Proceedings of the Edinburgh Mathematical Society (2023) **66**, 453–474
doi:10.1017/S0013091523000226

THE L^p CONVERGENCE OF FOURIER SERIES ON TRIANGULAR DOMAINS

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(Received 2 June 2022)

Abstract We prove L^p norm convergence for (appropriate truncations of) the Fourier series arising from the Dirichlet Laplacian eigenfunctions on three types of triangular domains in $\mathbb{R}^2$: (i) the 45-90-45 triangle, (ii) the equilateral triangle and (iii) the hemiequilateral triangle (i.e. half an equilateral triangle cut along its height). The limitations of our argument to these three types are discussed in light of Lamé's Theorem and the image method.

Keywords: L^p convergence; eigenfunctions; Fourier series; triangles; Dirichlet Laplacian

2020 Mathematics subject classification: Primary 42B08
Secondary 42B05; 34L10; 35P10; 42A10

1. Introduction

In one dimension, there is only one way to truncate the partial sums of a Fourier series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)\mathrm{e}^{2\pi \mathrm{i} nx},$$

namely

$$S_N(x) := \sum_{n=-N}^{N} \widehat{f}(n)\mathrm{e}^{2\pi \mathrm{i} nx}.$$

Then the Fourier series converges if and only if $S_N(x)$ converges as $N \to \infty$.

When moving to higher dimensions, we have

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \widehat{f}(m,n)\mathrm{e}^{2\pi \mathrm{i}(mx+ny)}$$

and an ambiguity arises.

Since the eigenvalue for $\mathrm{e}^{2\pi\mathrm{i}(mx+ny)}$ is proportional to m^2+n^2, the ‘natural choice’ of truncation for the partial sums is

$$\sum_{m^2+n^2 \leqslant N^2} \widehat{f}(m,n)\mathrm{e}^{2\pi \mathrm{i}(mx+ny)};$$

that is, we cut off the sum once we have picked out all eigenfunctions with eigenvalues $|\lambda_{m,n}| \lesssim N^2$. (Geometrically, this procedure corresponds to using a ‘circular cutoff’ in the frequency space by choosing frequencies (m,n) in the ball of radius N.) A celebrated result of Fefferman [5] implies, with the help of standard transference results [7, Chap. 4], that such ‘eigenvalue truncations’ of Fourier expansions fail, in general, to converge to f in the $L^p(\mathbb{T}^2)$ norm when $p \neq 2$.

We may, instead, truncate according to the labelling of the indices (or frequencies) $(m,n) \in \mathbb{Z}^2$:

$$\sum_{|m|,|n| \leqslant N} \widehat{f}(m,n)\mathrm{e}^{2\pi \mathrm{i}(mx+ny)}.$$

Happily, these ‘truncations by label’ always converge back to f in *all* $L^p(\mathbb{T}^2)$ spaces (provided that $1 < p < \infty$). [See 7, for a proof of this classical result.] Recently, Fefferman *et al.* [6] have noted that the general problem of finding ‘well-behaved’ truncations of eigenfunction expansions in L^p-based spaces is still open for more general bounded Euclidean domains.

A natural starting point is to consider the next ‘simplest’ domains, such as discs or triangles. The eigenfunctions of the disc are products of trigonometric and Bessel functions and so share a similar product structure to the classical Fourier series on $\mathbb{T}^2$. Specifically, the eigenfunctions are of the form

$$\mathrm{e}^{2\pi \mathrm{i}n\theta} J_{|n|}\left(j_m^{|n|} r\right),$$

giving rise to the multidimensional Bessel–Fourier series

$$\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} a_{m,n} \mathrm{e}^{2\pi \mathrm{i}n\theta} J_{|n|}\left(j_m^{|n|} r\right). \tag{1.1}$$

(Here, $J_{|n|}$ denotes a Bessel function of the first kind and $j_m^{|n|}$ its non-negative zeros.) The best result we were able to find in the literature is due to Balodis and Córdoba [3], who proved norm convergence in the mixed norm space $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ defined by the condition

$$\|f\|_{p,2} := \left[\int_0^1 \left(\sum_k |f_k(r)|^2\right)^{p/2} r\,\mathrm{d}r\right]^{1/p} < \infty,$$

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where $f_k(r)$ is the kth Fourier coefficient of the angular function $f(r,\cdot)$ for fixed r:

$$f_k(r) := \int_0^1 f(r,\theta)\mathrm{e}^{2\pi \mathrm{i}k\theta}\,\mathrm{d}\theta.$$

By truncating the series (1.1) in the ranges $|n| \leqslant N$, $m \leqslant M$, they were able to show that there is a constant $A > 0$ such that the Bessel–Fourier series of $f \in L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ converges to f in the $\|\cdot\|_{p,2}$ norm provided that $M \geqslant AN + 1$ and $4/3 < p < 4$. Furthermore, the endpoints for the range of p are sharp. [See 3, Theorem 2, of which our discussion is a special case when $d = 2$.]

By modifying their proof, we were able to improve the result to L^p convergence with respect to the usual measure $r\,\mathrm{d}r\,\mathrm{d}\theta$ on the disc, provided that $2 \leqslant p < 4$ and

$$\|f\|_{p,q} := \left[\int_0^1 \left(\sum_k |f_k(r)|^q\right)^{p/q} r\,\mathrm{d}r\right]^{1/p} < \infty, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

As far as we know, the problem of L^p convergence for functions $f \in L^p(r\,\mathrm{d}r\,\mathrm{d}\theta)$ in the range $4/3 < p < 4$ is still open; see [1].

