AN ANALOGUE OF THE WAVE EQUATION AND CERTAIN RELATED FUNCTIONAL EQUATIONS

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Consider the functional equation

$$(1) \qquad f(x+h, y) + f(x-h, y) - f(x, y+h) - f(x, y-h) = 0$$

assumed valid for all real x, y and h. Notice that (1) can be written

$$(2) \qquad \Delta^2_{h\,xx} f - \Delta^2_{h\,yy} f = 0,$$

a difference analogue of the wave equation, if we interpret $\Delta_{h\,x} f(x, y) = f(x + \frac{h}{2}, y) - f(x - \frac{h}{2}, y)$, etc., (i.e. symmetric differences), and that (1) has an interesting geometric interpretation. The continuous solutions of (1) were found by Saković [5]. In this paper the result of Saković is obtained using a method related to the distributional methods employed in [1] and [6]. This method is then used in dealing with related equations such as those considered in [1] and [6].

Throughout this paper R denotes the set of all real numbers and ϕ_n is a real valued function of two real variables satisfying the following conditions for each $n = 1, 2, 3, \ldots$

(i) $\phi_n(x, y) \geq 0$ for all real x, y and $\phi_n(x, y)$ depends only on $x^2 + y^2$ and is thus symmetric in both variables.

(ii) $\phi_n \in C^\infty(R^2)$ i.e. ϕ_n has continuous partial derivatives of every order.

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(iii) $\operatorname{supp}\phi_n = \overline{\{(x,y): \phi_n(x,y) \neq 0\}} \subset \{(x,y): x^2 + y^2 \leq \frac{1}{n^2}\}$.

(iv) $\iint_{R^2} \phi_n(x,y)\,dx\,dy = 1$.

A sequence $\{\phi_n\}$ with these properties can be constructed as follows. Let

$$\tau(t) = \begin{cases} 0 & |t| \geq 1 \\ \exp\left(\frac{1}{t^2 - 1}\right) & |t| < 1 \end{cases}$$

and define

$$\phi_n(x,y) = c_n\,\tau(n^2x^2 + n^2y^2)$$

where $c_n^{-1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau(n^2x^2 + n^2y^2)\,dx\,dy$.

Properties (i), (iii) and (iv) are obvious and (ii) follows from the fact that $\tau \in C^\infty(R)$. (See [7, page 3].)

Notice that if $f: R^2 \to R$ is continuous, then $\{f*\phi_n\}_{n=1}^{\infty}$ is a sequence of $C^\infty(R^2)$ functions which converges to f almost uniformly (uniformly on compact subsets of R^2). Here $f*\phi_n$ denotes the convolution of f and ϕ_n, i.e.

$$f*\phi_n(x,y) = \iint_{R^2} f(s,t)\phi_n(x-s,\ y-t)\,ds\,dt\,.$$

1. We now give a proof of the result of Sakovič.

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THEOREM 1. A continuous function $f: R^2 \to R$ satisfies (1) for all $x, y, h \in R$ if and only if there exist continuous functions $\alpha, \beta: R \to R$ such that for all $x, y \in R$,

$$(3) \qquad f(x,y) = \alpha(x+y) + \beta(x-y).$$

Proof. Suppose f is a continuous solution of (1). For each $n = 1, 2, 3, \dots$, let $f_n = f * \phi_n$. Multiply both sides of (1) by $\phi_n(s-x, t-y)$ and integrate with respect to x and y over R^2 to show that each f_n is also a solution of (1). That is,

$$(4) \qquad f_n(s+h, t) + f_n(s-h, t) - f_n(s, t+h) - f_n(s, t-h) = 0$$

for all real s, t, and h. But f_n has continuous partial derivatives of every order. Differentiate (4) twice with respect to h and then put $h = 0$ to obtain

$$\frac{\partial^2 f_n}{\partial x^2} - \frac{\partial^2 f_n}{\partial y^2} = 0.$$

Thus $f_n(x,y) = \alpha_n(x+y) + \beta_n(x-y)$ for all $x, y \in R$ where $\alpha_n, \beta_n \in C^\infty(R)$ for $n = 1, 2, 3, \dots$.

