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APPROXIMATION OF FUNCTIONS BY A BERNSTEIN-TYPE OPERATOR

BY
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1. **Introduction.** Various generalizations of the Bernstein operator, defined on $C[0, 1]$ by the relation

$$(1.1) \qquad B_n(f; x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$b_{n,k}(f; x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, 1, 2, \ldots, n$$

have been given. Note that $b_{n,k}(x)$ is the well-known binomial distribution.

The Bernstein operator as well as its various generalizations have been obtained by starting from the identity

$$(1.2) \qquad \sum_{k=0}^{n} b_{n,k}(x) = 1.$$

Recently Stancu [5] has considered a Bernstein-type operator $P_n^{(\alpha)}(f; x)$ defined as

$$(1.3) \qquad P_n^{(\alpha)}(f; x) = \sum_{k=0}^{n} p_{n,k}(x, \alpha) f\left(\frac{k}{n}\right)$$

based on the Polya distribution

$$p_{n,k}(x, \alpha) = \binom{n}{k} \prod_{\nu=0}^{k-1} (x+\nu\alpha) \prod_{\mu=0}^{n-k-1} (1-x+\mu\alpha) \Big/ \prod_{\lambda=0}^{n-1} (1+\lambda\alpha)$$

with

$$\sum_{k=0}^{n} p_{n,k}(x, \alpha) = 1.$$

As another generalization of (1.1) Cheney and Sharma [1] have defined an operator based on the following identity

$$(1.4) \qquad 1 = (1+n\alpha)^{-n} \sum_{k=0}^{n} \binom{n}{k} x(x+k\alpha)^{k-1}[1-x+(n-k)\alpha]^{n-k}.$$

The purpose of this paper is to consider a Bernstein-type operator as obtained by using the averaging method on the Meyer-König-Zeller operator.

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The results corresponding to Meyer-König-Zeller operator can easily be obtained from our operator as particular cases.

2. **The operator.** We consider the operator defined by

$$M_n^{(\alpha)}(f; x) = \sum_{k=0}^{\infty} m_{n+1,k}(x, \alpha) f\left[\frac{k}{n+k}\right], \tag{2.1}$$

where

$$m_{n,k}(x, \alpha) = \frac{\binom{n+k-1}{k} \prod_{\nu=0}^{k-1} (x+\nu\alpha) \prod_{\mu=0}^{n-1} (1-x+\mu\alpha)}{\prod_{\lambda=0}^{n+k-1} (1+\lambda\alpha)}, \qquad k = 1, 2, \ldots$$

and

$$m_{n,0}(x, \alpha) = \frac{\prod_{\mu=0}^{n-1} (1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1} (1+\lambda\alpha)}. \tag{2.2}$$

The parameter α in (2.1) may depend only on the natural number n. For $\alpha=0$, (2.1) reduced to Meyer-König and Zeller operator [3] defined on $C[0, 1]$. It may be further noted that (2.2) is the inverse Polya distribution [4].

The operator (2.1) can be looked upon as the average of the Meyer-König-Zeller operator with suitable weights chosen. We prove this in the following lemma:

LEMMA 2.1. *Let α be a positive parameter which may depend on the natural number n. Then for $x \in (0, 1)$*

$$M_n^{(\alpha)}(f; x) = \int_0^1 t^{(x/\alpha)-1}(1-t)^{[(1-x)/\alpha]-1} M_n(f; t)\, dt, \tag{2.3}$$

where

$$M_n(f; x) = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k (1-x)^{n+1} f\left(\frac{k}{n+k}\right)$$

is the Meyer-König and Zeller operator [3].

Proof. Writing the expression (2.2) as

$$m_{n,k}(x; \alpha) = \binom{n+k-1}{k} \frac{B\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}, \tag{2.4}$$

where

$$m_{n,k}(0; \alpha) = \begin{bmatrix} 1, & \text{for } k = 0 \\ 0, & \text{for } k = 1, 2, \ldots \end{bmatrix}$$

$$m_{n,k}(1, \alpha) = 0 \quad \text{for } k = 0, 1, 2, \ldots \tag{2.5}$$

$$B(a, b) = \int_0^1 z^{a-1}(1-z)^{b-1}\, dz, \qquad (a, b > 0)$$

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and evaluating the right-hand expression of (2.3) in conjunction with (2.1) and (2.4) gives the assertion immediately.

Indeed one can generalize some other operators in this manner.

As another example of the averaging principle we may obtain an operator $S_n^{(\alpha)}(f; x)$ described by

$$S_n^{(\alpha)}(f; x) = \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)} \int_0^\infty e^{-t} t^{(x/\alpha)-1} S_n(f; \alpha t)\, dt,$$

where

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

is the well-known Mirakyan operator. It may also be noted that the operator (1.3) defined by Stancu [5] is a weighted average of the Bernstein operator (1.1).

3. **Convergence property.**

THEOREM 3.1. *If* $f \in C[0, 1]$ *and* $0<\alpha=\alpha(n)\to 0$ *as* $n\to\infty$, *then the sequence* $\{M_n^{(\alpha)}(f; x)\}$ *converges to* $f(x)$ *uniformly on* $[0, 1]$.