We therefore turn our attention to triangular domains, which turn out to be much more amenable to analysis. Following early work of Lamé [12], other authors such as Práger [14] and McCartin [13] have derived explicit trigonometric expressions for Dirichlet eigenfunctions on the equilateral triangle. Neither of these authors, however, considers questions of convergence for the associated eigenfunction series, and subsequent work [2, 10, 16] restricts attention to numerical methods or convergence in L^2-based spaces such as H^k. [See 9, for a comprehensive survey of the literature on the Laplacian and its eigenfunctions.] To our knowledge, there is no treatment of the L^p convergence of series of eigenfunctions on the triangle for $p \neq 2$.

In this paper, we establish L^p convergence for trigonometric series of eigenfunctions for the Dirichlet Laplacian on three types of triangular domains: (i) the 45-90-45 triangle, (ii) the equilateral triangle and (iii) the hemiequilateral triangle (i.e. half an equilateral triangle cut along its height).

Eigenfunctions for the latter types, (ii) and (iii), were obtained in the abovementioned [14] and [13]. Using their insights, we prove that, on each of these domains, any L^p function can be written as a norm-convergent series of eigenfunctions. We also give a complete treatment of these issues for the simpler case (i). As far as we know, the result for type-(i) triangles is new.

In § 2, we derive the eigenfunctions for the type-(i) triangles, prove their completeness and establish L^p convergence of the associated series. This serves as a useful prototype for the arguments we will develop in the following sections.

The result for equilateral domains is obtained by a decomposition of a function into a symmetric and antisymmetric part with respect to the line $x = 0$. This reduces the problem to type-(iii) domains, which, once solved, easily yields the equilateral case.

Thus, in § 3 and § 5 we separately derive the antisymmetric and symmetric modes, following the ingenious approach of Práger [14] (see Figure 2). The hemiequilateral triangle T_1 is tiled into the rectangle $R = [0,\sqrt{3}] \times [0,1]$, where we exploit the classical

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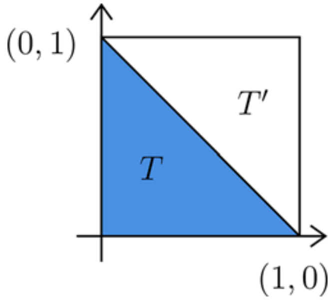

Figure 1. Reflecting T into the square $[0,1]^2$ along the diagonal $y = 1 - x$.

L^p convergence of 'double-sine' and 'cosine-sine' series on R to derive convergence on T_1. This is our main contribution, which is worked out in § 4 and § 6. These results are combined in § 7 to obtain L^p convergence on the equilateral triangle.

In § 8, we conclude with some remarks about the limitations of this argument: owing to a theorem of Lamé, the three domains (i)–(iii) listed above are the 'only' ones amenable to this procedure.

2. The 45-90-45 triangle

Taking our cue from Práger's analysis of the equilateral case, we reflect the 45-90-45 triangle along the hypotenuse to obtain a square (Figure 1).

Placing the triangle, T, at the vertices $(1,0)$, $(0,1)$ and $(0,0)$, the reflections have the simple form

$$x = 1 - \eta \quad \text{and} \quad y = 1 - \xi, \quad \text{where } (\xi, \eta) \in T.$$

Given a function $f : T \to \mathbb{R}$, we define the *prolongation* of f

$$\mathcal{P}f(x,y) := \begin{cases} f(x,y), & \text{if } (x,y) \in T; \\ -f(1-y, 1-x), & \text{if } (x,y) \in T'; \end{cases}$$

where T' denotes the reflected triangle, so that $[0,1]^2 = T \cup T'$.

We now expand $\mathcal{P}f$ as a double-sine series on $[0,1]^2$:

$$\begin{aligned} \widehat{\mathcal{P}f}(m,n) &:= 4 \int_0^1 \int_0^1 \mathcal{P}f(x,y) \sin(\pi m x)\, \sin(\pi n y)\, \mathrm{d}x\, \mathrm{d}y \\ &= 4 \int_T f(\xi,\eta)\, [\sin(\pi m \xi) \sin(\pi n \eta) \\ &\qquad - \sin(\pi m (1-\eta)) \sin(\pi n (1-\xi))]\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &= 4 \int_T f(\xi,\eta) u_{m,n}(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta, \end{aligned} \tag{2.1}$$

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where we defined the functions

$$u_{m,n}(\xi,\eta) := \sin(\pi m\xi)\sin(\pi n\eta) - \sin(\pi m(1-\eta))\sin(\pi n(1-\xi))$$

for all points $(\xi,\eta) \in T$.

Lemma 1. *We record the following facts about the functions $u_{m,n}$ (indexed by $m,n \in \mathbb{N}$).*

(1) For all $m \neq n$:

$$u_{m,n} = \sin(\pi m\xi)\sin(\pi n\eta) - (-1)^{m+n}\sin(\pi n\xi)\sin(\pi m\eta)$$

and clearly $u_{m,n} \equiv 0$ when $n = m$.

