Canad. Math. Bull. Vol. 61 (1), 2018 pp. 114–123
http://dx.doi.org/10.4153/CMB-2017-023-1

A Characterization of C^*-normed Algebras via Positive Functionals

Marina Haralampidou, Mohamed Oudadess, Lourdes Palacios, and Carlos Signoret

Abstract. We give a characterization of C^*-normed algebras, among certain involutive normed ones. This is done through the existence of enough specific positive functionals. The same question is also examined in some non normed (topological) algebras.

1 Introduction and Preliminaries

Some properties of C^*-algebras appear to be characterizations of such algebras in the class of involutive Banach algebras. This is the case, for example, with the Vidav–Palmer theorem. Here the focus is on positive functionals. It is known that a C^*-algebra always has a large supply of such functionals. There is even the following striking result (see [20, p. 227] and [21, Theorem 12.39]).

Theorem 1.1 *Let $(E, \|\cdot\|)$ be a unital C^*-algebra. Then, for every $z \in E$, there is a positive functional f such that $f(e) = 1$ and $f(zz^*) = \|z\|^2$.*

One first notices that this result is still valid in a non complete C^*-normed algebra. Moreover, it turns out that the property in the theorem is actually a characterization of C^*-normed algebras in the frame of involutive normed ones (Proposition 2.1). Corollary 2.2 concerns the Banach case.

Section 3 deals with locally m-convex algebras. We obtain the analogs of the results in Section 2, first for locally C^*-algebras (Proposition 3.5) and then for pre-locally C^*-algebras (Proposition 3.7). Notice that here we employ the perfectness property.

C^*-bornological algebras are examined in Section 4, the context being the one of *-pseudo-normed algebras (Proposition 4.2) and *-pseudo-Banach algebras (Remark 4.3).

In Section 5, we consider locally uniformly A-convex algebras (Proposition 5.4) and, more general, locally A-convex ones (Proposition 5.6). In the first case we do not need perfectness, while in the second case we do.

Received by the editors November 10, 2016; revised March 28, 2017.
Published electronically May 29, 2017.
AMS subject classification: **46H05**, 46K05.
Keywords: C^*-normed algebra, C^*-algebra, (pre-)locally C^*-algebra, pre-C^*-bornological algebra, positive functional, locally uniformly A-convex algebra, perfect locally m-convex algebra, C^*-subnormable algebra, *-subnormable algebra.

In Section 6 we examine locally uniformly convex algebras. Without completeness, we need a somewhat stronger condition (Proposition 6.1). Here the matter is C^*-subnormability. In the complete case, we get rid of it. Again the expected conclusion is *-subnormability (Proposition 6.2).

Perfectness plays an important role, but we did not find examples and counterexamples in the literature. We give some examples in Section 3 (see also [9]).

The interest of the kind of the results obtained here is that the existence of a particular positive functional implies the existence of many of them. This is very important for representation theory, and may have applications in field theory.

In the sequel, we employ the following terminology. A linear form (functional) f on an involutive algebra E is *positive* if $f(xx^*) \geq 0$, for all $x \in E$. A *C^*-normed algebra* is an involutive normed algebra $(E, \|\cdot\|)$ satisfying the *C^*-condition* ($\|x^*x\| = \|x\|^2$, for all $x \in E$, *viz.* $\|\cdot\|$ is a C^*-norm). If, moreover, E is complete, it is called a *C^*-algebra*. In the non-normed case, a *pre-locally C^*-algebra* is an involutive topological algebra, the topology of which is defined by a (saturated) family $(p_\lambda)_{\lambda\in\Lambda}$ of C^*-seminorms. A *locally C^*-algebra* is a complete pre-locally C^*-algebra. Let (E,τ) be a locally convex algebra with a separately continuous multiplication whose topology τ is given by a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms. The algebra (E,τ) is said to be a *locally A-convex algebra* [4, 5] if, for every x and every λ, there is $M(x,\lambda) > 0$ such that $\max[p_\lambda(xy), p_\lambda(yx)] \leq M(x,\lambda)p_\lambda(y)$, for all $y \in E$. In the case of a single space norm, $(E, \|\cdot\|)$ is called an *A-normed algebra*. If $M(x,\lambda) = M(x)$ depends only on x, we say that (E,τ) is a *locally uniformly A-convex algebra* [4]. If it happens that, for every λ, $p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y)$, for all $x, y \in E$, then (E,τ) is named a *locally m-convex algebra* [13]. Recall also that a locally convex algebra has a continuous multiplication if, for every λ, there is λ'such that $p_\lambda(xy) \leq p_{\lambda'}(x)p_{\lambda'}(y)$, for all $x, y \in E$.

