POINT (COUNTABLE) PARACOMPACTNESS

J. M. BOYTE

1. Introduction

This paper introduces two new separation axioms, point paracompactness and point countable paracompactness, both somewhat weaker than regularity, and shows that they can replace regularity in several standard theorems about paracompact or absolutely H-closed or Lindelöf spaces. Thus we obtain sharpened versions of these theorems. We also show that under certain hypotheses the new properties are equivalent to regularity.

When we speak of a topological space we are not assuming that the space satisfies any of the separation axioms. and by a paracompact space X we mean a space X such that every open cover of X has a locally finite open refinement. We shall denote the closure of a set A by cl A, and we shall let $N(a)$, $V(a)$, etc. be open sets containing the point a. Similarly, if A is a set we shall let $N(A)$, $V(A)$, etc. denote open sets containing the set A.

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DEFINITION 1. A collection of sets $\{G_\alpha \,|\, \alpha \in A\}$ is *locally finite* with respect to a set S or point S if and only if there exists $V(S)$ such that $V(S) \cap G_\alpha \neq \varnothing$ for only finitely many $\alpha \in A$.

DEFINITION 2. A topological space X is said to be *point (countably) paracompact* if and only if every (countable) open cover of X has, for each $a \in X$, an open refinement which is locally finite with respect to a.

The concept point paracompactness is not to be confused with the usual pointwise paracompactness.

We remark that every paracompact space is point paracompact, and every countably paracompact space is point countable paracompact.

THEOREM 1. *A topological space X is point (countably) paracompact if and only if every (countable) open cover of X has, for each $a \in X$, a finite subset whose union contains the closure of some neighborhood of a.*

PROOF. We shall only give a proof for point paracompactness, since a similar proof holds for point countable paracompactness.

Suppose X is point paracompact. Let $C = \{G_\alpha \,|\, \alpha \in A\}$ be an open cover of X with a an arbitrary point of the space. Now there exists an open refinement, $\{V_\lambda \,|\, \lambda \in \Lambda_a\}$, of C for which there exists $N(a)$ such that $N(a) \cap V_\lambda \neq \varnothing$ for only finitely many $\lambda \in \Lambda_a$. It follows that there exists a finite subset, $\{G_{\alpha(k)}\}$, $k = 1, \cdots, n$, of C such that

$$\operatorname{cl} N(a) \subset \bigcup_{k=1}^{n} \{G_{\alpha(k)}\}$$

Conversely, suppose $\{G_\alpha \,|\, \alpha \in A\}$ is an open cover of X. If $a \in X$ then there exist $N(a)$, and a finite subset, $\{G_{\alpha(k)}\}$, $k = 1, \cdots, n$, for which

$$\operatorname{cl} N(a) \subset \bigcup_{k=1}^{n} \{G_{\alpha(k)}\} .$$

Let

$$C = \{(X - \operatorname{cl} N(a)) \cap G_\alpha \,|\, \alpha \in A\} \cup \{G_{\alpha(k)}\}, k = 1, \cdots, n .$$

Clearly C is an open refinement of $\{G_\alpha \,|\, \alpha \in A\}$ with $N(a)$ intersecting only finitely many elements of C. So X is point paracompact.

The following corollary follows immediately from Theorem 1.

COROLLARY 1. *Every regular space is point paracompact.*

THEOREM 2. *In a T_2 space point paracompactness is equivalent to regularity.*

PROOF. We have by Corollary 1 that every regular space is point paracompact.

Conversely, suppose X is a T_2 point paracompact space. Let F be a closed set, and suppose $x \in X - F$. Then for each $z \in F$ there exists $V(z)$ such that $x \notin \operatorname{cl} V(z)$. If

$$C = \{X - F\} \cup \{V(z) \,|\, z \in F\},$$

then C is an open cover of X. Because X is point paracompact, it follows from Theorem 1 that there exists $N(x)$ such that $\operatorname{cl} N(x) \subset (X - F)$. The desired result now follows.

The following example shows that we need T_2 in the above theorem.

EXAMPLE 1. If X is an infinite set with the cofinite topology, then X is T_1 and point paracompact but not regular.