(2) Symmetry of the indices:

$$u_{n,m} = -(-1)^{m+n}u_{m,n} \quad \textit{for all } m \neq n;$$

therefore, it suffices to consider the set with $0 < m < n$.

(3) Letting (x,y) range over $[0,1]^2$,

$$\mathcal{P}\left[u_{m,n}|_T\right](x,y) = u_{m,n}(x,y) \quad \textit{for all } m \neq n.$$

In other words, restricting $u_{m,n}$ from $[0,1]^2$ to T and prolonging by $\mathcal{P}$ yields $u_{m,n}$ once more, as a function on $[0,1]^2$.

(4) The set $\{u_{m,n} : 0 < m < n\}$ is a complete, orthogonal set in $L^2(T)$ and

$$\|u_{m,n}\|_{L^2(T)} = \frac{1}{2}.$$

Proof. Statements (i)–(iii) are easily verified. For the completeness claim in (iv), suppose that $f \in L^2(T)$ is such that

$$\int_T f u_{m,n} = 0 \quad \text{for all } 0 < m < n.$$

It follows from (i), (iii), and Equation (2.1) that the Fourier coefficients of $\mathcal{P}f$ on $[0,1]^2$ vanish, so $\mathcal{P}f = 0$ in $L^2([0,1]^2)$, and therefore $f = \mathcal{P}f|_T = 0$. This proves the completeness of the set.

From (iii) and Equation (2.1), it is clear that

$$\int_T u_{m,n}u_{k,l} = \int_0^1\int_0^1 u_{m,n}(x,y)\sin(\pi kx)\sin(\pi ly)\,\mathrm{d}x\,\mathrm{d}y.$$

Ordering the indices as $0 < m < n$ and $0 < k < l$, it follows from the formula in (i) that the functions $u_{m,n}$ and $u_{k,l}$ are pairwise orthogonal; setting $m = k$ and $n = l$ easily yields the norm of $u_{m,n}$. $\square$

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Corollary 1. *The function $u_{m,n}$ with $0 < m < n$ is an eigenfunction of (minus) the Dirichlet Laplacian on T with eigenvalue $\pi^2(m^2+n^2)$.*

Proof. It is immediate from (i) and (iii) in Lemma 1 that

$$-\Delta u_{m,n}(x,y) = \pi^2(m^2+n^2)u_{m,n}(x,y)$$

pointwise for all $(x,y) \in [0,1]^2$ and $u_{m,n} \equiv 0$ on the boundary of the square. Thus, restricting to $T \subset [0,1]^2$ yields eigenfunctions of the Laplacian on T that clearly vanish along the edges parallel to the axes. It remains to show that $u_{m,n}$ also vanishes on the hypotenuse $y = 1 - x$, but this is immediately verified on substitution into the formula from (i) above. $\square$

We thus have a complete orthogonal set of eigenfunctions for the Dirichlet Laplacian on T. We can define the Fourier coefficients as usual:

$$f^{\triangle}(m,n) := \frac{1}{\|u_{m,n}\|^2_{L^2(T)}} \int_T f u_{m,n} = \widehat{\mathcal{P}f}(m,n);$$

the last equality holds by Equation (2.1).

The partial sum operators, truncated by indices, for the Fourier series on T and $[0,1]^2$ are

$$S^T_N f := \sum_{0<m<n\leqslant n} f^{\triangle}(m,n)u_{m,n} \quad \text{and}$$
$$s^{[0,1]^2}_n f := \sum_{0<m,n\leqslant n} \widehat{\mathcal{P}f}(m,n)\sin(\pi m x)\sin(\pi n y),$$

respectively.

The next proposition tells us that the prolongation operation commutes with the Fourier partial sums, allowing us to 'push forward' the L^p convergence results from the square to the triangle.

Proposition 1. *Let $f \in L^p(T)$. Then*

$$\mathcal{P}\left[S^T_N f\right](x,y) = S^{[0,1]^2}_N\left[\mathcal{P}f\right](x,y) \quad \text{for all } (x,y) \in [0,1]^2.$$

Proof. Break up the lattice $[1,N]^2 \cap \mathbb{N}^2$ into the following regions:

$$I : 0 < m < n \leqslant N \quad \text{and} \quad II : 0 < n < m \leqslant N.$$

(Recall that, by (i) in Lemma 1, the diagonal $m = n$ yields no eigenfunctions.) Hence,

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$$
\begin{aligned}
S_N^{[0,1]^2}\left[\mathcal{P}f\right](x,y) &= \sum_I \widehat{\mathcal{P}f}(m,n)\sin(\pi m x)\sin(\pi n y) \\
&\quad + \sum_{II} \widehat{\mathcal{P}f}(m,n)\sin(\pi m x)\sin(\pi n y) \\
&= \sum_I \widehat{\mathcal{P}f}(m,n)\left[\sin(\pi m x)\sin(\pi n y)\right. \\
&\quad \left. -(-1)^{m+n}\sin(\pi n x)\sin(\pi m y)\right] && \text{Lemma 1 (ii)} \\
&= \sum_I \widehat{\mathcal{P}f}(m,n) u_{m,n}(x,y) && \text{Lemma 1 (i)} \\
&= \sum_I \widehat{\mathcal{P}f}(m,n) \mathcal{P}u_{m,n}(x,y) && \text{Lemma 1 (iii)} \\
&= \mathcal{P}\left[S_N^T f\right](x,y),
\end{aligned}
$$

where in the last line we make use of the linearity of $\mathcal{P}$ and the identity $f^{\triangle}(m,n) = \widehat{\mathcal{P}f}(m,n)$ for all (m,n) in I. $\square$

We now have everything we need to prove our result.