If $(E, (p_\lambda)_{\lambda\in\Lambda})$ is a unital locally-A-convex algebra, then it can be endowed with a stronger m-convex topology $M(\tau)$, where τ is the topology on E. It is determined by the family $(q_\lambda)_{\lambda\in\Lambda}$ of seminorms given by

$$q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \leq 1\}. \tag{1.1}$$

If $(E, (p_\lambda)_{\lambda\in\Lambda})$ is a locally uniformly A-convex algebra, then there is yet [14,15] an algebra norm $\|\cdot\|_0$ which induces a topology $\tau_{\|\cdot\|_0}$ stronger than $M(\tau)$. It is given by

$$\|x\|_0 = \sup\{q_\lambda(x) : \lambda \in \Lambda\}. \tag{1.2}$$

2 Normed Algebras

It is known that unital involutive Banach algebras do not necessarily admit nonzero positive functionals.

R. S. Doran [6, p. 475] asked if there exists a Banach *-algebra without identity that admits no nonzero positive functionals. He also remarks that for C^*-algebras (unital or not) there are enough positive functionals. Relative to this, N. V. Gorbachev [8] gave an example of a Banach $\star$-algebra without identity and with no nonzero positive functionals. The following result shows that, indeed, the existence of a large number of continuous positive functionals is closely related to the C^*-structure. The proof goes along the lines of standard techniques in representation theory.

https://doi.org/10.4153/CMB-2017-023-1 Published online by Cambridge University Press

Proposition 2.1 *Let $(E, \|\cdot\|)$ be a unital involutive normed algebra. Then the following are equivalent.*

(i) *$(E, \|\cdot\|)$ is a C^*-normed algebra.*

(ii) *For every z there is a continuous positive functional $f_z = f$ such that $f(e) = 1$ and $f(zz^*) = \|z\|^2$.*

Proof (i) ⇒ (ii). Just take the completion $\widetilde{E}$ of E, which is a C^*-algebra and apply Theorem 1.1. The continuity of the positive functionals comes from a result of Varopoulos [22].

(ii) ⇒ (i). Take $u \in E$, $u \neq 0$, and $f_u = f$ as in (ii). Put

$$Y_u = Y = \{y \in E : f(xy) = 0, \text{ for all } x \in E\},$$

which is obviously a closed subspace (in fact, a left ideal) of E. Consider the quotient space E/Y. One defines a scalar product on it, by putting $\langle a', b'\rangle = f(b^* a)$ with $a' = a + Y, b' = b + Y, a, b \in E$. It is well defined and $\langle a', a'\rangle = f(a^* a) \geq 0$. One also shows that $\langle a', a'\rangle = 0$ implies that $a' = Y$. Hence $(E/Y, \|\cdot\|)$ with $\|a'\|^2 = f(a^* a)$ is a pre-Hilbert space. Denote by $H_u = \widetilde{E/Y} = \widetilde{E/Y_u}$ the Hilbert space, the completion of E/Y_u.

Now to every $x \in E$, one associates the operator

$$T_x : E/Y \to E/Y, a' \mapsto T_x(a') = (xa') \equiv xa + Y.$$

It is well defined, and one shows that

- $x \mapsto T_x$ is linear.
- $T_{x_2} T_{x_1} = T_{x_2 x_1}$, for every $x_1, x_2 \in E$.
- $T_e = \mathrm{Id}_{E/Y}$.
- $\|T_x\| \leq \|x\|$, for every $x \in E$.
- $T_{x^*} = (T_x)^*$, for every $x \in E$.
- $\|T_u\| = \|u\|$.

So T_x is extended to a bounded operator still denoted by $T_x : H_u \to H_u$ with the same properties.

