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GROUPS WITH COMMUTING POWERS

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Abstract

A group G satisfies the second Engel condition $[X, Y, Y] = 1$ if and only if x commutes with x^y, for all $x, y \in G$. This paper considers the generalization of this condition to groups G such that, for fixed positive integers r and s, x^r commutes with $(x^s)^y$ for all $x, y \in G$. Various general bounds are proved for the structure of groups in the corresponding variety, defined by the law $[X^r, (X^s)^Y] = 1$.

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1. Introduction

Let G be a group. Suppose that in every conjugacy class of G, every two elements commute. Hence $[x, x^y] = 1$ for all $x, y \in G$. This clearly is equivalent so saying then every element $x \in G$ generates an Abelian normal subgroup of G, which in turn is equivalent to the second Engel Condition $[y, x, x] = 1$ for all $x, y \in G$. It is known that groups satisfying one of these equivalent conditions is nilpotent of class at most three.

The above notion has been generalized in various directions. For example, every two-Engel group for every integer n satisfies the following conditions discussed in [7] and [2]:

$$\begin{aligned} [x^n, y] &= [x, y]^n \quad (n\text{-Levi property}),\\ [x^n, y] &= [x, y^n] \quad (n\text{-Bell property}). \end{aligned}$$

In this paper, we consider another generalization of two-Engel groups. Clearly, if x and x^y commute, then so do all powers x^r and $(x^y)^s$ where r, s are integers.

For positive integers r, s, let

$$\mathcal{O}(r, s) = \{G \mid G \text{ satisfies the law } [X^r, (X^s)^Y] = 1\}$$

denote the class of all groups with this property. This class clearly is a variety defined by one single law.

In the second section, we present typical examples for groups in $\mathcal{O}(r, s)$ for various values of r and s, while the third one contains various general bounds for the structure of groups in $\mathcal{O}(r, s)$. In the final section, structural results on *finite* groups in $\mathcal{O}(r, s)$ are proved in case where the defining parameters r and s are 'small' in some sense.

We use standard notation throughout. All commutators are left-normed. In addition, by $o(x)$ we denote the order of the element x, and $G = [N]Q$ indicates that the group G is a split extension of a normal subgroup N of G by a complement Q. The class of all Abelian groups will be denoted by $\mathfrak{A}$, and the class of all *finite* nilpotent groups by $\mathcal{N}$.

2. Examples

Assume that the order of every element $x \in G$ either divides r or s or maybe both (where r and s are fixed positive integers). Then we have the disjunction $x^r = 1$ OR $x^s = 1$ for all elements $x \in G$. In particular, $G \in \mathcal{O}(r, s)$. This idea was used in [3] to determine a basis for the laws of of PSL(2, 5).

Clearly, $\mathcal{O}(r, s)$ contains all groups of exponent dividing r or s. Moreover, it contains all Abelian groups.

If x^r commutes with *all* conjugates of x^s, then conversely, *all* conjugates of x^r commute with x^s. Thus $\mathcal{O}(r, s) = \mathcal{O}(s, r)$ for all r, s.

EXAMPLE 1. $\mathcal{O}(1, 1)$ is the class of all groups with the law $[X, X^Y] = 1$, thus $\mathcal{O}(1, 1)$ is the class of all groups satisfying the second Engel condition.

If G is second Engel, then for all $x, y \in G$ we have that x^r commutes with $(x^s)^y$, and so $G \in \mathcal{O}(r, s)$. Hence $\mathcal{O}(r, s)$ contains all second Engel groups. Thus, the classes $\mathcal{O}(r, s)$ generalize the second Engel groups.

EXAMPLE 2. We have $\mathfrak{A}\mathfrak{A}_r \subseteq \mathcal{O}(r, s)$ and $\mathfrak{A}\mathfrak{A}_s \subseteq \mathcal{O}(r, s)$ for all r, s.