Theorem 1. *Let $f \in L^p(T)$ with $1 < p < \infty$. Then, $S_N^T f \to f$ in $L^p(T)$.*

Proof. Note that, for $f \in L^p$, we have

$$
\|\mathcal{P}f\|_{L^p([0,1]^2)}^p = 2\,\|f\|_{L^p(T)}^p .
$$

Thus, by successively applying this equality and Proposition 1, we have

$$
\begin{aligned}
\left\|S_N^T f - f\right\|_{L^p(T)}^p &= \frac{1}{2}\left\|\mathcal{P}\left[S_N^T f\right] - \mathcal{P}f\right\|_{L^p([0,1]^2)}^p \\
&= \frac{1}{2}\left\|S_N^{[0,1]^2}\mathcal{P}f - \mathcal{P}f\right\|_{L^p([0,1]^2)}^p \to 0
\end{aligned}
$$

by the L^p convergence of Fourier double-sine series on $[0,1]^2$. $\square$

3. Antisymmetric eigenfunctions on the equilateral triangle

To tackle convergence on the equilateral triangle T, we divide it along its midline $x = 0$ into two congruent hemiequilateral triangles and call the right-hand one T_1 (Figure 2). Any function $f : T \to \mathbb{R}$ may be decomposed into a symmetric and antisymmetric part with respect to x, i.e. $f = f_a + f_s$, where

$$
f_a(x,y) := \frac{f(x,y) - f(-x,y)}{2} \quad \text{and} \quad f_s(x,y) := \frac{f(x,y) + f(-x,y)}{2}
$$

for all $(x,y) \in T$. Clearly, each of f_a and f_s is determined by its values on $T_1 \subset T$.

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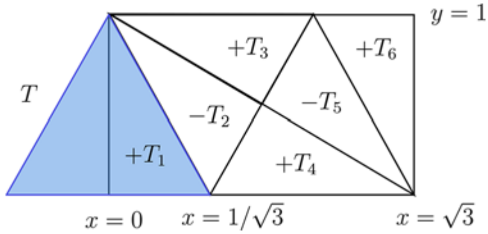

Figure 2. The equilateral triangle T (shaded), the right-angled triangle T_1 and six congruent copies T_i arranged into the rectangle $R = [0, \sqrt{3}] \times [0, 1]$. The '$\pm$' signs indicate a symmetric or antisymmetric reflection, respectively, in the definition of $\mathcal{P}_a$. Assuming zero boundary conditions on T_1, the extension by $\mathcal{P}_a$ vanishes on all lines drawn in R and its boundary; see [14].

Let us begin with the L^p convergence of the Fourier series of f_a on the triangle T_1. Denote by (ξ, η) the coordinates of a point in T_1. Then, for each coordinate pair (x, y) in the rectangle $R = [0, \sqrt{3}] \times [0, 1]$, we write [following 14, p. 312]

$$(x, y) = (x_i(\xi, \eta), y_i(\xi, \eta)) \quad \text{for exactly one } (x_i, y_i) \in T_i.$$

Note that the i subscript serves just as a reminder that the point (x_i, y_i) lies in the region T_i of the Cartesian plane.

The *antisymmetric prolongation* of $u\colon T_1 \to \mathbb{R}$ to the rectangle R is defined by

$$\mathcal{P}_a u(x, y) := \begin{cases} u(\xi, \eta) & \text{if } (x, y) = (x_1, y_1) \in T_1, \\ -u(\xi, \eta) & \text{if } (x, y) = (x_2, y_2) \in T_2, \\ u(\xi, \eta) & \text{if } (x, y) = (x_3, y_3) \in T_3, \\ u(\xi, \eta) & \text{if } (x, y) = (x_4, y_4) \in T_4, \\ -u(\xi, \eta) & \text{if } (x, y) = (x_5, y_5) \in T_5, \\ u(\xi, \eta) & \text{if } (x, y) = (x_6, y_6) \in T_6. \end{cases}$$

More succinctly (albeit less precisely), we express this relation as

$$\mathcal{P}_a u(x, y) = \sum_{i=1}^{6} c_i u(\xi(x_i, y_i), \eta(x_i, y_i)), \tag{3.1}$$

where c_i are the appropriate '$\pm$' signs.

