreqless 0$, are (G, H)-projective, all the maps ∂_n, $n \gtreqless 0$, are G-homomorphic, and the sequence is (G, H)-exact, in the sense of Hochschild [2]. We observe also that each X_n is in fact G-isomorphic to its dual X_{-n-1} (and is, hence, (G, H)-injective too); for $n \geqq 0$ the isomorphism is given by associating $(\sigma_0 H, \ldots, \sigma_n H)$ with the element $\{\sigma_0 H, \ldots, \sigma_n H\}$ of $\mathrm{Hom}_Z(X_n, Z)$ which maps $(\sigma_0 H, \ldots, \sigma_n H)$ into 1 but other $(n+1)$-tuples to 0.

(Relative) cohomology groups $H^n(G, H; N)$, $n \gtreqless 0$, on a G-module N is defined by the sequence

$$\cdots \xleftarrow{\partial} \mathrm{Hom}_G(x_1, N) \xleftarrow{\partial} \mathrm{Hom}_G(X_0, N) \xleftarrow{\partial} \mathrm{Hom}_G(X_{-1}, N) \xleftarrow{\partial} \mathrm{Hom}_G(X_{-2}, N) \xleftarrow{\partial} \cdots \tag{5}$$

while (relative) homology groups $H_n(G, H; M)$, $n \gtreqless 0$, on a G-module M is defined by the sequence

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$$(6)\qquad \cdots \xrightarrow{\partial} M\otimes_G X_1 \xrightarrow{\partial} M\otimes_G X_0 \xrightarrow{\partial} M\otimes_G X_{-1} \xrightarrow{\partial} M\otimes_G X_{-2}\cdots,$$

both sequences being derived from (4) in natural manner. As X_{-n-1} is $\mathrm{Hom}_Z(X_n, Z)$ (and is in fact isomorphic to X_n itself) we have an G-isomorphism

$$\nu : N\otimes_Z X_n \approx \mathrm{Hom}_Z(X_{-n-1}, N)$$

where $v\otimes x$ $(v\in N,\ x\in X)$ in the left-hand side is mapped to the element $\nu(v\otimes x)$ of the right-hand side such that $\nu(v\otimes x)f=(fx)v$ for $f\in X_{-n-1}$ $=\mathrm{Hom}_Z(X, Z)$; the map is not only (G-)homomorphic but isomorphic since X_n has an (independent) finite Z-basis. Hence we have an isomorphism, denoted also by ν,

$$(7)\qquad (N\otimes_Z X_n)^G \approx \mathrm{Hom}_G(X_{-n-1}, N).$$

Now, we first consider the case $n\geqq 0$. There is a system of $(n+1)$-tuples

$$(8)\qquad s=(\sigma_0 H, \sigma_1 H, \ldots, \sigma_n H),\qquad s'=(\sigma'_0 H, \sigma'_1 H, \ldots, \sigma'_n H), \ldots$$

such that for every $(n+1)$-tuple $t=(\tau_0 H, \tau_1 H, \ldots, \tau_n H)$ there is one, and only one, among them, say $s^{(\mu)}$, from which the given $(n+1)$-typle t is obtained by the operation of an element of G, thus $t=\tau s^{(\mu)}$ $(\tau\in G)$; here $\tau\equiv\tau_0\sigma_0^{(\mu)^{-1}}$ r. mod $\sigma_0^{(\mu)}H\sigma_0^{(\mu)^{-1}}$, $\equiv\tau_1\sigma_1^{(\mu)^{-1}}$ r. mod $\sigma_1^{(\mu)}H\sigma_1^{(\mu)^{-1}}, \ldots,$ and τ is determined uniquely up to r. mod $\sigma_0^{(\mu)}H\sigma_0^{(\mu)^{-1}}\cap\sigma_1^{(\mu)}H\sigma_1^{(\mu)^{-1}}\cap\cdots\cap\sigma_n^{(\mu)^{-1}}H\sigma_n^{(\mu)^{-1}}$. We see readily that every element of the tensor product $M\otimes_G X_n$ is expressed in a form