PROOF. Let $G \in \mathfrak{A}\mathfrak{A}_r$, and let N be an Abelian normal subgroup of G such that G/N is Abelian of exponent dividing r. For all $x, y \in G$,

$$[x^r, (x^s)^y] = [x^r, x^s[x^s, y]] = [x^r, [x^s, y]].$$

As $x^r \in N$ and $[x^s, y] \in G' \le N$ and N is Abelian, we get $[x^r, (x^s)^y] = 1$. This shows $G \in \mathcal{O}(r, s)$. The second claim follows in the same way. □

We now consider another typical example in which one can read off the exponents r and s from the group.

EXAMPLE 3. Let $G = [N]Q$ be a Frobenius group, and set $r = \exp(N)$ and $s = \exp(Q)$. As every element of $G \setminus N$ is contained in some conjugate of Q, we see that $G \in \mathcal{O}(r, s)$. If N is Abelian, then $G \in \mathcal{O}(1, s)$. Indeed, if $x \in N$, then $x^y \in N$, and so $[x, x^y] = 1$. If $x \in G \setminus N$, then $x^s = 1$.

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3. Some general bounds

We first collect some absolutely basic properties of groups in $\mathcal{O}(r, s)$.

LEMMA 4. *Let G be a group in $\mathcal{O}(r, s)$. Then we have the following results.*

(a) *For every element $x \in G$, we have $x^r \in C_G(\langle x^s \rangle^G)$ and $x^s \in C_G(\langle x^r \rangle^G)$.*
(b) *For all $x, y \in G$, we have $[y, x^s, x^r] = 1$.*
(c) *For all positive integers λ, μ, we have $G \in \mathcal{O}(\lambda r, \mu s)$.*
(d) *Let $n = \operatorname{lcm}(r, s)$. For all $x, y \in G$, we have $[y, x^n, x^n] = 1$ and $[x^n, (x^n)^y] = 1$.*
(e) *If $o(x)$ is finite and coprime to rs, then $[x, x^g] = 1$ and $[g, x, x] = 1$ for all elements $g \in G$.*
(f) *Every torsion $(rs)'$-subgroup of G is a second Engel group, hence nilpotent of class less than or equal to three.*
(g) *Assume that G has a unique minimal normal subgroup N. If $x^r \neq 1$ for some element $x \in G$, then $x^s \in C_G(N)$.*

PROOF. Part (a) is clear. For (b), note that we have

$$[y, x^s, x^r] = [(x^{-s})^y x^s, x^r] = [(x^{-s})^y, x^r]^{x^s} = 1.$$

For (c), let $r' = \lambda r$ and $s' = \mu s$. As $G \in \mathcal{O}(r, s)$, for all $x, y \in G$ the elements $a = x^r$ and $b = (x^s)^y$ commute. Clearly, also the powers $a^\lambda = x^{r'}$ and $b^\mu = (x^{s'})^y$ of a and b commute. This shows $G \in \mathcal{O}(r', s')$.

For (d), write $n = \lambda r$ and $n = \mu s$ for some positive integers λ, μ. By (c), we have $G \in \mathcal{O}(n, n)$, and this by (b) implies $[y, x^n, x^n] = 1$ for all $x, y \in G$.

For (e), we can find integers λ, μ such that $r\lambda \equiv s\mu \equiv 1 \bmod (o(x))$. As x^r and $(x^g)^s$ commute, so do their powers $(x^r)^\lambda = x$ and $((x^g)^s)^\mu = x^g$. Part (f) is obvious from (e).

For (g), let $R = \langle (x^r)^G \rangle$. As $x^r \neq 1$, we have $R \neq 1$, and so, from the hypothesis, we get $N \leq R$. By (a), we have $x^s \in C_G(R) \leq C_G(N)$. □

As we have seen in Example 2, the class $\mathcal{O}(2, 3)$ contains $\mathfrak{A}\mathfrak{A}_2$ and $\mathfrak{A}\mathfrak{A}_3$. This, in some sense, is worst possible because of the following result.