To finish, one puts together the Hilbert spaces H_u, by considering the standard direct sum $H = \bigoplus H_u$. One finally obtains $\|T_x\| = \|x\|$, for every $x \in E$. ■

The continuity condition in Proposition 2.1 (ii) becomes redundant if $(E, \|\cdot\|)$ is a Q-*algebra*, *viz.*, the group of its invertible elements is open. So it is, in particular, worthwhile to give the following statement. It is the characterization alluded to in the introduction (Theorem 1.1).

Corollary 2.2 *Let $(E, \|\cdot\|)$ be a unital involutive Banach algebra. Then the following are equivalent.*

(i) *$(E, \|\cdot\|)$ is a C^*-algebra.*

(ii) *For every z there is a positive functional $f_z = f$ such that $f(e) = 1$ and $f(zz^*) = \|z\|^2$.*

https://doi.org/10.4153/CMB-2017-023-1 Published online by Cambridge University Press

3 Locally m-convex Algebras

The main result of the previous section extends to the locally m-convex case, modulo an additional condition, *i.e.*, perfectness (see Definition 3.1). The term "perfect" has also been employed by C. Apostol in a different sense [2, Definition 1.1]. We now define the term "perfect" as it will be used throughout this section [9, Definition 2.7].

Definition 3.1 A projective system $\{(E_\lambda, f_{\lambda\mu})\}_{\lambda\in\Lambda}$ of topological algebras is called *perfect* if the restrictions to the projective limit algebra

$$E = \varprojlim E_\lambda = \Big\{ (x_\lambda) \in \prod_{\lambda\in\Lambda} E_\lambda : f_{\lambda\mu}(x_\mu) = x_\lambda, \text{ if } \lambda \le \mu \text{ in } \Lambda \Big\}$$

of the canonical projections $\pi_\lambda: \prod_{\lambda\in\Lambda} E_\lambda \to E_\lambda$, $\lambda \in \Lambda$, namely, the continuous algebra morphisms $f_\lambda = \pi_\lambda|_{E=\varprojlim E_\lambda}: E \to E_\lambda$, $\lambda \in \Lambda$, are onto maps. The resulting projective limit algebra $E = \varprojlim E_\lambda$ is then called a *perfect (topological) algebra*.

A *perfect locally m-convex algebra* is a locally m-convex algebra $(E, (p_\lambda)_{\lambda\in\Lambda})$ for which the respective Arens–Michael projective system $\{(E_\lambda, f_{\lambda\mu})\}_{\lambda\in\Lambda}$ is perfect.

In the previous definition, by the term *Arens–Michael projective system* we mean that which corresponds to the Arens–Michael decomposition (see also [13, Definition 2.1, p. 86, (3.9), and Definition 3.1]).

The notion of perfectness as in Definition 3.1 is algebraic, however we give examples in the topological context in which we are working.

Example 3.2 If the E_λ's, $\lambda \in \Lambda$, are C^*-normed algebras and if the preorder of Λ is the equality, then the only connecting maps are $f_{\lambda\lambda} = \mathrm{Id}_{E_\lambda}$ and thus $\varprojlim E_\lambda = \Pi E_\lambda$ (see [3, Example 1]). Therefore, the latter is a perfect pre-locally C^*-algebra. If it is complete, then it is a locally C^*-algebra.

Example 3.3 If E is a C^*-normed algebra and $E_\lambda = E$, $\lambda \in \Lambda$ and $f_{\lambda\mu}: E_\mu \to E_\lambda$, $\lambda \le \mu$ are the identity maps, then $\varprojlim E_\lambda$ is the diagonal Δ of ΠE_λ and $\Delta = \varprojlim E_\lambda$ is a perfect locally m-convex algebra [10, Example 2.3]. In particular, it is a perfect pre-locally C^*-algebra. If E is complete, then it is a locally C^*-algebra.

Example 3.4 The classical locally C^*-algebra $(C(\mathbb{R}), (\|\cdot\|_K)_K)$ is perfect.