$$(9)\qquad u\otimes_G s+u'\otimes_G s'+\cdots\qquad (u, u', \ldots\in M).$$

Here the classes in the residue-modules $M_K, M_{K'}, \ldots$ of the elements $u, u', \ldots,$ respectively, are uniquely determined, where we put

$$(10)\qquad K=\sigma_0 H\sigma_0^{-1}\cap\sigma_1 H\sigma_1^{-1}\cap\cdots\cap\sigma_n H\sigma_n^{-1},\qquad K'=\sigma'_0 H\sigma_0'^{-1}\cap\sigma'_1 H\sigma_1'^{-1}\cap\cdots\cap \sigma'_n H\sigma_n'^{-1}, \ldots$$

for brevity.

Now, let $\kappa=\kappa(K)$, $\kappa'=\kappa(K')$, ... be the homomorphisms of M into N which are described in the condition (A') of our theorem and thus in particular induce isomorphisms $M_K\approx N^K$, $M_{K'}\approx N^{K'}$, ... respectively. We associate with (9) the element

$$(11)\qquad \sum_{\rho\ \mathrm{r.\,mod}\ K}\rho(\kappa u)\otimes\rho s+\sum_{\rho'\ \mathrm{r.\,mod}\ K'}\rho'(\kappa' u')\otimes\rho' s'+\cdots$$

of $N\otimes_Z X_n$, where in the first sum ρ runs over a representative system of right

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cosets modulo K in G, in the second sum ρ' runs over a representative system of right cosets modulo K' in G, and so on. This element (11) of $N\otimes_Z X_n$ is independent of the choices of these representative systems of cosets, as follows from the definitions of κ, K, etc., and is, moreover, determined uniquely by the element (9), irrespective of the special choices of elements u, u', . . . from their classes in M_K, $M_{K'}$, ...[2] The former of the last remarks entails that (11) belongs in fact to $(N\otimes_Z X_n)^G$. Thus, by (9) → (11) we obtain a homomorphism

$$\varphi:\ M\otimes_G X_n \to (N\otimes_Z X_n)^G.$$

We contend that this homomorphism is an isomorphism. Thus, suppose that (11) is 0. Since $\{\rho s\}$, $\{\rho' s'\}$, . . . are altogether the set of distinct $(n+1)$-tuples, this implies that each single term in the sum (11) is 0, i.e. $\rho(\kappa u)=0$ for every ρ r. mod K, etc. It follows then that u, u', . . . belong to the kernels of κ, κ', . . . , i.e. to the 0-classes in M_K, $M_{K'}$, . . . , respectively. This implies however that the element (9) is 0, showing that φ is monomorphic.

To prove further that φ is epimorphic, consider any element of $(N\otimes_Z X_n)^G$, which we can express in a form

$$\sum_{\rho\ \mathrm{r.\,mod}\ K} v^{(\rho)}\otimes\rho s+\sum_{\rho'\ \mathrm{r.\,mod}\ K'} v'^{(\rho')}\otimes\rho' s'+\cdots \tag{12}$$

$(v^{(\rho)}, v'^{(\rho')}, \ldots \in N)$; we assume that the unit element 1 appears in each of the representative systems $\{\rho\}$, $\{\rho'\}$, ... From its invariance by the elements of K we deduce that $v^{(1)}$ belongs to N^K. Its invariance by ρ implies $v^{(\rho)}=\rho v^{(1)}$. Similarly we have $v'^{(1)}\in N^{K'}$, $v'^{(\rho')}=\rho' v'^{(1)}$, etc. The elements $v^{(1)}$, $v'^{(1)}$, . . . of N^K, $N^{K'}$, . . . may be expressed as κu, $\kappa' u'$, . . . with u, u', . . . $\in M$. So the element (12) assumes a form (11) and is contained in the image of $M\otimes_G X_n$ by φ. This proves that φ is an isomorphism.

In case $n<0$ we consider $\{\sigma_0 H, \ldots, \sigma_m H\}$ $(m=-n-1)$ in place of $(\sigma_0 H, \ldots)$ and obtain similarly an isomorphism φ: $M\otimes_G X_n \approx (N\otimes X_n)^G$.