THEOREM 5. *Let G be a group.*

(a) *Let $G \in \mathcal{O}(r, s)$, and set $n = \operatorname{lcm}(r, s)$. Then $\exp(G/F(G))$ divides n, where $F(G)$ denotes the Hirsch–Plotkin radical of G.*
(b) *Let $r = p^e$ and $s = q^f$ be prime powers. If $G \in \mathcal{O}(r, s)$ is finite, then G is soluble, and $\exp\big(G/F(G)\big)$ divides rs.*

PROOF. For (a), let $x \in G$. By Lemma 4(d), every two conjugates of x^n in G commute. Hence x^n generates an Abelian normal subgroup of G, thus $x^n \in F(G)$. Thus, $G/F(G)$ is of exponent dividing n. For (b), note that $G/F(G)$ by (a) is of exponent dividing $rs = p^e q^f$. Hence G is soluble. □

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COROLLARY 6. *Every group in $\mathcal{O}(1, 2)$ is locally supersoluble.*

PROOF. Let $G \in \mathcal{O}(1, 2)$. By Theorem 5, G^2 is locally nilpotent. Let H be a finitely generated subgroup of G. Then H/H^2 is finite, so that H^2 is finitely generated hence nilpotent. This implies that H is supersoluble. □

In view of Corollary 6 one might ask whether every group in $\mathcal{O}(1, 2)$ is even metabelian. This, however, is not the case as we shall now show the following result.

EXAMPLE 7. Let p be an odd prime, and let $N = \langle a, b\rangle$ be the relatively free two-generator group of exponent p and class two. Then $|N| = p^3$, and N has an automorphism z with $a^z = a^{-1}$ and $b^z = b^{-1}$. Note that z is of order two, and z centralizes $c := [a, b]$. Let $G = [N]\langle z\rangle$ be the canonic split extension. Then $G' = N$ is nonabelian, so that G is not metabelian.

Note that z inverts $N/\langle c\rangle$, so that every subgroup of N containing c is normal in G. We claim $G \in \mathcal{O}(1, 2)$. For this, let $x, y \in G$. If $x \in N$, then the above comment shows $\langle x^G\rangle \le \langle x, c\rangle$, so that $\langle x^G\rangle$ is Abelian. Hence $[x, (x^2)^y] = 1$. Next, assume $x \in G \setminus N$. Then $x = nz$ for some $n \in N$. As z inverts $N/\langle c\rangle$, this yields $x^2 \in \langle c\rangle$. As $\langle c\rangle = Z(G)$, we arrive at $[x, (x^2)^y] = 1$ as claimed.

QUESTION Is every 2-group in $\mathcal{O}(1, 2)$ metabelian? Is every group in $\mathcal{O}(1, 2)$ centre-by-metabelian?

COROLLARY 8. *All finite groups in $\mathcal{O}(r, s)$ are supersoluble if and only if $r \le 2$ and $s \le 2$.*

PROOF. Let $G \in \mathcal{O}(r, s)$. If $r \le 2$ and $s \le 2$, then by Theorem 5(a), we have that $G/F(G)$ is of exponent two, and hence G is supersoluble. For the converse, assume $r \ge 3$. By Example 2, we have $\mathfrak{A}\mathfrak{A}_r \subseteq \mathcal{O}(r, s)$. As $\mathfrak{A}\mathfrak{A}_r$ contains finite groups which are nonsupersoluble (for example the wreath product $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}_r$ for a prime p not dividing r with $p \not\equiv 1 \bmod r$), we have a contradiction. The result follows. □

COROLLARY 9. *All groups (finite or infinite) in $\mathcal{O}(r, s)$ are nilpotent if and only if $r = s = 1$.*

PROOF. Clearly, every group in $\mathcal{O}(1, 1)$ is nilpotent of class ≤ 3. For the converse, Corollary 8 implies that we have $r \le 2$ and $s \le 2$. It suffices to note that $S_3 \in \mathcal{O}(1, 2) = \mathcal{O}(2, 1)$ and $S_3 \in \mathcal{O}(2, 2)$. □

In the situation of Theorem 5(a), we conjecture that G^n is even a second Engel group. We can prove this in a special case.