Proposition 3.5 *Let $(E, (p_\lambda)_{\lambda\in\Lambda})$ be a unital involutive complete, perfect locally m-convex algebra. Then the following are equivalent.*

(i) *$(E, (p_\lambda)_{\lambda\in\Lambda})$ is a locally C^*-algebra.*
(ii) *For every λ and every z, there is a continuous positive functional $f_{\lambda,z} = f$ such that $f(e) = 1$ and $f(zz^*) = p_\lambda(z)^2$.*

Proof (i) ⇒ (ii). See Remark 3.6.

(ii) ⇒ (i). By perfectness, f_λ is onto. So for every λ and $\dot{u} \equiv u + \ker(p_\lambda)$ in E_λ (the normed factor in the Arens–Michael decomposition of E [13]), there is a $z_{\dot{u}} = z$ in E such that $\dot{u} = f_\lambda(z)$. Consider a positive form g, as in the hypothesis. Then a positive form $g_{\dot{u}}: E_\lambda \to \mathbb{C}$ is defined by $g_{\dot{u}}(\dot{v}) \equiv g_{\dot{u}}(v + \ker(p_\lambda)) = g(y)$, where $\dot{v} = f_\lambda(y)$.

https://doi.org/10.4153/CMB-2017-023-1 Published online by Cambridge University Press

One has $g_{\dot{u}}(e_\lambda) = 1$, since $f_\lambda(e) = e_\lambda$. Also

$$g_{\dot{u}}(\dot{v}\dot{v}^*) = g(yy^*) = p_\lambda(y)^2 = \dot{p}_\lambda(f_\lambda(y))^2 = \dot{p}_\lambda(\dot{v})^2.$$

One can then apply Proposition 2.1 to get C^*-normed factors, and passing to their completions, we can take an inverse limit of C^*-algebras, which assures that E is finally a locally C^*-algebra (see [12, Theorem 2.1] and [19, Proposition 1.2]). ■

Remark 3.6 Condition (ii) is natural since it is fulfilled in any unital locally C^*-algebra [7, Proposition 14.21].

Without completeness, one still has an interesting statement.

Proposition 3.7 *Let $(E,(p_\lambda)_{\lambda\in\Lambda})$ be a unital Hausdorff involutive and perfect locally m-convex algebra. Then the following are equivalent.*

(i) *$(E,(p_\lambda)_{\lambda\in\Lambda})$ is a pre-locally C^*-algebra.*
(ii) *For every λ and every z, there is a continuous positive functional $f_{\lambda,z} = f$ such that $f(e) = 1$ and $f(zz^*) = p_\lambda(z)^2$.*

Proof (i) ⇒ (ii). Consider the completion $\widetilde{E}$ of E and then apply Proposition 3.5.

(ii) ⇒ (i). Without loss of generality, we can assume that $p_\lambda(x^*) = p_\lambda(x)$, for every x and every λ. Then $(E,(p_\lambda)_{\lambda\in\Lambda})$ is the projective limit of the normed *-algebras $E_\lambda = E/N_\lambda$, where $N_\lambda = \{x : p_\lambda(x) = 0\}$. Now, arguing as in Proposition 3.5, one shows that the factors E_λ satisfy the same property as E. To finish, apply Proposition 3.5. ■

4 Pre-C^*-bornological Algebras

A *pre-C^*-bornological algebra* is a bornological inductive limit of C^*-normed algebras (see Definition 4.1). This must be a *-pseudo-normed algebra, according to the terminology of [1]. When it is complete, it is a *-pseudo-Banach algebra. A *C^*-bornological algebra* is a bornological inductive limit of C^*-algebras. These algebras have been introduced in [11] under the name "Algèbres bornologiques stellaires complètes". They have been reexamined in [17], where a C^*-norm has been exhibited, and thus one gets an improvement of the results and a simplification of the proofs.

Definition 4.1 Let E be a complex algebra which is the union of subalgebras E_i, each one being a C^*-normed algebra $(E_i, \|\cdot\|_i)$ such that (E_i, f_{ji}), $i \le j$, is an inductive system, where f_{ji} is the injection of E_i in E_j. Endowed with the bornological inductive limit $\mathcal{B}$, E is named a *pre-C^*-bornological algebra* and it is symbolized by $(E,\mathcal{B}) = \varinjlim(E_i, \|\cdot\|_i)$.