Before proceeding further, we observe that in deriving (11) from (9), to define φ, we need not assume that s, s', . . . in (9) are the specific $(n+1)$-tuples in (8). Thus we consider (9) to be an arbitrary expression of a given element of $M\otimes_Z X_n$, in which s, s', . . . are allowed to be any $(n+1)$-tuples of right cosets of H in G, and show that the element (11) is determined uniquely

[2] We shall soon observe that the element (9) is independent of the choice of s, s', ... Indeed it is determined by the element (9) itself, irrespective of its special form.

by the given element (9) itself, independent of its particular form (9). Indeed, since any other similar expression for the same element (9) is devived from (9) by addition of differences like $u\otimes_G s-\sigma u\otimes_G \sigma s$ (up to trivial rules like cancelling equal terms with opposite signatures), we have only to prove the equality $\sum_{\rho\ \mathrm{r.mod}\ K}\rho(\kappa u)\otimes\rho s=\sum_{\rho_1\ \mathrm{r.mod}\ K_1}\rho_1(\kappa_1\sigma u)\otimes\rho_1\sigma s$, where $K=\sigma_0H\sigma_0^{-1}\cap\cdots\cap\rho_nH\rho_n^{-1}$ $(s=(\sigma_0H,\ldots,\sigma_nH))$, $K_1=\sigma H\sigma^{-1}$, $\kappa=\kappa(K)$, $\kappa_1=\kappa(K)$. But this is certainly the case since $\kappa_1=\kappa(\sigma K\sigma^{-1})=\sigma\kappa(K)\sigma^{-1}$, as was observed before, and $\rho_1\sigma$ runs over a representative system of right cosets modulo K, in G, when ρ_1 runs over a such of right cosets modulo $K_1=\sigma K\sigma^{-1}$. A similar remark holds also in case $n<0$.

Now we consider the diagram

$$(13)\qquad \begin{array}{ccccc} M\otimes_G X_n & \longrightarrow & (N\otimes_Z X_n)^G & \approx & \mathrm{Hom}_G(X_{-n-1},\ N) \\ \partial\downarrow & & & & \downarrow\delta \\ M\otimes_G X_{n-1} & \longrightarrow & (N\otimes_Z X_{n-1})^G & \approx & \mathrm{Hom}_G(X_{-n},\ N) \end{array}$$

where the horizontal arrows are the isomorphisms φ defined above. We contend that the diagram is commutative. First consider the case $n\geqq 1$, and consider an element $u\otimes_G s$ of $M\otimes_G X$ $(s=(\sigma_0H,\ldots,\sigma_nH))$. We have

$$\partial(u\otimes_G s)=\sum_{i=0}^{n}(-1)^i u\otimes_G(\sigma_0H,\ldots,\sigma_{i-1}H,\sigma_{i+1}H,\ldots,\sigma_nH)\in M\otimes_G X_{n-1}.$$

By our above remark about φ, we have

$$\varphi(\partial(u\otimes_G s))=\sum_{i=0}^{n}(-1)^i\sum_{\rho\ \mathrm{r.mod}\ K_i}\rho(\kappa_i u)\otimes\rho(\sigma_0H,\ldots,\sigma_{i-1}H,\sigma_{i+1}H,\ldots,\sigma_nH)$$

where $K_i=K(\sigma_0,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n)=\sigma_0H\sigma_0^{-1}\cap\ldots\cap\sigma_{i-1}H\sigma_{i-1}^{-1}\cap\sigma_{i+1}H\sigma_{i+1}^{-1}$ $\cap\cdots\cap\sigma_nH\sigma_n^{-1}$ and $\kappa_i=\kappa(K_i)$. This element $\varphi(\partial(u\otimes_G s))$ of $(N\otimes_Z X_{n-1})^G$ is associated by ν, (7), with the element of $\mathrm{Hom}_G(X_{-n},N)$ mapping $\{\tau_1H,\ldots,\tau_nH\}\in X_{-n}=\mathrm{Hom}_Z(X_{n-1},Z)$ to

$$(14)\qquad \sum_{i=0}^{n}(-1)^i\sum_{\rho\ \mathrm{r.mod}\ K_i}\delta_{(\tau_1H,\ldots,\tau_nH),(\rho\sigma_0H,\ldots,\rho\sigma_{i-1}H,\rho\sigma_{i+1}H,\ldots,\rho\sigma_nH)}\rho(\kappa_i u)\in N$$