Proof. Since $M_n^{(\alpha)}(f; x)$ is a $+$ve linear operator, we can apply a well-known theorem of Korovkin [2]. By this theorem, it is sufficient to prove that $Lt_{n\to\infty} M_n^{(\alpha)}(t^r, x)=x^r$, $r=0, 1, 2$. By setting $f(t)=1$, this is obvious for $r=0$ from the fact that

$$\sum_{k=0}^{\infty} m_{n,k}(x, \alpha) = 1. \tag{3.1}$$

Putting $f(t)=t$ and using (2.4) and (2.5), (2.1) can be written as

$$\begin{aligned} M_n^{(\alpha)}(t; x) &= \int_0^1 \frac{\left\{\sum_{k=1}^{\infty} \binom{n+k}{k} \frac{k}{n+k} z^k [z^{(x/\alpha)-1}(1-z)^{[(1-x)/\alpha]+n}]\right\} dz}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \\ &= \int_0^1 \sum_{k=1}^{\infty} \binom{-n-1}{k-1} (-)^{k-1} \frac{z^{k-1}[z^{x/\alpha}(1-z)^{[(1-x)/\alpha]+n}]\, dz}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \\ &= \int_0^1 \frac{z^{x/\alpha}(1-z)^{[(1-x)/\alpha]-1}\, dz}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} = x; \end{aligned} \tag{3.2}$$

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where we have used

$$\binom{-n}{k}(-)^k = \binom{n+k-1}{k}.$$

Now putting $f(t)=t^2$, we have

$$M_n^{(\alpha)}(t^2; x) = \sum_{k=0}^{\infty} \binom{n+k}{k} \frac{k^2}{(n+k)^2} \frac{B\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n+1\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}$$

$$< \int_0^1 \sum_k \binom{n+k-2}{k-2} \frac{k}{k-1} \left[z^k \frac{z^{(x/\alpha)-1}(1-z)^{[(1-x)/\alpha]+n}}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \right] dz.$$

Now summing over k and using the values of beta functions the right-hand expression easily yields

$$\frac{x(x+\alpha)}{1+\alpha} + \frac{x(1-x)}{n(1+\alpha)}.$$

Similarly by obtaining a lower bound we can write

$$\frac{x(x+\alpha)}{1+\alpha} < M_n^{(\alpha)}(t^2, x) < \frac{x(x+\alpha)}{1+\alpha} + \frac{x(1-x)}{n(1+\alpha)}. \tag{3.3}$$

Combining (3.1), (3.2), and (3.3), the proof of the Theorem 3.1 is complete.

4. Order of approximation.

THEOREM 4.1. *If $f \in C[0, 1]$ and $\alpha > 0$, then*

$$|f(x) - M_n^{(\alpha)}(f, x)| < \frac{3}{2} w\left(\sqrt{\frac{1+\alpha n}{n+\alpha n}}\right), \tag{4.1}$$

where $w(\delta) = w(f; \delta) = \sup |f(x'') - f(x')|$; $x', x'' \in [0, 1] \ni |x'' - x'| < \delta$; δ *being a* +ve *number.*

Proof Note the following properties of the modulus of continuity:

$$|f(x'') - f(x')| \le w(|x'' - x'|); \tag{4.2}$$

$$w(\lambda\delta) \le (\lambda+1) w(\delta); \qquad (\lambda > 0). \tag{4.3}$$

Now since

$$\sum_{k=0}^{\infty} m_{n,k}(x, \alpha) = 1$$

and

$$m_{n,k}(x, \alpha) \ge 0, \qquad \forall n, k$$

we have

$$
\begin{aligned}
|f(x)-M_n^{(\alpha)}(f;x)| &\le \sum_{k=0}^{\infty} m_{n+1,k}(x,\alpha)\left|f(x)-f\left(\frac{k}{k+n}\right)\right| \\
&\le \sum_{k=0}^{\infty} m_{n+1,k}(x,\alpha)w\left(\left|x-\frac{k}{k+n}\right|\right) \\
&\le \sum_{k=0}^{\infty} m_{n+1,k}(x,\alpha)\left(1+\frac{1}{\delta}\left|x-\frac{k}{k+n}\right|\right)w(\delta) \\
&\le \left(1+\frac{1}{\delta}\sum_{k=0}^{\infty} m_{n+1,k}(x,\alpha)\left|x-\frac{k}{k+n}\right|\right)w(\delta).
\end{aligned}
\tag{4.4}
$$