From Figure 2 it is clear that the prolongation $\mathcal{P}_a u$ vanishes along the sides of the rectangle R. We therefore express $\mathcal{P}_a f_a$ as a double-sine series on R:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \widehat{\mathcal{P}_a f_a}(m, n) \sin\left(\frac{\pi m x}{\sqrt{3}}\right) \sin(\pi n y).$$

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The Fourier coefficients of $\mathcal{P}_a f_a$ are

$$\begin{aligned}\widehat{\mathcal{P}_a f_a}(m,n) &= \frac{4}{\sqrt{3}}\int_R \mathcal{P}_a f_a(x,y)\sin\left(\frac{\pi m x}{\sqrt{3}}\right)\sin(\pi n y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{4}{\sqrt{3}}\int_{T_1} f_a(\xi,\eta)\left[\sum_{i=1} c_i \sin\left(\frac{\pi m x_i}{\sqrt{3}}\right)\sin(\pi n y_i)\right]\mathrm{d}\xi\,\mathrm{d}\eta \qquad \text{by (3.1)} \\ &= \frac{4}{\sqrt{3}}\int_{T_1} f_a(\xi,\eta)u_{m,n}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{3.2}$$

where we defined the antisymmetric eigenfunctions as

$$u_{m,n}(\xi,\eta) := \sum_{i=1} c_i \sin\left(\frac{\pi m x_i}{\sqrt{3}}\right)\sin(\pi n y_i).$$

(Recall that each pair $(x_i, y_i) \in R$ is a function of $(\xi,\eta) \in T_1$.)

By explicit computation using the parametrizations of the T_i, Práger obtained the following.

Lemma 2. *For integers $0 < m < n$ with the same parity (i.e. both odd or both even),*

$$\begin{aligned} u_{m,n}(x,y) = {}& 2\sin\left(\frac{\pi m x}{\sqrt{3}}\right)\sin(\pi n y) \\ &- 2(-1)^{(m+n)/2}\sin\left(\frac{\pi x}{2\sqrt{3}}(m+3n)\right)\sin\left(\frac{\pi y}{2}(m-n)\right) \\ &+ 2(-1)^{(m-n)/2}\sin\left(\frac{\pi x}{2\sqrt{3}}(m-3n)\right)\sin\left(\frac{\pi y}{2}(m+n)\right). \end{aligned} \tag{3.3}$$

Furthermore, $u_{m,n} \equiv 0$ whenever n and m have different parity, or $n = m$ or $m = 3n$. Finally, with $0 < m < n$ as above, define the pairs

$$\begin{cases} m' = \frac{1}{2}(3n-m), \\ n' = \frac{1}{2}(m+n); \end{cases} \quad \text{and} \quad \begin{cases} m'' = \frac{1}{2}(m+3n), \\ n'' = \frac{1}{2}(m-n). \end{cases}$$

Then,

$$u_{m',n'} = -(-1)^{(m-n)/2}u_{m,n} \quad \text{and} \quad u_{m'',n''} = (-1)^{(m+n)/2}u_{m,n}. \tag{3.4}$$

The next technical result is all that stands between us and the L^p convergence of a $u_{m,n}$-expansion of f_a on T_1.

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Lemma 3. *The functions* $\{u_{m,n} : 0 < m < n$ *and* $m \equiv n \mod 2\}$ *are pairwise orthogonal. Furthermore,*

$$\mathcal{P}_a u_{m,n}(x,y) = u_{m,n}(x,y) \quad \textit{for all } (x,y) \in R, \tag{3.5}$$

from which it follows that

$$\|u_{m,n}\|^2_{L^2(T_1)} = \frac{\sqrt{3}}{2}. \tag{3.6}$$

Corollary 2. *The functions* $u_{m,n}$ *are eigenfunctions of (minus) the Dirichlet Laplacian on* T_1*, with eigenvalue* $\pi^2\left(\frac{m^2}{3} + n^2\right)$.

Proof. Since each $\mathcal{P}_a u_{m,n}$ is a finite combination of eigenfunctions on R, we have

$$-\Delta \mathcal{P}_a u_{m,n}(x,y) = \pi^2\left(\frac{m^2}{3} + n^2\right)\mathcal{P}_a u_{m,n}(x,y) \quad \text{for all } (x,y) \in R$$

by direct calculation. This equation holds pointwise owing to the smoothness of the functions; therefore, it holds when restricted to T_1. But $\mathcal{P}_a u_{m,n} \equiv u_{m,n}$ on T_1, whence

$$-\Delta u_{m,n}(\xi,\eta) = \pi^2\left(\frac{m^2}{3} + n^2\right)u_{m,n}(\xi,\eta) \quad \text{for all } (\xi,\eta) \in T.$$

These functions clearly vanish on the boundary of the triangle by construction. □

So far, our discussion is a recap of the results in [14], although we have added a more explicit proof of Corollary 2 than can be found there. We will now use these results to obtain L^p convergence of the eigenfunction expansions.

4. L^p convergence of the Fourier series of f_a

Equation (3.6) tells us that the 'Fourier coefficients' of f_a (with respect to $u_{m,n}$) are

$$f_a^{\triangle}(m,n) := \frac{2}{\sqrt{3}}\int_{T_1} f_a(\xi,\eta)u_{m,n}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

and it follows from Equation (3.2) that

$$\widehat{\mathcal{P}_a f_a}(m,n) = 2f_a^{\triangle}(m,n). \tag{4.1}$$

To help keep track of our indices, we will introduce the sets

$$\mathcal{U}_N := \{(m,n) : 0 < m < n \leqslant N \quad \text{and} \quad m \equiv n \mod 2\}.$$

Denote the Nth 'component-wise' partial sums on the triangle T_1 and rectangle R, respectively, by

$$S_N^{T_1} f_a(\xi,\eta) := \sum_{(m,n)\in\mathcal{U}_N} f_a^{\triangle}(m,n) u_{m,n}(\xi,\eta),$$

$$S_N^R \mathcal{P}_a f_a(x,y) := \sum_{0<m,n\leqslant N} \widehat{\mathcal{P}_a f_a}(m,n) \sin\left(\frac{\pi m}{\sqrt{3}} x\right) \sin(\pi n y).$$

The next theorem is a key ingredient in the proof of our main result.