EXAMPLE 2. If X is the space of ordinals $\geq \omega_1$, in which points have their usual order neighborhoods except that from the neighborhoods of ω_1 we omit

all limit ordinals, then X is a countably compact T_2 space which is not regular; hence X is point countably paracompact but not point paracompact.

THEOREM 3. *In a Lindelöf space, point countable paracompactness is equivalent to paracompactness.*

PROOF. It is clear that paracompactness implies point countable paracompactness.

Suppose X is a Lindelöf space which is point countably paracompact. Let $C_1 = \{G_\alpha \mid \alpha \in A\}$ be an open cover of X. There exists a countable subcover $C_2 = \{G_{\alpha k}\}$, $k = 1, 2, \cdots$. For each $x \in X$, there exists $N(x)$ and a finite subset D_x of C_2 such that

$$\text{cl } N(x) \subset M(x) = \cup D_x.$$

Let $C_3 = \{N(x) \mid x \in X\}$. Now C_3 is an open cover of X; so there exists a countable subcover $\{N(x_1), N(x_2), \cdots\}$ of C_3. Put $H_1 = M(x_1)$ and for each positive integer $n > 1$,

$$H_n = M(x_n) - \bigcup_{i=1}^{n-1} (\text{cl } N(x_i)).$$

Suppose $x \in X$ and $x \notin H_1$. Clearly $x \in M(x_n)$ for some $n \neq 1$. Let k be the least, positive integer such that $x \in \text{cl } N(x_k)$; then clearly $x \in H_k$. Therefore $C_4 = \{H_1, H_2, \cdots\}$ is an open cover of X. Put

$$C_5 = \{H_n \cap G_{\alpha(k)} \mid G_{\alpha(k)} \in D_{xn}, n = 1, 2, \cdots\}.$$

Then C_5 is open cover of X which is a refinement of C_1. To show that C_5 is locally finite, let $x \in X$. Then there exists a positive integer k such that $x \in N(x_k)$. If $n > k$, then $N(x_k) \cap H_n = \phi$. It now follows that $N(x_k)$ intersects only finitely many members of C_5. Thus C_5 is locally finite. So X is paracompact.

It is obvious that every point paracompact space is point countably paracompact. Therefore from Corollary 1 and Theorem 3 we have the following well known corollary [5].

COROLLARY 2. *Every regular Lindelöf space is paracompact.*

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The main theorem of this section is a generalization of one of Alexandrov's and Urysohn's theorems [1].

DEFINITION 3. A sequence $\{x_n\}$ *converges openly* to a point a if and only if, for each $N(a)$, there exists a positive integer k such that $j \geq k$ implies $x_j \in \text{cl} N(a)$.

We shall denote a sequence $\{x_n\}$ converging openly to a point a by $\{x_n\} \overset{0}{\to} a$.

Because the proof of Theorem 4 is trivial and the proof of Theorem 5 is not very difficult, we state the following two theorems without proofs.

THEOREM 4.   *If x is a regular space, then a sequence $\{x_n\}$ converges openly to a point a if and only if it converges to a.*

THEOREM 5.   *If X is a first countable space, then X is regular if and only if $\{x_n\} \xrightarrow{0} a$ implies $\{x_n\} \to a$.*

THEOREM 6.   *If X is a first countable T_2 topological space, then the following are equivalent.*

(a)   *X is point countably paracompact.*
(b)   *If F is a countable closed set and a $\in X - F$, then there exists $N(a)$ and $V(F)$ such that $N(a) \cap V(F) = \varnothing$.*
(c)   *X is regular.*
(d)   *X is point paracompact.*

PROOF.   ($a \Rightarrow b$). Let F be a countable closed set and suppose $a \in X - F$. Then for each $x \in F$ there exists $V(x)$ such that $a \notin \text{cl} \ V(x)$. If we put

$$C = \{X - F\} \cup \{V(x) \mid x \in F\}$$

then C is a countable open cover of X, and because X is point countable paracompact it follows from Theorem 1 that there exists $N(a)$ such that $\text{cl} \ N(a)$ is contained in $X - F$. The result follows since $N(a) \cap (X - \text{cl} \ N(a)) = \varnothing$.