(Kronecker δ's). On the other hand, we have

$$\varphi(u\otimes_G s)=\sum_{\rho\ \mathrm{r.mod}\ K}\rho(\kappa u)\otimes\rho(\sigma_0H,\sigma_1H,\ldots,\sigma_nH),$$

which corresponds by ν to the element of $\mathrm{Hom}_G(X_{-n-1},N)$ mapping $\{\tau_0H,\tau_1H,\ldots,\tau_nH\}\in X_{-n-1}=\mathrm{Hom}_Z(X_n,Z)$ to $\sum_{\rho\ \mathrm{r.mod}\ K}\delta_{(\tau_0H,\ldots,\tau_nH),(\rho\sigma_0H,\ldots,\rho\sigma_nH)}\rho(\kappa u)\in N$.

Its image by the coboundary operation δ is the element of $\mathrm{Hom}_G(X_{-n}, N)$ which maps $\{\tau_1 H, \ldots, \tau_n H\} \in X_{-n} = \mathrm{Hom}_Z(X_{n-1}, Z)$ to

$$(15) \quad \sum_{\rho \text{ r. mod } K} \sum_{i=0}^{n} (-1)^i \sum_{\sigma \text{ r. mod } H} \delta_{(\tau_1 H, \ldots, \tau_i H, \sigma H, \tau_{i+1} H, \ldots, \tau_n H), (\rho\sigma_0 H, \ldots, \rho\sigma_n H)} \rho(\kappa u) \in N,$$

since $\partial\{\tau_1 H, \ldots, \tau_n H\} = \sum_{i=0}^{n} (-1)^i \sum_{\rho \text{ r. mod } H} \{\tau_1 H, \ldots, \tau_i H, \sigma H, \tau_{i+1} H, \ldots, \tau_n H\}$. In the summation $\sum_{\sigma \text{ r. mod } H}$ for fixed ρ, i in (15) the only effective term is the one with $\sigma H = \rho\sigma_i H$, and (15) may be rewritten as

$$(16) \quad \sum_{\rho \text{ r. mod } K} \sum_{i=0}^{n} (-1)^i \delta_{(\tau_1 H, \ldots, \tau_n H), (\rho\sigma_0 H, \ldots, \rho\sigma_{i-1} H, \rho\sigma_{i+1} H, \ldots, \rho\sigma_n H)} \rho(\kappa u).$$

Now, for each i we have $K_i \supset K$ and

$$\sum_{K_i \ni \rho \text{ r. mod } K} \rho(\kappa u) = \kappa_i u$$

by (2). Thus (14) is in fact equal to (16) whence to (15), which shows that the diagram (13) is commutative.

Next consider the case $n < 0$. Set $m = -n \geqq 1$. With $s = \{\sigma_1 H, \ldots, \sigma_m H\}$ we have

$$\partial(u \otimes_G s) = \sum_{i=0}^{m} (-1)^i u \otimes_G \sum_{\sigma \text{ r. mod } H} \{\sigma_1 H, \ldots, \sigma_i H, \sigma H, \sigma_{i+1} H, \ldots, \sigma_m H\},$$

$$\varphi(\partial(u \otimes_G s)) = \sum_{i=0}^{m} (-1)^i \sum_{\sigma \text{ r. mod } H} \sum_{\rho \text{ r. mod } K_\sigma} \rho(\kappa_\sigma u) \otimes \rho\{\sigma_1 H, \ldots, \sigma_i H, \sigma H, \sigma_{i+1} H, \ldots, \sigma_m H\}$$

where $K_\sigma = \sigma H \sigma^{-1} \cap \sigma_1 H \sigma_1^{-1} \cap \cdots \cap \sigma_m H \sigma_m^{-1}$ and $\kappa_\sigma = \kappa(K_\sigma)$. Thus $\varphi(\partial(u \otimes_G s))$ is associated, by ν, with the element of $\mathrm{Hom}_G(X_m, N)$ mapping $(\tau_0 H, \tau_1 H, \ldots, \tau_m H)$ to

$$(17) \quad \sum_{i=0}^{m} (-1)^i \sum_{\sigma \text{ r. mod } H} \sum_{\rho \text{ r. mod } K_\sigma} \delta_{(\tau_0 H, \ldots, \tau_m H), (\rho\sigma_1 H, \ldots, \rho\sigma H, \rho\sigma_{i+1} H, \ldots, \rho\sigma_m H)} \rho(\kappa_\sigma u) \in N.$$