Theorem 2. *Let $f \in L^p(T)$ with $1 < p < \infty$ and denote its antisymmetric part by f_a. Then $f_a \in L^p(T_1)$ and $S_N^{T_1} f_a \to f_a$ in $L^p(T_1)$.*

Proof. Since $f \in L^p(T)$ and f_a is a linear combination of f and its reflections, clearly $f_a \in L^p(T)$ and, by symmetry (and abuse of notation), $f_a = f_a|_{T_1} \in L^p(T_1)$.

Next, we break up the set of indices with $0 < m, n \leqslant N$ into three sets:

$$0 < m < n, \quad n < m < 3n \quad \text{and} \quad 3n < m,$$

noting that the coefficients when $m = n$ and $m = 3n$ vanish. From Lemma 2, we may label these groups as (m,n), (m',n') and (m'',n''), respectively. Note that such labelling divides $[1,N]\times[1,N]\cap\mathbb{Z}^2$ into three disjoint regions, which we label as $I = \mathcal{U}_N$, II and III (Figure 3).

It follows from Equation (3.4) that

$$\widehat{\mathcal{P}_a f_a}(m,n) = -(-1)^{(m-n)/2}\widehat{\mathcal{P}_a f_a}(m',n') = (-1)^{(m+n)/2}\widehat{\mathcal{P}_a f_a}(m'',n'').$$

Hence,

$$\begin{aligned}
S_N^R \mathcal{P}_a f_a(x,y) &= \sum_{(m,n)\in\mathcal{U}_N} \widehat{\mathcal{P}_a f_a}(m,n) \sin\left(\frac{\pi m x}{\sqrt{3}}\right) \sin(\pi n y) \\
&\quad + \sum_{(m',n')\in II} \widehat{\mathcal{P}_a f_a}(m',n') \sin\left(\frac{\pi m' x}{\sqrt{3}}\right) \sin(\pi n' y) \\
&\quad + \sum_{(m'',n'')\in III} \widehat{\mathcal{P}_a f_a}(m'',n'') \sin\left(\frac{\pi m'' x}{\sqrt{3}}\right) \sin(\pi n'' y) \\
&= \sum_{(m,n)\in\mathcal{U}_N} \widehat{\mathcal{P}_a f_a}(m,n) \Bigg[\sin\left(\frac{\pi m x}{\sqrt{3}}\right) \sin(\pi n y) \\
&\quad - (-1)^{(m+n)/2} \sin\left(\frac{\pi x}{2\sqrt{3}(m+3n)}\right) \sin\left(\frac{\pi y}{2}(m-n)\right) \\
&\quad + (-1)^{(m-n)/2} \sin\left(\frac{\pi x}{2\sqrt{3}}(m-3n)\right) \sin\left(\frac{\pi y}{2}(m+n)\right)\Bigg]
\end{aligned}$$

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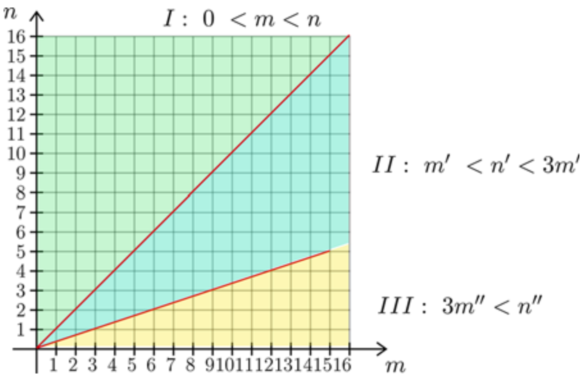

Figure 3. The three regions in the lattice $[1,16]\times[1,16]$. The lines correspond to the 'degenerate cases' $m = n$ and $m = 3n$, in which $u_{m,n} \equiv 0$, and thus the Fourier coefficients vanish. Note that all points where m and n have opposite parity will also be excluded.

$$= \sum_{(m,n)\in\mathcal{U}_N} \frac{\widehat{\mathcal{P}_a f_a}(m,n)}{2} u_{m,n}(x,y) \quad \text{by (3.3)}$$
$$= \sum_{(m,n)\in\mathcal{U}_N} f_a^{\triangle}(m,n)\mathcal{P}_a u_{m,n}(x,y).$$

The last line follows from Equations (3.5) and (4.1). Since $\mathcal{P}_a$ is a linear operator, we have therefore proved that

$$\mathcal{P}_a \left[S_N^{T_1} f_a\right](x,y) = S_N^R \mathcal{P}_a f_a(x,y).$$

Furthermore, since each transformation $T_1 \to T_i$ is an isometry, we have

$$\|\mathcal{P}_a u\|_{L^p(R)} = 6^{1/p}\, \|u\|_{L^p(T_1)} \quad \text{for all } u \in L^p(T_1).$$