COROLLARY 10. *Let $G \in \mathcal{O}(r, s)$, and set $n = \operatorname{lcm}(r, s)$. If G is torsion free, then G^n is nilpotent of class two.*

PROOF. Let $x, y, z \in G^n$, and let $H = \langle x, y, z\rangle$. By Theorem 5(a), we know that H is nilpotent Let p be a prime with $p \nmid 3n$. Then H is residually a finite p-group. By

Lemma 4(f), every such finite quotient is second-Engel, hence of class two (here we use $p \neq 3$). This shows $[x, y, z] = 1$. Hence G^n is of class two. □

Note that the derived length of groups in $\mathcal{O}(r, s)$, in general is not bounded by some function of r and s. For example, every group of exponent four is in $\mathcal{O}(1, 4)$, and by [8] there are such groups of arbitrary derived length.

COROLLARY 11. *There exists a function f with the following property. If $G \in \mathcal{O}(r, s)$ is finite and soluble, then the Fitting length of G is less than or equal to $f(r, s)$.*

PROOF. By Theorem 5(a), it suffices to show that the Fitting length of a finite soluble group H of exponent rs is bounded. Let K be a two-generator subgroup of H. By the solution of the restricted Burnside problem (see [9, 10]), K is of bounded order, thus of bounded Fitting length less than or equal to $f_0(r, s)$, say. By [4], also H is of Fitting length less than or equal to $f_0(r, s)$. □

The following shows that simple groups $G \in \mathcal{O}(r, s)$ have bounded structure. In particular, the element orders of G can be recovered from r and s.

COROLLARY 12 (See [3]). *Let $G \in \mathcal{O}(r, s)$ be a nonabelian simple group (possibly infinite). Then every element of G has finite order dividing r or s. In particular, the exponent of G is finite.*

PROOF. Suppose the corollary is false. Then there exists $x \in G$ with $x^r \neq 1$ and $x^s \neq 1$. Let $N = \langle (x^s)^G \rangle$. By Lemma 4(a), we have $x^r \in C_G(N)$. As $x^s \neq 1$, we have $N \neq 1$. As N is normal in G and G is simple, we get $N = G$, and so $1 \neq x^r \in C_G(N) = Z(G)$. Hence we have a contradiction. □

COROLLARY 13. *For every pair r, s of positive integers, the class $\mathcal{O}(r, s)$ contains only finitely many nonabelian finite simple groups.*

PROOF. By Corollary 12, the exponent of a simple group in $\mathcal{O}(r, s)$ is bounded by rs. It is well known from the classification, that there are only finitely many nonabelian finite simple groups of exponent dividing rs. □

4. Small values for r and s

We now consider finite groups in $\mathcal{O}(r, s)$ where r and s are particularly simple. For this, we need to consider some very specific groups that occur naturally as minimal counterexamples.

LEMMA 14 (See [1]). *Let $\mathcal{K}$ be a subgroup-closed formation of finite soluble groups, and let G be a finite soluble group all of whose proper subgroups and quotients are in $\mathcal{N}\mathcal{K}$, but $G \notin \mathcal{N}\mathcal{K}$. Then $G = [N]Q$ where $N = C_G(N) = F(G)$ is the unique minimal normal subgroup of G. Moreover, $Q \notin \mathcal{K}$, but every proper subgroup of Q belongs to $\mathcal{K}$.*

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The following result determines further properties of a group in $\mathcal{O}(r, s)$ which has the structure as in Lemma 14.

LEMMA 15. *Let $G = [N]Q$ be group where $N = C_G(N)$ is the unique minimal normal subgroup of G. Assume that $G \in \mathcal{O}(r, s)$. Then we have the following results.*

(a) *Let $x \in G$. If $x^r \neq 1$, then $x^s \in N$.*
(b) *For every $x \in Q$, we have $x^r = 1$ or $x^s = 1$.*
(c) *Assume that r and s are coprime. Then for all $x \in G \setminus N$, we have $x^r = 1$ or $x^s = 1$.*

PROOF. (a) By Lemma 4(g), we have $x^s \in C_G(N) = N$.