By linearity of the operator and by using (3.1), (3.2), and (3.3) we have

$$
\begin{aligned}
\sum_{k=0}^{\infty} m_{n+1,k}(x,\alpha)\left[x-\frac{k}{k+n}\right]^2 &= x^2M_n^{(\alpha)}(1;x)-2xM_n^{(\alpha)}(t;x)+M_n^{(\alpha)}(t^2;x) \\
&< -x^2+\frac{x(1-x)}{n(1+\alpha)}+\frac{x(x+\alpha)}{(1+\alpha)} = \frac{1+\alpha n}{1+\alpha}\,\frac{x(1-x)}{n},
\end{aligned}
$$

so that

$$
\sum_{k=0}^{\infty} m_{n+1,k}(x;\alpha)\left(x-\frac{k}{k+n}\right)^2 < \frac{1+\alpha n}{4(n+n\alpha)}. \tag{4.5}
$$

Finally, Cauchy's inequality gives

$$
\sum_{k=0}^{\infty} m_{n+1,k}(x,\alpha)\left|x-\frac{k}{k+n}\right| \le \left[\sum_{k=0}^{\infty} m_{n+1,k}(x,\alpha)\left(x-\frac{k}{k+n}\right)^2\right]^{1/2},
$$

and the substitution of (4.5) in (4.4) gives

$$
|f(x)-M_n^{(\alpha)}(f;x)| < \left\{1+\frac{1}{2\delta}\left[\frac{1+\alpha n}{n+\alpha n}\right]^{1/2}\right\}w(\delta). \tag{4.6}
$$

Hence, choosing $\delta=\sqrt{(1+\alpha n)/(n+\alpha n)}$, Theorem (4.1) is proved.

Since the expression (4.1) for our operator is a strict inequality for all n while in the case of Stancu's operator the equality may also hold for some n, the approximation of functions given by our operator for some n is better than that given by Stancu's operator.

Remark. The degree of approximation for our operator $M_n^{(\alpha)}(f;x)$ cannot be improved further. For the value of the constant C for which

$$
|f(x)-M_n^{(\alpha)}(f;x)| < Cw\left[\left(\frac{1+\alpha n}{n+\alpha n}\right)^{1/2}\right] \tag{4.7}
$$

is always less than $\frac{3}{2}$. Moreover $C\ge 1$, which can be seen from the following example. Let $\delta_n=0\sqrt{(1+\alpha n)/(n+\alpha n)}$ and suppose that $f_n(x)$ is the function which is

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equal to zero at x_0, $0<x_0<1$, equal to 1 in $[0, x_0-\delta_n]$ and $[x_0+\delta_n, 1]$ and linear in the rest of [0, 1]. For large n, we have $w(\delta_n)=1$ for f_n; also

$$(4.8)\quad |f_n(x_0)-M_n^{(\alpha)}(f; x)| = M_n^{(\alpha)}(f; x_0) \geq \sum_{|k/(k+n)-x_0|\geqslant\delta_n} m_{n,k}(x_0; \alpha) = 1-\varepsilon_n.$$

Therefore (4.7) cannot be true if $C<1$.

The function $w(\delta)$ cannot, therefore, be replaced in (4.6) by any other function decreasing to zero more rapidly.

Further, by using (3.1), (3.2), and (3.3) and following Stancu [5], the following theorem can easily be proved.

THEOREM 4.2. *If $f\in C^1[0, 1]$, then the inequality*

$$|f(x)-M_n^{(\alpha)}(f; x)| < \frac{3}{4}\left(\frac{1+\alpha n}{n+\alpha n}\right)^{1/2} w_1\left(\sqrt{\frac{1+\alpha n}{n+\alpha n}}\right)$$

holds, where $w_1(\delta)$ is the modulus of continuity of f'.

5. **Application to summability.** As in the case of Bernstein polynomials the operator $M_n^{(\alpha)}(f; x)$ also suggests a summability method. For $0<r\leq 1$, $\alpha\geq 0$, let the matrix $A=(\alpha_{nk})$ be defined by

$$\alpha_{nk} = \binom{n+k}{k}\prod_{\nu=0}^{k-1}(r+\nu\alpha)\prod_{\mu=0}^{n}(1-r+\mu\alpha)\Big/\prod_{\lambda=0}^{n+k}(1+\lambda\alpha), \qquad k = 0, 1, 2, \ldots$$

We shall say that a sequence $\{S_k\}$ is A-summable to the value S if each of the series

$$\sigma_n = \sum_{k=0}^{\infty}\alpha_{nk}S_k, \qquad n = 0, 1, 2, \ldots$$

is convergent and $\sigma_n\to S$.

In order to prove that the summation method is regular, it is enough to show on using the theorem of Toeplitz that

(i) $\lim_{n\to\infty}\alpha_{nk} = 0$

(ii) $\lim_{n\to\infty}\sum_{k=0}^{\infty}\alpha_{nk} = 1$ and

(iii) $\sum_{k=0}^{\infty}|\alpha_{nk}| \leq M$,

where M is a constant.

Now the coefficients α_{nk} are such that $\alpha_{nk}\geq 0$ and $\sum_k \alpha_{nk}=1$ for any n implying conditions (ii) and (iii). Further it can easily be seen that

$$\alpha_{nk} = 0(n^k(1-r)^n) = o(1), \quad \text{as } n\to\infty.$$

This proves the regularity of summation method for the operator $M_n^{(\alpha)}(f; x)$.

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