($b \Rightarrow c$). If we assume X is not regular, then by Theorem 5 there exists a sequence $\{x_n\}$ converging openly to a point a such that $\{x_n\}$ does not converge to a. Clearly there exists a subsequence $\{x_{n(j)}\}$ of $\{x_n\}$ for which $\{x_{n(j)}\} \xrightarrow{0} a$ and

$$a \notin \text{cl} \left(\bigcup_{j=1}^{\infty} \{x_{n(j)}\} \right).$$

Because X is T_2, $\text{cl}(\bigcup_{j=1}^{\infty} \{x_{n(j)}\})$ is countable which implies there exist disjoint open sets $V(a)$ and $N(\text{cl}[\bigcup_{j=1}^{\infty} \{x_{n(j)}\}])$. This is a contradiction to $\{x_{n(j)}\} \xrightarrow{0} a$. Therefore X is regular.

($c \Rightarrow d$). c implies d is true by Corollary 1.
($d \Rightarrow a$).   Trivial.

The following corollary for countable compactness was proven by Alexandrov and Urysohn [1], and for countable paracompactness, was proven by Aull [2].

COROLLARY 3. *If X is a first countable T_2 space which is countably compact or more generally, countably paracompact, then X is regular.*

We also have the following as a corollary to Theorem 6.

COROLLARY 4. *If X is a first countable T_2 space, then X is regular if and only if given $a \in X$ and $M(a)$ such that $X - M(a)$ is countable then there exists $V(a)$ for which* $\mathrm{cl}(V(a) \subset M(a))$.

REMARK. We remark that the definition of open convergence extends straightforwardly to nets; the proof of Theorem 5 goes over to non-first-countable spaces; and the proof $b \Rightarrow c$ of Theorem 6 can be modified into an alternative proof of Theorem 2. But the proof actually given for Theorem 2 is of course simpler.

A topological space X is said to be *absolutely H-closed* [6] if and only if every open cover of X has a finite subset whose union is dense in X.

THEOREM 7. *A topological space X is compact if and only if it is point paracompact and absolutely H-closed.*

PROOF. Assume X is point paracompact and absolutely H-closed. Let $C = \{G_\alpha \,|\, \alpha \in A\}$ be an open cover of X, and let

$C_1 = \{N(x) \,|\, x \in X$ and $\mathrm{cl}\, N(x)$ is contained in the union of some finite subset of $C\}$.

By Theorem 1, C_1 is an open cover of X. Thus there exists a finite subset, $\{N(x)_i\}$ $i = 1, \cdots, n$, of C_1 such that

$$X = \mathrm{cl}\left(\bigcup_{i=1}^{n} \{N(x)_i\}\right).$$

We have $\mathrm{cl}(\bigcup_{i=1}^{n}\{N(x)_i\})$ contained in the union of finitely many elements of C. It follows from this that X is compact.

It is clear that compactness implies point paracompactness and absolutely H-closedness.

The following result, which is well known [6], follows from Corollary 1, Theorem 7, and the fact that every compact T_2 space is regular.

COROLLARY 5. A T_2 space X is compact if and only if it is regular and absolutely H-closed.

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We shall generalize in this section one of Michael's well known theorems [4] on paracompactness.

THEOREM 8. *If X is normal, then each open cover $\{G_\alpha \,|\, \alpha \in A\}$ of X has for each $\beta \in A$ and each closed set $F \subset G_\beta$ an open refinement which is locally finite with respect to F. Conversely, if X is T_2 and each open cover $\{G_\alpha \,|\, \alpha \in A\}$ of X has for each $\beta \in A$ and each closed set $F \subset G_\beta$, an open refinement which is locally finite with respect to F, then X is normal.*