One obtains a C^*-norm $\|\cdot\|$ on E by putting $\|x\| = \|x\|_i$, $i \in \{j : x \in E_j\}$ (see [17]). It is easy to see that every positive functional on $(E,\mathcal{B})$ is bounded, and since $\mathbb{B}_{\tau_{\|\cdot\|}} \subset \mathcal{B}$, it is continuous on $(E,\|\cdot\|)$. Here, $\mathbb{B}_{\tau_{\|\cdot\|}}$ stands for the collection of all subsets B of E, which are bounded in the sense of Kolmogorov–von Neumann *viz.*, B is absorbed by every neighborhood of zero. Arguing in the C^*-algebra $\widetilde{E}$, the completion of E, one gets the conclusion of Theorem 1.1. Thus, we are led to the following statement.

Proposition 4.2 *Let $(E, \mathcal{B}) = \varinjlim(E_i, \|\cdot\|_i)$ be a unital *-pseudo-normed algebra such that*

(i) *There is an algebra norm $\|\cdot\|$ on E.*
(ii) $\|x\| = \|x\|_i$, $i \in \{j : x \in E_j\}$.
(iii) *For every z there is a positive functional $f_z = f$ such that $f(e) = 1$ and $f(zz^*) = \|z\|^2$.*

Then $(E, \mathcal{B})$ is a pre-C^*-bornological algebra.

Proof Apply Proposition 2.1 to each factor, which then becomes a *-normed algebra. ∎

Remark 4.3 Proposition 4.2 applies, of course, to *-pseudo-Banach algebras, in particular, to involutive p-Banach algebras [1].

Remark 4.4 Condition (iii) in Proposition 4.2 is also necessary. Indeed, if $(E, \mathcal{B})$ is a pre-C^*-bornological algebra, then $\|\cdot\|$ is a C^*-norm. The claim follows by arguing in the C^*-algebra $\widetilde{E}$.

5 Locally A-convex Algebras

We begin with locally uniformly A-convex algebras. We produce the appropriate statement of Theorem 1.1. To justify it, let us first recall two examples.

Example 5.1 Let $C[0,1]$ be the complex algebra of continuous functions on the interval $[0,1]$. Endow it with the seminorms p_{K_d}, with $p_{K_d}(f) = \sup\{f(x) : x \in K_d\}$, where K_d is running over all denumerable compact subsets of $[0,1]$. It is a uniformly A-convex algebra, and the supremum norm $\|\cdot\|_\infty$ makes it a C^*-algebra.

Example 5.2 Let $C_b(\mathbb{R})$ be the algebra of complex continuous bounded functions on the real field $\mathbb{R}$, with the usual pointwise operations and the complex conjugation as an involution. Denote by $C_0^+(\mathbb{R})$ the family of strictly positive elements of $C_b(\mathbb{R})$ tending to zero at infinity. Endow $C_b(\mathbb{R})$ with the seminorms p_φ, $\varphi \in C_0^+(\mathbb{R})$, given by $p_\varphi(f) = \sup\{\|\varphi(x)\|f(x) : x \in \mathbb{R}\}$. The family (p_φ) determines a locally convex topology β. The space $(C_b(\mathbb{R}), \beta)$ is a complete locally convex algebra, which is not a locally m-convex one [5]. Actually, it is a locally uniformly A-convex algebra, and under the supremum norm $\|\cdot\|_\infty$ it turns out to be a C^*-algebra.

The examples above suggest the following definition and statement.

Definition 5.3 Let (E, τ) be a unital locally uniformly A-convex algebra. It is said to be *C^*-subnormable* (respectively, **-subnormable*) if there is a C^*-norm (respectively, *-norm) the topology of which is stronger than τ.

See (1.1) for the q_λ appearing in the following propositions. We also recall that a locally convex algebra is *pseudo-complete* if every bounded and closed idempotent (alias multiplicative) disk B is Banach, *i.e.*, the vector space generated by it is a Banach space when endowed with the respective gauge $\|\cdot\|_B$ of B.