Let $g(u,v) = f(\frac{u+v}{2}, \frac{u-v}{2})$ and $g_n(u,v) = f_n(\frac{u+v}{2}, \frac{u-v}{2}) = \alpha_n(u) + \beta_n(v)$ for all $u, v \in R$ and $n = 1, 2, 3, \dots$. Since $f_n \to f$ almost uniformly, $g_n \to g$ almost uniformly. Thus

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$$g(u,v) - g(u,0) - g(0,v) + g(0,0)$$

$$= \lim_{n\to\infty}(\alpha_n(u) + \beta_n(v) - \alpha_n(u) - \beta_n(0) - \alpha_n(0) - \beta_n(v) + \alpha_n(0) + \beta_n(0))$$

$$= 0 \text{ for all } u, v \in R.$$

Let $\alpha(u) = g(u,0) - \frac{1}{2}g(0,0)$ and $\beta(v) = g(0,v) - \frac{1}{2}g(0,0)$ for $u, v \in R$. Since g is continuous so are α and β. Moreover $f(x,y) = g(x+y, x-y) = \alpha(x+y) + \beta(x-y)$ for all $x, y \in R$.

Since the converse is trivial, this completes the proof.

Remark. Notice that any f of the form (3), with arbitrary α and β, is a solution of (1). It is thus easy to find solutions of (1) which are nonmeasurable, measurable solutions which are nowhere continuous and solutions of class C^r which are not of class C^{r+1} for each $r = 0, 1, 2, \ldots$. This is in marked contrast to the equation

$$f(x+t, y+t) + f(x+t, y-t) + f(x-t, y+t) + f(x-t, y-t) = 4f(x,y)$$

whose only measurable solutions are harmonic polynomials of degree ≤ 4. See [1, Theorem 3].

We remark that not every solution of (1) is of the form (3). In fact it is easily verified that any biadditive antisymmetric function f satisfies (1). (An example of a non-trivial function of this type is supplied by

$$f(x,y) = xa(y) - ya(x)$$

where a is additive and discontinuous.) However if f is biadditive and antisymmetric then f cannot be of the form (3) unless $f = 0$. To see this, suppose, on the contrary, that

$$f(x,y) = \alpha(x+y) + \beta(x-y).$$

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Since f is antisymmetric and additive in each variable $f(x, x) = -f(x, x) = f(x, -x)$ for all $x \in R$. Thus

$$0 = f(x, x) = \alpha(2x) + \beta(0)$$

and

$$0 = f(x, -x) = \alpha(0) + \beta(2x)$$

which implies that α and β are constant which in term implies that f is constant and finally, $f = 0$.

If we interpret $\Delta_h^x f(x, y) = f(x+h, y) - f(x, y)$, etc., then (2) becomes

(5) $$f(x+2h, y) - 2f(x+h, y) - f(x, y+2h) + 2f(x, y+h) = 0.$$

THEOREM 2. A continuous function $f: R^2 \to R$ satisfies (5) for all $x, y, h \in R$ if and only if there exists a continuous $\alpha: R \to R$ and real constants a, b and c such that for all $x, y \in R$,

(6) $$f(x, y) = \alpha(x+y) + a(x-y)^2 + b(x-y) + c.$$

Proof. Suppose f is continuous and satisfies (5). As in the proof of Theorem 1 we find that $f(x, y) = \alpha(x+y) + \beta(x-y)$ for all $x, y \in R$ where α and β are continuous. Then from (5) we obtain $\beta(x+2h-y) - 2\beta(x+h-y) - \beta(x-y-2h) + 2\beta(x-y-h) = 0$ for all $x, y, h \in R$. With $y = 0$ this becomes

(7) $$\beta(x+2h) - 2\beta(x+h) - \beta(x-2h) + 2\beta(x-h) = 0$$

for all $x, h \in R$. According to [2, Theorem 6.1], since β is continuous, β must be a polynomial. Differentiating (7) three times with respect to h and setting $h = 0$ we find that $\beta'''(x) = 0$ for all $x \in R$ and thus conclude that (6) holds.

It is easy to show that if f is of the form (6), then f satisfies (5). This completes the proof.