(b) Assume $x^r \neq 1$. By (a), we have $x^s \in Q \cap N = 1$.

(c) Let $x \in G \setminus N$. By way of contradiction, suppose $x^r \neq 1$ and $x^s \neq 1$. By Lemma 4(a), we have $x^s \in N$ and $x^r \in N$. By hypothesis, r and s are coprime, so there exist $\lambda, \mu \in \mathbb{Z}$ with $\lambda r + \mu s = 1$. We arrive at $x = x^{\lambda r + \mu s} = (x^r)^\lambda (x^s)^\mu \in N$, against the hypothesis $x \in G \setminus N$. $\square$

COROLLARY 16. *Under the hypothesis of Lemma 15, let $x \in Q$, $x \neq 1$. If $o(x)$ and $|N|$ are coprime then x acts fixed point freely on N, that is, $C_N(x) = 1$.*

COROLLARY 17. *Under the hypothesis of Lemma 15, assume that r and s are powers of two distinct primes. Then every element of G is of prime power order.*

We now discuss when one of the classes $\mathcal{O}(r, s)$ is contained in another. Note that a sufficient condition is contained in Lemma 4(c): for all r, s and all λ, μ, we have $\mathcal{O}(r, s) \subseteq \mathcal{O}(\lambda r, \mu s)$. We now consider whether the converse of this might be true.

COROLLARY 18. *Let r and s be positive integers. If $\mathcal{O}(r, s) \subseteq \mathcal{O}(r', s')$ for some positive integers r', s', then r divides r' or s', and s divides r' or s'.*

PROOF. The result is clear for $r = 1$, so let $r > 1$. Let $G = [N]Q$ be a Frobenius group where $Q = \langle x \rangle$ is cyclic of order r, and $|N| = p \equiv 1 \bmod r$ where p is a prime.

By Example 3, we have $G \in \mathcal{O}(r, 1) \subseteq \mathcal{O}(r, s)$, hence the hypothesis yields $G \in \mathcal{O}(r', s')$. From Lemma 15(b), we get $x^{r'} = 1$ or $x^{s'} = 1$. Hence $r = o(x)$ divides either r' or s' as claimed. The second assertion follows from an analogous argument. $\square$

QUESTION Suppose that $\mathcal{O}(r, s) \subseteq \mathcal{O}(r', s')$. Does it follow that $r|r', s|s'$ or $r|s', s|r'$?

We can now set out to determine properties of the class $\mathcal{O}(r, s)$ where r and s are 'small' in some sense. First we consider the case when $r = 1$.

THEOREM 19. *Let p^e be a prime power, and let $G \in \mathcal{O}(1, p^e)$ be a finite group.*

(a) *If $p \neq 2$, then $G/F(G)$ is an Abelian p-group.*
(b) *If $p = 2$, then $G/F(G)$ is a 2-group of class $\leq e$.*

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PROOF. Note that by Theorem 5, we have that $\exp(G/F(G))$ divides p^e, so that G is soluble. Let $\mathcal{K}$ be the class of all finite Abelian p-groups if $p \neq 2$ and the class of all finite 2-groups of nilpotent class $\leq e$ if $p = 2$. In both cases, $\mathcal{K}$ is QSD-closed.

Let G be a counterexample to (a) or (b) of least possible order. By minimality, every proper subgroup and quotient of G is in $\mathcal{N}\mathcal{K}$. As G is soluble, it has the form $G = [N]Q$ as described in Lemma 14.