Proposition 5.4 *Let $(E,(p_\lambda)_{\lambda\in\Lambda})$ be a unital involutive pseudo-complete locally uniformly A-convex algebra. If, for every z, there is a positive functional $f_z = f$ such that $f(e) = 1$ and $q_\lambda(z)^2 \le f(zz^*)$, for every λ, then $(E,(p_\lambda)_{\lambda\in\Lambda})$ is C^*-subnormable.*

Proof On E we consider the algebra norm defined by (1.2). Namely,

$$\|x\|_0 = \sup\{q_\lambda(x) : \lambda \in \Lambda\},$$

where $q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \le 1\}$. Without loss of generality, one may assume that $p_\lambda(x^*) = p_\lambda(x)$, for every λ. Then it is easily checked that, $q_\lambda(x^*) = q_\lambda(x)$, for every λ. Whence $\|x^*\|_0 = \|x\|_0$. Now if f is as in the hypothesis, then $\|zz^*\|_0 = \sup\{q_\lambda(zz^*) : \lambda\} \le f(zz^*)$. On the other hand, E being pseudo-complete, $(E,\|x\|_0)$ is a Banach algebra. Hence f is continuous. So $f(zz^*) \le \|zz^*\|_0$. Thus $f(zz^*) = \|zz^*\|_0$. An application of Proposition 2.1 gives the assertion. ■

Remark 5.5 The previous result is still valid for a pseudo-complete Q-algebra. Without any of these two properties, one has to ask for the continuity of the positive form.

Concerning the general non uniformly A-convex case, one has the following.

Proposition 5.6 *Let $(E,(p_\lambda)_{\lambda\in\Lambda})$ be a unital involutive locally A-convex algebra such that $(E,(q_\lambda)_\lambda)$ is perfect (see* (1.1)*). If, for every z, there is a positive functional $f_z = f$ such that $f(e) = 1$ and $q_\lambda(zz^*) \le f(zz^*)$, for every λ, then $(E,(p_\lambda)_\lambda)$ is a pre-locally C^*-algebra.*

We take this opportunity to give two examples of locally uniformly convex algebras which are not uniformly A-convex ones.

Example 5.7 Let $(E,\|\cdot\|)$ be a commutative unital C^*-algebra. Endow it with the weak topology, say σ, given by the convex hull $co(M(E))$ of its carrier space. Then (E,σ) is a locally uniformly convex algebra which is not A-convex [18, Example 11]. The same topological algebra is obviously not uniformly A-convex.

Example 5.8 Let $C[X]$ be the commutative unital algebra of complex polynomials, and $(z_m)_m$ a sequence of complex numbers such that $|z_m| \to +\infty$. Then $C[X]$ endowed with the topology, say τ, given by the seminorms $P \mapsto |P|_m = |P(z_m)|$ becomes a metrizable locally m-convex algebra. Let $\|\cdot\|$ be a vector space norm stronger than τ. Then, for every m, there is a $k_m > 1$ such that $|PQ|_m = |P|_m|Q|_m \le \|P\|k_m|Q|_m$, for all Q. For the local uniform convexity of $C[X]$, we consider the family $(|\cdot|_m) \cup (\alpha|\cdot|_m)_{m,\alpha\ge1}$ of seminorms which also define the topology. Besides, it cannot be uniformly A-convex; otherwise we should have a stronger algebra norm than τ, say $\|\cdot\|_0$. But then the characters $P \mapsto P(z_m)$ should be continuous, which contradicts $|P(z_m)| \to +\infty$ (see [16, Example 6.4]).

Example 5.9 Let $C_b(\mathbb{R})$ be the algebra of Example 5.2 and $C(\mathbb{R})$ the very classical locally C^*-algebra (see Example 3.4). Then the standard cartesian product algebra

$C_b(\mathbb{R}) \times C(\mathbb{R})$ is a pre-locally C^*-algebra. It is locally A-convex, but not locally uniformly A-convex.