PROOF. Suppose X is a normal space and $\{G_\alpha \mid \alpha \in A\}$ is an open cover of X. Let F be a closed set contained in G_β for an arbitrary $\beta \in A$. Since X is normal there exists $N(F)$ such that cl $N(F) \subset G_\beta$. Clearly a refinement satisfying the desired property is

$$C = (X - \text{cl }N(F)) \cap G_\alpha \mid \alpha \in A\} \cup \{G_\beta\}.$$

Conversely, let F be a closed set with $N(F)$ an open set containing F. We shall show that X is normal by showing there exists $O(F)$ such that cl $O(F) \subset N(F)$. By Theorem 2, X is regular from which it follows that

$$C = \{N(F)\} \cup \{X - \text{cl }M(F) \mid M(F) \text{ is an open set containing } F\}$$

is an open cover of X. There exists an open refinement which is locally finite with respect to F. Thus there exists a finite subset

$$C_1 = \{N(F)\} \cup \{X - \text{cl }M(F)_i \mid i = 1, \cdots, n\}$$

of C and $V(F)$ for which cl $V(F) \subset \cup C_1$. Put

$$O(F) = V(F) \cap \left[\bigcap_{i=1}^{n} (M(F)_i) \right].$$

Now cl $O(F) \subset N(F)$. So X is normal.

We observe that the condition of T_2 in the converse part of Theorem 8 cannot be replaced by T_1, since an infinite space X with the cofinite topology is not normal, but it satisfies the given condition if we only require T_1.

The following theorem is a slight generalization of Michael's Theorem [4] which states that a regular space is paracompact if and only if every open cover of X has a σ-locally finite open refinement.

THEOREM 9. *A topological space X is paracompact if and only if it is point paracompact and each open cover of X has a σ-locally finite open refinement.*

PROOF. It is clear that if X is paracompact then it is point paracompact and each open cover of X has a σ-locally finite open refinement.

Suppose X is point paracompact and that each cover of X has a σ-locally finite open refinement (which covers X). Let $C = \{U_\alpha \mid \alpha \in A\}$ be an open cover of X. For each $x \in X$ there exist $N(x)$ and a finite subset $\{U_{\alpha(x,k)} \mid k = 1, \cdots, n(x)\}$ of C such that

$$\text{cl }N(x) \subset \bigcup_{k=1}^{n(x)} \{U_{\alpha(x,k)}\}.$$

Put $C_1 = \{N(x) \mid x \in X\}$. Michael has shown [4] that if each open cover of X has a σ-locally finite open refinement, then each open cover of X has a locally

finite refinement which consists of sets not necessarily open or closed. Since C_1 is an open cover of X, there exists a locally finite refinement C_2 of C_1 where $\cup C_2 = X$, and $M \in C_2$ is not necessarily open. If we put $C_3 = \{\text{cl } M/M \in C_2\}$ then C_3 is also locally finite. For each cl $M \in C_3$, cl M is contained in a finite union of elements of C. There exists, for each $x \in X$, $V(x)$ such that $V(x)$ intersects only a finite number of elements of C_3. Now $C_4 = \{V(x) \mid x \in X\}$ is an open cover of X. Duplicating what we have done above, we can obtain a locally finite collection of closed sets, C_5, where $\cup C_5 = X$, and for each $F \in C_5$, F is contained in a finite union of elements of C_4. Thus for each $F \in C_5$, we have F intersecting only a finite number of elements of C_3. Because $F \in C_5$ implies F intersects only a finite number of elements of C_3, there exists an open locally finite covering C_6 of X such that, for each cl $M \in C_3$, there exists $V(\text{cl } M) \in C_6$ for which cl $M \subset V(\text{cl } M)$. See Theorem 1.5 in [3, p.162]. Because C_2 refines C_1, we choose for each $M \in C_2$ a finite subset $\{U_{\alpha(M,k)} \mid k = 1, 2, \cdots, n(M)\}$ whose union contains cl M. If

$$C_7 = \{V(\text{cl } M) \cap U_{\alpha(M,k)} \mid M \in C_2, k = 1, 2, \cdots, n(M)\},$$

then clearly C_7 is an open locally finite refinement of C which covers X. So X is paracompact.

The result of Michael [4] now follows as a corollary.

References

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Appalachian State University,
Boone, North Carolina, 28607
U.S.A.