By Lemma 15(c), every element in $G \setminus N$ has order dividing p^e. Now N is an elementary Abelian q-group for some prime q, say. Clearly, $q \neq p$. Hence every element $\neq 1$ of Q acts fixed point freely on N (otherwise, $G \setminus N$ would contain elements of order pq), so Q is a Frobenius complement. We now use the classification of Frobenius complements (see [5, p. 505]). If $p \neq 2$, then the p-group Q is cyclic thus $G/F(G) = Q$ is Abelian against minimality. If $p = 2$, the 2-group Q is cyclic or generalized quaternion of exponent dividing 2^e. Thus, Q is nilpotent of class $\leq e$ and $Q \in \mathcal{K}$. But this is against minimality. □

The following shows that the bound given in (b) of the above theorem is best possible.

EXAMPLE 20. Let q be a prime with $q \equiv 1 \bmod 2^e$, and let Q be a Sylow 2-subgroup of $\mathrm{SL}(2, q)$. Then Q is a generalized quaternion group of exponent 2^e. Let N be the elementary Abelian group of order q^2, and let $G = [N]Q$ be the natural split extension. Then G is a Frobenius group with kernel N and complement Q. In particular, every element of $G \setminus N$ is of order dividing 2^e. Example 3 yields $G \in \mathcal{O}(1, 2^e)$, and $G/F(G)$ is of class e precisely.

The next case is when $r = 1$, and $s = pq$ is a product of two primes.

THEOREM 21. *Let p and q be primes, and let $G \in \mathcal{O}(1, pq)$, G finite. Then G is metanilpotent.*

PROOF. For $p = q$, the result follows immediately from Theorem 5, so let $p \neq q$. Note that G by Theorem 5 is soluble. Let G be a counterexample of least possible order. By Lemma 14, we have $G = [N]Q$ where $N = C_G(N)$ is the unique minimal normal subgroup of G, and Q is minimal nonnilpotent.

By Lemma 15(c), every element of $G \setminus N$ is of order dividing pq. Let r be the prime dividing the order of N. First, assume $r \notin \{p, q\}$. As $G \setminus N$ does not contain any element of order divisible by r, the group Q must act fixed point freely on N. As $\exp(Q)$ divides pq, the structure of Frobenius complements (see [5, p. 505]) shows that Q is cyclic, hence Abelian, so that we have a contradiction. Hence $r \in \{p, q\}$. By symmetry, we may assume $r = p$. Let P be a Sylow p-subgroup of G. By Lemma 15(c), every element of $P \setminus N$ is of order p, so $\exp(P) = p$. An appeal to Hall and Higman's Theorem B (see [6, p. 451f]) shows that G is of p-length one. As $N = C_G(N)$, we have $O_{p'}(G) = 1$, so $P \trianglelefteq G$, and G/P is a q-group. Here, G is metanilpotent, a final contradiction. □

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Note that groups in $\mathcal{O}(1, 2p)$ generally need not be metabelian as can be seen from Example 7. Moreover, every faithful extension of an elementary Abelian 2-group by the nonabelian group of exponent $p \neq 2$ and order p^3 is of exponent $2p$, hence lies in $\mathcal{O}(1, 2p)$. But this group does not have a nilpotent derived subgroup.

We now consider the case where r and s are prime powers.

THEOREM 22. *Let $r = p_1^{e_1}$ and $s = p_2^{e_2}$ be prime powers, and let $G \in \mathcal{O}(r, s)$ be a finite group.*

(a) *$G/F(G)$ is supersoluble.*
(b) *If $p_1 \nmid p_2 - 1$ and $p_2 \nmid p_1 - 1$, then G is metanilpotent.*
(c) *If $e_1 = e_2 = 1$ and $p_1 \neq p_2$, then $G/F(G)$ is Abelian.*

PROOF. If $p_1 = p_2$, then by Theorem 5(a), the quotient $G/F(G)$ is a p_1-group, and so (a) and (b) hold. We thus may assume $p_1 \neq p_2$.

(a) By Theorem 5(b), the group G is soluble. Let G be a counterexample of least possible order. By Lemma 14, we have that $G = [N]Q$ where $N = F(G)$ is the unique minimal normal subgroup of G. Moreover, $N = C_G(N)$, and every proper subgroup of Q is supersoluble, but Q is not supersoluble. A result of Doerk (see [5, p. 721]) yields that $Q = [A]B$ where A is a noncyclic normal Sylow subgroup of Q, and B acts irreducibly on $A/\Phi(A)$ which is a noncyclic chief factor of G.