6 Locally Uniformly Convex Algebras

We first recall the basics. A locally convex algebra $(E, (p_\lambda)_{\lambda\in\Lambda})$ is said to be *locally uniformly convex* if, for all x and for all λ, there exist $M(x) > 0$ and λ' such that $p_\lambda(xy) \le M(x)p_{\lambda'}(y)$, for all y. When E has a unit element e, one may suppose that $p_{\lambda'}(e) \ne 0$, for every λ. The family

$$(q_\lambda)_\lambda = ([p_{\lambda'}(e)]^{-1}p_\lambda)_\lambda \tag{6.1}$$

of seminorms determines on E a topology, say τ'. We have

$$\sup_\lambda [p_{\lambda'}(e)]^{-1}p_\lambda(x) \le M(x), \quad \text{for all } x. \tag{6.2}$$

But then

$$x \mapsto \|x\| = \sup_\lambda [p_{\lambda'}(e)]^{-1}p_\lambda(x) \tag{6.3}$$

is a vector space norm stronger than τ'. What we do have is

$$\|xy\| \le \sup\Big\{ \frac{p_{\lambda''}(e)}{p_{\lambda'}(e)} M(x) \frac{1}{p_{\lambda''}(e)} p_{\lambda'}(y) \Big\}.$$

So to proceed, we are led to put a condition on $([p_\lambda(e)]^{-1}p_{\lambda'}(e))_\lambda$, in the sense of the next result.

Proposition 6.1 *Let $(E, (p_\lambda)_{\lambda\in\Lambda})$ be a unital involutive locally uniformly convex algebra such that $([p_\lambda(e)]^{-1}p_{\lambda'}(e))_\lambda$ is bounded by 1. If, for every z, there is a positive functional $f_z = f$ such that $f(e) = 1$ and $q_\lambda(zz^*) \le f(zz^*)$, for every λ, then $(E, (p_\lambda)_{\lambda\in\Lambda})$ is C^*-subnormable.*

Proof In view of (6.2) and (6.3), we have $\|xy\| \le M(x)\|y\|$, for every y. So $(E, \|\cdot\|)$ is an A-normed algebra. One then gets an algebra norm $\|\cdot\|_0$ given by

$$\|x\|_0 = \sup\{\|xu\| : \|u\| \le 1\}$$

(see also (6.3)). Without loss of generality, we may assume $p_\lambda(x^*) = p_\lambda(x)$, for every x and every λ. But, then the same is true for the q_λ's (*cf.* (6.1)) and $\|\cdot\|_0$. ∎

In the case of completeness, we do not need the somewhat artificial condition that $([p_\lambda(e)]^{-1}p_{\lambda'}(e))_\lambda$ is bounded by 1. But we obtain less, as the following statement indicates.

Proposition 6.2 *Let $(E, (p_\lambda)_{\lambda\in\Lambda})$ be a unital involutive and complete locally uniformly convex algebra. If, for every z, there is a positive functional $f_z = f$ such that $f(e) = 1$ and $q_\lambda(zz^*) \le f(zz^*)$, for every λ (see (6.1)), then $(E, (p_\lambda)_{\lambda\in\Lambda})$ is *-subnormable.*

Proof Since $(E, (p_\lambda)_{\lambda\in\Lambda})$ is complete, $(E, \|\cdot\|)$ is a Banach space (with $\|\cdot\|$ as in (6.3)). One has $\mathbb{B}\tau_{\|\cdot\|} \subset \mathbb{B}\tau'$, since τ' is coarser than the topology given by $\|\cdot\|$. On the other hand, the latter has a fundamental basis of zero consisting of τ'-barrels, and then they are τ'-bornivorous (namely, each one absorbs all τ'-bounded subsets). Hence $\mathbb{B}\tau_{\|\cdot\|} \subset \mathbb{B}\tau$. So the multiplication $(x, y) \to xy$ is $\|\cdot\|$-separately bounded. Thus it is jointly continuous. Therefore $(E, \|\cdot\|)$ is a *-normed algebra for an equivalent norm. ■

Remark 6.3 The difference between the conclusions of the previous propositions shows the strength of the condition $([p_\lambda(e)]^{-1}p_{\lambda'}(e))_{\check{\lambda}}$ is bounded by 1.

Acknowledgement We are greatly indebted to the referee for careful reading of the paper and for valuable remarks.