2. The method employed above may be used to solve or to

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obtain qualitative information concerning functional equations of the form

(8) $$f(x, y) = \sum_{i=1}^{m} \gamma_i f(x + \rho_i h, y + \sigma_i h)$$

where ρ_i, σ_i and γ_i are real constants. Formally differentiating (8) twice with respect to h one obtains, upon setting $h = 0$,

(9) $$\frac{A\partial^2 f}{\partial x^2} + \frac{2B\partial^2 f}{\partial x \partial y} + \frac{C\partial^2 f}{\partial y^2} = 0$$

where $A = \sum_{i=1}^{m} \gamma_i \rho_i^2$, $B = \sum_{i=1}^{m} \gamma_i \rho_i \sigma_i$ and $C = \sum_{i=1}^{m} \gamma_i \sigma_i^2$.

THEOREM 3. _Suppose_ f _is a continuous solution of_ (8) _and at least one of_ A, B _or_ C _is non zero._

(i) _If_ (9) _is hyperbolic_ $(AC - B^2 < 0)$ _then_ $f(x, y) = \alpha(ax + by) + \beta(cx + dy)$ _where_ $\alpha, \beta: R \to R$ _are continuous and_ a, b, c _and_ d _are real constants with_ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

(ii) _If_ (9) _is parabolic_ $(AC - B^2 = 0)$ _then_ $f(x, y) = (ax + by)\alpha(cx + dy) + \beta(cx + dy)$ _where_ $\alpha, \beta: R \to R$ _are continuous and_ a, b, c _and_ d _are real constants with_ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

(iii) _If_ (9) _is elliptic_ $(AC - B^2 > 0)$ _then_ $f(x, y) = g(ax + by, cx + dy)$ _where_ g _is harmonic and_ a, b, c _and_ d _are real constants with_ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. _Thus_ $f \in C^\infty(R^2)$ _if_ (9) _is elliptic._

Proof. As above we let $f_n = f * \phi_n$ and find that f_n also satisfies (8) and $f_n \in C^\infty(R^2)$. Thus f_n is a solution of (9).

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(i) If (9) is hyperbolic then, since $f_n \in C^\infty(R^2)$, a suitable change of variable gives rise to the wave equation. Thus there are continuous functions $\alpha_n, \beta_n : R \to R$ and real constants a, b, c and d with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ such that $f_n(x,y) = \alpha_n(ax+by) + \beta_n(cx+dy)$ for all real x, y. Let $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ and

$g(u,v) = f(a'u + b'v, c'u + d'v)$ and
$g_n(u,v) = f_n(a'u + b'v, c'u + d'v) = \alpha_n(u) + \beta_n(v)$ for all real u and v. Since $f_n \to f$ almost uniformly, $g_n \to g$ almost uniformly.
Thus $g(u,v) - g(u,0) = \lim_{n\to\infty}(g_n(u,v) - g_n(u,0))$
$= \lim_{n\to\infty}(\alpha_n(u) + \beta_n(v) - \alpha_n(u) - \beta_n(0)) = \lim_{n\to\infty}(\beta_n(v) - \beta_n(0))$.
We can thus write

$$g(u,v) = \alpha(u) + \beta(v)$$

where $\alpha(u) = g(u,0)$ and $\beta(v) = \lim_{n\to\infty}(\beta_n(v) - \beta_n(0))$. Since g is continuous, so are α and β and
$f(x,y) = g(ax+by, cx+dy) = \alpha(ax+by) + \beta(cx+dy)$ for all $x, y \in R$.

(ii) If (9) is parabolic we similarly find that
$f_n(x,y) = (ax+by)\alpha_n(cx+dy) + \beta_n(cx+dy)$. Let

$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ and $f_n(a'u + b'v, c'u + d'v) = u\alpha_n(v) + \beta_n(v)$

and $f(a'u + b'v, c'u + d'v) = g(u,v)$ for all real u and v.
Then since $f_n \to f$, $g_n \to g$ and
$g(u,v) - g(0,v) = \lim_{n\to\infty}(u\alpha_n(v) + \beta_n(v) - \beta_n(v)) = u \lim_{n\to\infty}\alpha_n(v)$.
If we let $\alpha(v) = \lim \alpha_n(v)$ and $\beta(v) = g(0,v)$ then
$g(u,v) = u\alpha(v) + \beta(v)$ for all real u and v. Hence
$f(x,y) = g(ax+by, cx+dy) = (ax+by)\alpha(cx+dy) + \beta(cx+dy)$
for all real x and y.