As Q acts faithfully and irreducibly on N, we have $(|N|, |A|) = 1$. By Lemma 15(c), every element in $G \setminus N$ is of prime power order. Hence $C_N(a) = 1$ for all $a \in A$, $a \neq 1$. By the classification of Frobenius complements (see [5, p. 505]) A is either cyclic or generalized quaternion. However, $A/\Phi(A)$ is noncyclic, so that the first case does not occur. Let A be generalized quaternion. As B acts irreducibly on $A/\Phi(A)$, we see that B induces on A an automorphism of order three which clearly centralizes the involution in A. In particular, Q contains elements of order six. But all elements of $G \setminus N$ are of prime power order, so we have a contradiction.

(b) Let G be a counterexample of least possible order. As in (a), we have $G = [N]Q$ where $N = C_G(N)$, and every proper subgroup of Q is nilpotent. By (a), the complement Q is supersoluble.

By the structure of minimal nonnilpotent groups (see [5, p. 281]), we get $Q = [A]B$ where A and B are cyclic. By Lemma 15(c), every element in $G \setminus N$ is of order dividing $p_1^{e_1}$ or $p_2^{e_2}$. So $\pi(Q) = \{p_1, p_2\}$. As Q is nonnilpotent, this implies $p_1 | p_2 - 1$ or $p_2 | p_1 - 1$ against the hypothesis.

(c) Let G be a counterexample of least possible order. As in (a), we have $G = [N]Q$ where $N = C_G(N)$, and every proper subgroup of Q is Abelian. By (a), we know that Q is supersoluble.

First, assume that Q is nilpotent, thus a p-group for some prime p. By Lemma 15(c), every element in $G \setminus N$ is of order p_1 or p_2, so $\exp(Q) = p \in \{p_1, p_2\}$, and Q does not contain any subgroup $\cong \mathbb{Z}_p \times \mathbb{Z}_p$. This implies that $Q \cong \mathbb{Z}_p$ is Abelian, so we have a contradiction.

We may assume $p_1 > p_2$. As $\exp(Q)$ divides $p_1 p_2$, the minimal nonabelian group Q is isomorphic to the nonabelian split extension of $\mathbb{Z}_{p_1}$ by $\mathbb{Z}_{p_2}$. Assume that N

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is a q-group for the prime q. If $q \notin \{p_1, p_2\}$, then Q would be a Frobenius complement of G, but this is against the well-known structure of these. As Q acts faithfully and irreducibly on N, we have $q \neq p_1$. Hence N is a p_2-group. Now consider a Sylow p_2-subgroup P of G. By Lemma 15(c), every element of $G \setminus N$ is of order p_1 or p_2, so every element of $P \setminus N$ is of order p_2. Hence $\exp(P) = p_2$. By Hall and Higman's Theorem B (see [6, p. 451f]), G is of p_2-length one. As $O_{p_1}(G) = 1$, the group G is p_2-closed, so we have a contradiction. □

REMARK 23.

(a) Part (c) of Theorem 22 is no longer true in the case when $p = q$. Indeed, let G be an extension of an Abelian normal subgroup by a group of exponent p. Then $G \in \mathcal{O}(p, p)$, but $G/F(G)$ is not necessarily Abelian if $p > 2$. Moreover, for large p, there exist infinite simple groups of exponent p, and so, in general, G need not even be soluble. Here, one cannot dispose with that hypothesis that G is finite.

(b) Part (c) of Theorem 22 is no longer true for arbitrary e_1, e_2. Indeed, the symmetric group $G = S_4$ has elements of orders one, two, three and four. Hence we have $G \in \mathcal{O}(3, 4)$. But $G/F(G) \cong S_3$ is nonabelian.

QUESTION Assume that all finite groups in $\mathcal{O}(r, s)$ are metanilpotent. What can be said about r and s?

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