References

[1] G. R. Allan, H. G. Dales, and J. P. McClure, *Pseudo-Banach algebras*. Studia Math. **40**(1971), 55–69.
[2] C. Apostol, *b^*-algebras and their representations*. J. London Math. Soc. 3(1971), 30–38. http://dx.doi.org/10.1112/jlms/s2-3.1.30
[3] N. Bourbaki, *Théorie des Ensembles*. Chap. 3, Hermann, Paris, 1967.
[4] A. C. Cochran, *Representation of A-convex algebras*. Proc. Amer. Math. Soc. **41**(1973), 473–479. http://dx.doi.org/10.2307/2039116
[5] A. C. Cochran, R. Keown, and C. R. Williams, *On a class of topological algebras*. Pacific J. Math. **34**(1970), 17–25. http://dx.doi.org/10.2140/pjm.1970.34.17
[6] R. S. Doran, *Research problems: Does there exist a Banach *- algebra without identity with no nonzero positive functionals?* Amer. Math. Monthly **87**(1980), 474–475. http://dx.doi.org/10.2307/2320257
[7] M. Fragoulopoulou, *Topological algebras with involution*. North Holland Mathematics Studies, 200. Elsevier Science, Amsterdam, 2005.
[8] N. V. Gorbachev, *An example of a Banach $\ast$-algebra without identity and with no nonzero positive functionals*. Uspekhi Mat. Nauk **38**(1983), 189.
[9] M. Haralampidou, *The Krull nature of locally C^*-algebras*. Function Spaces (Edwardsville IL, 2002), pp. 195–200, Contemp. Math. 328, Amer. Math. Soc., Providence, RI, 2003. http://dx.doi.org/10.1090/conm/328/05781
[10] M. Haralampidou, L. Palacios, and C. Signoret, *Multipliers in perfect locally m-convex algebras*. Banach J. Math. Anal. **9**(2015), no. 1, 137–143. http://dx.doi.org/10.15352/bjma/09-1-11
[11] H. Hogbé-Nlend, *Les fondements de la théorie spectrale des algèbres bornologiques*. Bol. Soc. Brasil Mat. **3**(1972), 19–56. http://dx.doi.org/10.1007/BF02584840
[12] A. Inoue, *Locally C^*-algebras*. Mem. Faculty Sci. Kyushu Univ. SerA **25**(1971), 197–235. http://dx.doi.org/10.2206/kyushumfs.25.197
[13] A. Mallios, *Topological algebras. Selected topics*. North-Holland Mathematics Studies, 124. North-Holland, Amsterdam, 1986.
[14] M. Oudadess, *Théorème du type Gelfand-Naimark dans les algèbres uniformément A-convexes*. Ann. Sc. Math. Québec, **9**(1985), 73–82.
[15] ———, *Une norme d'algèbre de Banach dans les algèbres uniformément A-convexes*. Africa Math. **9**(1987), 15–22.
[16] ———, *Bounded structures in locally A-convex algebras*. Math. Stud. (Tartu), 4. Est. Math. Soc., Tartu, 2008, pp. 80–88.
[17] ———, *C^*-bornological algebras*. African Diaspora J. Math. **9**(2010), 87–95.
[18] ———, *Subnormable A-convex algebras*. De Gruyter Proceedings in Mathematics (to appear).
[19] N. C. Phillips, *Inverse limits of C^*-algebras*. J. Operator Theory **19**(1988), 159–195.
[20] C. E. Rickart, *General theory of Banach algebras*. D. Van Nostrand, Princeton, N.J., 1960.
[21] W. Rudin, *Functional analysis*. McGraw-Hill, New York, 1973.
[22] N. Th. Varopoulos, *Sur les formes positives d' une algèbre de Banach*. C. R. Acad. Sci. Paris **258**(1964), 2465–2467.

Department of Mathematics, University of Athens, Panepistimioupolis, Athens 15784, Greece
e-mail: mharalam@math.uoa.gr

Ecole Normale Supérieure, B.P 5118, Takaddoum, 10105 Rabat, Morocco
e-mail: oudadesshha@gmail.com

Universidad Autónoma Metropolitana Iztapalapa, Mexico City 09340, Mexico
e-mail: pafa@xanum.uam.mx casi@xanum.uam.mx

https://doi.org/10.4153/CMB-2017-023-1 Published online by Cambridge University Press