Part (iii) follows in a similar manner from the well-known theorem of Harnack which asserts that the almost uniform limit of harmonic functions is harmonic. (See, for example, [4]).

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COROLLARY. If $\gamma_i > 0$ for $i = 1, 2, \ldots, m$ and the ρ_i's and σ_i's are not proportional then every continuous solution of (8) belongs to $C^\infty(R^2)$.

Proof. By the Schwarz inequality

$$B^2 = \left(\sum_{i=1}^{m} (\sqrt{\gamma_i}\,\rho_i)(\sqrt{\gamma_i}\,\sigma_i)\right)^2 < \left(\sum_{i=1}^{m} (\sqrt{\gamma_i}\,\rho_i)^2\right)\left(\sum_{i=1}^{m} (\sqrt{\gamma_i}\,\sigma_i)^2\right) = AC$$

and the result follows from (iii) of Theorem 3.

3. As a further example let us show that the present method can be used to solve some equations of the type considered in [6]. The following equation was solved in [6],

$$(10) \qquad f(x+h,y) + h^2 f(x,y+h) - (h^2+1)f(x,y) = h(2x+h+h^3+2yh^2).$$

Let us show that f is continuous and satisfies (10) if and only if $f(x,y) = x^2 + y^2 + C$ where C is a real constant.

Suppose f is a continuous solution of (10). Multiply both sides of (10) by $\phi_n(u-x, v-y)$ and integrate with respect to x and y over R^2. By (i), $\phi_n(s,t)$ is an even function of s for each fixed t. Thus $\int_{-\infty}^{\infty} s\phi_n(s,t)ds = 0$ for every t so that

$$\iint_{R^2} x\phi_n(u-x, v-y)dx\,dy = \iint_{R^2} (u-s)\phi_n(s,t)ds\,dt = u - \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} s\phi_n(s,t)ds\right)dt = u.$$

Similarly $\iint_{R^2} y\phi_n(u-x, v-y)\,dx\,dy = v$ and so

$$(11) \qquad f_n(u+h, v) + h^2 f_n(u, v+h) - (h^2+1)f_n(u,v) = h(2u+h+h^3+2vh^2)$$

for all real u, v and h. Now $f_n \in C^\infty(R^2)$. Differentiating (11) once with respect to h and setting $h = 0$ we find $\frac{\partial f_n}{\partial x}(u,v) = 2u$. Thus $f_n(u,v) = u^2 + b_n(v)$ where $b_n \in C^\infty(R)$. Since $f_n \to f$ we find, as before, that $f(u,v) = u^2 + b(v)$ for all real u and v where $b: R \to R$ is continuous. This, together with (10), implies that

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$h^2[b(y+h) - b(y)] = h^4 + 2yh^3$ for all real y and h. Thus

$$\frac{b(y+h) - b(y)}{h} = h + 2y$$

for all $y \in R$ and all real $h \neq 0$. Thus $b'(y) = 2y$ so that $b(y) = y^2 + C$ where C is constant. We thus have $f(x,y) = x^2 + y^2 + C$ and every function of this form is readily shown to satisfy (10).

In [1] and [6] continuous solutions of functional equations of a type similar to those considered here were found using results concerning the regularity of distributional solutions to certain elliptic and hypoelliptic partial differential equations. Such methods clearly cannot be applied to equations like (1) which do not give rise to elliptic or hypoelliptic partial differential equations.

M.A. McKiernan [3] has found the general solution of (8) (and its natural generalization to R^n) under the assumption that $\sum_{i \in J} \gamma_i \neq 0$ whenever $\emptyset \neq J \subset \{1, 2, \ldots, m\}$. Thus our Corollary is a particular case of McKiernan's result. Again (1) and (5) are not of this type.

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