Combining the last two displayed equations with the L^p convergence of double-sine series on R, we obtain

$$\left\|S_N^{T_1} f_a - f_a\right\|_{L^p(T_1)}^p = \frac{1}{6}\left\|\mathcal{P}_a\left[S_N^{T_1} f_a\right] - \mathcal{P}_a f_a\right\|_{L^p(R)}^p$$
$$= \frac{1}{6}\left\|S_N^R \mathcal{P}_a f_a - \mathcal{P}_a f_a\right\|_{L^p(R)}^p \to 0$$

as $N \to \infty$. □

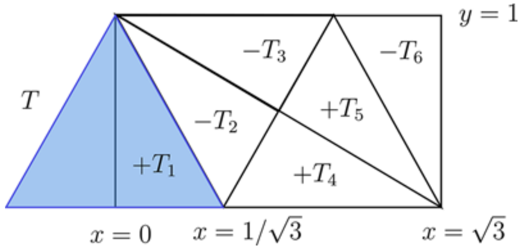

Figure 4. The sign arrangements for the symmetric prolongation $\mathcal{P}_s$; see [14].

5. Symmetric eigenfunctions on the equilateral triangle

It is now time to deal with the symmetric part: f_s. To do so, we introduce a *symmetric prolongation* of u to R:

$$\mathcal{P}_s u(x,y) := \sum_{i=1}^{6} d_i u(\xi(x_i), \eta(y_i)) \quad \text{where } d_i := \begin{cases} 1 & \text{if } i = 1,4,5; \\ -1 & \text{if } i = 2,3,6. \end{cases} \tag{5.1}$$

This notation uses the same 'shorthand' as in Equation (3.1) but uses a different combination of reflections and anti-reflections (Figure 4).

From Figure 4, it is clear that the prolongation $\mathcal{P}_s u$ only vanishes on the horizontal sides $y = 0$ and $y = 1$ of the rectangle, since for an x-symmetric function, we cannot assume that $u(x, 0) \equiv 0$. We therefore express $\mathcal{P}_s f_s$ as a cosine-sine series on R:

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \widehat{\mathcal{P}_s f_s}(m,n) \cos\left(\frac{\pi m x}{\sqrt{3}}\right) \sin\left(\pi n y\right).$$

When $m, n > 0$, the Fourier coefficients of $\widehat{\mathcal{P}_s f_s}$ are given by

$$\begin{aligned} \widehat{\mathcal{P}_s f_s}(m,n) &= \frac{4}{\sqrt{3}} \int_R \mathcal{P}_s f_s(x,y) \cos\left(\frac{\pi m x}{\sqrt{3}}\right) \sin\left(\pi n y\right) \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{4}{\sqrt{3}} \int_{T_1} f_s(\xi,\eta) \left[\sum_{i=1} d_i \cos\left(\frac{\pi m x_i}{\sqrt{3}}\right) \sin\left(\pi n y_i\right)\right] \mathrm{d}\xi \,\mathrm{d}\eta \qquad \text{by (3.1)} \\ &= \frac{4}{\sqrt{3}} \int_{T_1} f_s(\xi,\eta) v_{m,n}(\xi,\eta) \,\mathrm{d}\xi \,\mathrm{d}\eta; \end{aligned} \tag{5.2}$$

when $m = 0$ and $n > 0$, they are

$$\widehat{\mathcal{P}_s f_s}(0,n) = \frac{2}{\sqrt{3}} \int_{T_1} f_s(\xi,\eta) v_{0,n}(\xi,\eta) \,\mathrm{d}\xi \,\mathrm{d}\eta. \tag{5.3}$$

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Here we have defined the symmetric eigenfunctions as

$$v_{m,n}(\xi,\eta) := \sum_{i=1} d_i \cos\left(\frac{\pi m x_i}{\sqrt{3}}\right) \sin(\pi n y_i).$$

(Recall that each pair $(x_i, y_i) \in R$ is a function of $(\xi,\eta) \in T_1$.)

By explicit computation, Práger obtained the following.

Lemma 4. *For integers $0 \leqslant m < n$ with the same parity,*

$$\begin{aligned} v_{m,n}(x,y) = 2 \cos\left(\frac{\pi m x}{\sqrt{3}}\right) \sin(\pi n y) \\ + 2(-1)^{(m+n)/2} \cos\left(\frac{\pi x}{2\sqrt{3}}(m+3n)\right) \sin\left(\frac{\pi y}{2}(m-n)\right) \\ - 2(-1)^{(m-n)/2} \cos\left(\frac{\pi x}{2\sqrt{3}}(3n-m)\right) \sin\left(\frac{\pi y}{2}(m+n)\right). \end{aligned} \tag{5.4}$$

Furthermore, $v_{m,n} \equiv 0$ whenever n and m have different parity. Finally, with $0 < m < n$ as mentioned earlier, define the pairs

$$\begin{cases} m' = \frac{1}{2}(3n-m), \\ n' = \frac{1}{2}(m+n); \end{cases} \quad \text{and} \quad \begin{cases} m'' = \frac{1}{2}(m+3n), \\ n'' = \frac{1}{2}(m-n). \end{cases}$$

Then,

$$v_{m',n'} = -(-1)^{(m-n)/2} v_{m,n}, \quad v_{m'',n''} = -(-1)^{(m+n)/2} v_{m,n} \tag{5.5}$$

and, for all $n > 0$,

$$v_{0,n} = -(-1)^{n/2} v_{3n/2,n/2}. \tag{5.6}$$

Remark 1. Note that we no longer exclude the cases $m=0$ and $m=3n$ since the corresponding $v_{m,n}$ do not vanish and must therefore be carefully accounted for when we compute the partial sums. Note also that m and n must have the same parity, so that n is even when $m=0$, and, consequently, the right-hand side of Equation (5.6) is well-defined. We can thus regard $(m',n') = (3n/2, n/2)$ when $m=0$.