In the summation $\sum_{\rho \text{ r. mod } K_\sigma}$ for fixed i, σ in (17) the only effective terms are the ones with $\rho\sigma H = \tau_i H$. Hence (17) may be rewritten as

$$(18) \quad \sum_{i=0}^{m} (-1)^i \sum_{\sigma \text{ r. mod } H} \sum_{\tau_i H \sigma^{-1} \ni \rho \text{ r. mod } K_\sigma} \delta_{(\tau_0 H, \ldots, \tau_{i-1} H, \tau_{i+1} H, \ldots, \tau_m H), (\rho\sigma_1 H, \ldots, \rho\sigma_m H)} \rho(\kappa_\sigma u)$$

$$= \sum_{i=0}^{m} \sum_{\rho \text{ r. mod } K_\sigma} \delta_{(\tau_0 H, \ldots, \tau_{i-1} H, \tau_{i+1} H, \ldots, \tau_m H), (\rho\sigma_1 H, \ldots, \rho\sigma_m H)} \rho(\kappa_\sigma u).$$

On the other hand, we have, with $K = \sigma_1 H \sigma_1^{-1} \cap \cdots \cap \sigma_m H \sigma_m^{-1}$,

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$$\varphi(u\otimes_G s)=\sum_{\rho\ \mathrm{r.\,mod}\ K}\rho(\kappa u)\otimes\rho\{\sigma_1 H,\ \ldots,\ \sigma_m H\}$$

and this corresponds by ν to the element of $\mathrm{Hom}_G(X_{m-1}, N)$ mapping $(\tau_1 H, \ldots, \tau_m H)$ to

$$\sum_{\rho\ \mathrm{r.\,mod}\ K}\delta_{(\tau_1 H,\ldots,\tau_m H),(\rho\sigma_1 H,\ldots,\rho\sigma_m H)}\rho(\kappa u)\in N.$$

Hence $\delta(\nu(\varphi(u\otimes_G s)))$ is the element of $\mathrm{Hom}_G(X_m, N)$ mapping $(\tau_0 H, \tau_1 H, \ldots, \tau_m H)$ to

$$\sum_{i=0}^{m}(-1)^i\sum_{\rho\ \mathrm{r.\,mod}\ K}\delta_{(\tau_0 H,\ldots,\tau_{i-1}H,\tau_{i+1}H,\ldots,\tau_m H),(\rho\sigma_1 H,\ldots,\rho\sigma_m H)}\rho(\kappa u). \tag{19}$$

But, since $\kappa u=\sum_{K\ni\rho\ \mathrm{r.\,mod}\ K_\sigma}\rho(\kappa_\sigma u)$, for each σ, the right-hand side of (18) coincides with (19). This proves the commutativity of (13) for $n<0$.

We finally consider the case $n=0$. For $s=(\sigma_0 H)$ we have

$$\partial(u\otimes_G s)=\partial(u\otimes_G(\sigma_0 H))=u\otimes_G\sum_{\sigma\ \mathrm{r.\,mod}\ H}\{\sigma_0\sigma H\}=u\otimes_G\sum_{\sigma\ \mathrm{r.\,mod}\ H}\{\sigma H\}$$
$$=\sum_{\sigma\ \mathrm{r.\,mod}\ H}\sigma^{-1}u\otimes_G\{H\},$$
$$\varphi(\partial(u\otimes_G s))=\sum_{\sigma\ \mathrm{r.\,mod}\ H}\ \sum_{\rho\ \mathrm{r.\,mod}\ H}\rho(\kappa\sigma^{-1}u)\otimes\rho\{H\}$$

with $\kappa=\kappa(H)$. Hence $\nu\varphi(\partial(u\otimes_G s))$ is the element of $\mathrm{Hom}_G(X_0, N)$ which maps $(H)\in X_0$ onto

$$\sum_{\sigma\ \mathrm{r.\,mod}\ H}\ \sum_{\rho\ \mathrm{r.\,mod}\ H}\{\rho H\}(H)\rho(\kappa\sigma^{-1}u)=\sum_{\sigma\ \mathrm{r.\,mod}\ H}\kappa\sigma^{-1}u. \tag{20}$$

On the other hand, we have

$$\varphi(u\otimes_G s)=\varphi(u\otimes_G(\sigma_0 H))=\varphi(\sigma_0^{-1}u\otimes_G(H))=\sum_{\rho\ \mathrm{r.\,mod}\ H}\rho(\kappa\sigma_0^{-1}u)\otimes\rho(H),$$

and $\nu\varphi(u\otimes_G s)$ is the element of $\mathrm{Hom}_G(X_{-1}, N)$ mapping $\{\tau H\}\in X_{-1}$ onto $\sum_{\rho\ \mathrm{r.\,mod}\ H}\{\tau H\}(\rho H)\rho(\kappa\sigma_0^{-1}u)\tau(\kappa\sigma_0^{-1}u)$. Hence $\delta\nu\varphi(u\otimes_G s)$ is the element of $\mathrm{Hom}_G(X_0, N)$ mapping (H) onto

$$\sum_{\tau\ \mathrm{r.\,mod}\ H}\tau(\kappa\sigma_0^{-1}u). \tag{21}$$

Here, since $\kappa=\sum_{H\ni\rho\ \mathrm{mod}\ K_0}\rho\kappa_0$ with $\kappa_0=\kappa(K)$, K_0 being the intersection of all conjugates of H in G, the sum (21) is equal to $\sum_{\sigma\ \mathrm{r.\,mod}\ K_0}\sigma\kappa_0\sigma_0^{-1}u$, and this is in turn equal to $\sum_{\sigma\ \mathrm{r.\,mod}\ K_0}\sigma\sigma_0^{-1}\kappa_0 u=\sum_{\sigma\ \mathrm{r.\,mod}\ K_0}\sigma\kappa_0 u$, as κ_0 is a G-homomorphism. Similarly the right-hand side of (20) is equal to $\sum_{\sigma\ \mathrm{r.\,mod}\ H}\ \sum_{H\ni\rho\ \mathrm{mod}\ K_0}\kappa_0\rho\sigma u^{-1}=\sum_{\sigma\ \mathrm{mod}\ K_0}\kappa_0\sigma u=\sum_{\sigma\ \mathrm{mod}\ K_0}\sigma\kappa_0 u$:

Thus (20) and (21) are equal and this proves the commutativity of (13) for $n=0$.

The isomorphism of φ and the commutativity of (3) shows that the complexes $\sum M\otimes_G X_n$, $\sum \mathrm{Hom}_G(X_{-n-1}, N)$ with differentiations ∂, δ are isomorphic, as was asserted.

Remark. If H is (not only of finite index in G but) of finite order and if $H^{-1}(K, M)=H^0(K, M)=0$ for every subgroup K of H which is an intersection of conjugates of H in K, then the condition (A) (or equivalently (A')) is satisfied with $N=M$. For, we have then in particular $H^{-1}(K_0, M)=H^0(K_0, M)=0$, which implies that the G-endomorphism κ_0 of M defined by $\kappa_0 u=\sum_{\rho\in K_0}\rho u$ induces an isomorphism of M_{K_0} onto M^{K_0}, and furthermore $\kappa(K)$ defined by (1) with our κ_0, just defined, induces an isomorphism of M_K onto M^K, because of $H^{-1}(K, M)=H^0(K, M)=0$, for each K.

The application alluded to in the foot-note 1) will be under this stronger assumption.

References

[1] I. T. Adamson, Cohomology theory for non-normal subgroups and non-normal fields, Proc. Glasgow Math. Assoc. **2** (1954-6), 67-76.

[2] G. Hochschild, Relative homological algebra, Trans. Amer. Math. Soc. **82** (1956), 246-269.

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