It is also clear that $v_{m,n}$ are the symmetric eigenfunctions of the Dirichlet Laplacian on T, restricted to T_1; the proof is identical to that of Corollary 2 [See 13, for a fuller treatment.]

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Lemma 5. *The eigenfunctions* $\{v_{m,n} : 0 \leqslant m < n,\ n > 0,\ m \equiv n \mod 2\}$ *are pairwise orthogonal. Furthermore,*

$$\mathcal{P}_s v_{m,n}(x,y) = v_{m,n}(x,y) \quad \textit{for all } (x,y) \in R, \tag{5.7}$$

from which it follows that

$$\|v_{m,n}\|^2_{L^2(T_1)} = \frac{\sqrt{3}}{2} \qquad \textit{for all } n, m > 0; \tag{5.8}$$

$$\|v_{0,n}\|^2_{L^2(T_1)} = \sqrt{3} \qquad \textit{for all } n > 0. \tag{5.9}$$

Once again, these results are contained in [14]. They are crucial to our proof of L^p convergence in the next section.

6. L^p convergence of the Fourier series of f_s

Writing

$$f_s^{\triangle}(m,n) := \frac{1}{\|v_{m,n}\|^2_{L^2(T_1)}} \int_{T_1} f_s v_{m,n},$$

it immediately follows from Equations (5.2), (5.8) and (5.3), (5.9) that

$$\widehat{\mathcal{P}_s f_s}(m,n) = 2 f_s^{\triangle}(m,n) \quad \text{for all } m \geqslant 0, n > 0. \tag{6.1}$$

The next theorem is the other key ingredient of our main result.

Theorem 3 *Let* $f \in L^p(T)$ *with* $1 < p < \infty$ *and denote its symmetric part by* f_s. *Then* $f_s \in L^p(T_1)$ *and* $S_N^{T_1} f_s \to f_s$ *in* $L^p(T_1)$.

Proof. The proof is essentially the same as that of Theorem 2, the only modifications arising from the 'degenerate' indices $(0,n)$ and $(3n,n)$.

As before, we break up the set of indices with $0 \leqslant m \leqslant N$, $0 < n \leqslant N$ into three sets:

$$0 < m < n, \quad n < m < 3n \quad \text{and} \quad 3n < m$$

sparing the cases $m = 0$ and $m = 3n$. From Lemma 4, we may label these groups as (m,n), (m',n') and (m'',n''), respectively, dividing the lattice $[0,N] \times [1,N] \cap \mathbb{Z}^2$ into three disjoint regions; see once again Figure 3.

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It follows from Equation (5.5) that, for all positive indices,

$$\widehat{\mathcal{P}_s f_s}(m,n) = -(-1)^{(m-n)/2}\widehat{\mathcal{P}_s f_s}(m',n') = (-1)^{(m+n)/2}\widehat{\mathcal{P}_s f_s}(m'',n''),$$

and from Equation (5.6) that

$$\widehat{\mathcal{P}_s f_s}(0,n) = -(-1)^{n/2}\widehat{\mathcal{P}_s f_s}(3n/2,n/2) \quad \text{for all } n>0. \tag{6.2}$$

Recall that we identify $(0',n') = (3n/2,n/2)$ and that this is precisely the case $m' = 3n'$; see Remark 1.

We begin by grouping the partial sums on R as follows:

$$S_N^R \mathcal{P}_s f_s(x,y) = \sum_{\substack{0<m,n\leqslant N\\ m\neq n, m\neq 3n}} \widehat{\mathcal{P}_s f_s}(m,n)\cos\left(\frac{\pi m x}{\sqrt{3}}\right)\sin(\pi n y) \tag{6.3}$$

$$+\left[\sum_{(0,n)} \widehat{\mathcal{P}_s f_s}(0,n)\sin(\pi n y)\right.$$

$$\left. + \sum_{(0',n')} \widehat{\mathcal{P}_s f_s}(0',n')\cos\left(\frac{0'\pi x}{\sqrt{3}}\right)\sin(\pi n' y)\right]. \tag{6.5}$$

The calculations for the term (6.3) proceed in the same way as for the antisymmetric part $S_N^R\mathcal{P}_a f_a$, using Equation (5.4) instead of Equation (3.3) after grouping the various cosine-sine terms. We will therefore only consider the bracketed term (6.5) in detail.

The sums in Equation (6.5) are taken over even $n \leqslant N$ and pairs $(0',n') = (3n/2,n/2)$ corresponding to $m = 3n$. Using Equation (6.2), we have

$$\begin{aligned}
&\sum_{(0,n)} \widehat{\mathcal{P}_s f_s}(0,n)\sin(\pi n y) + \sum_{(0',n')} \widehat{\mathcal{P}_s f_s}(0',n')\cos\left(\frac{0'\pi x}{\sqrt{3}}\right)\sin(\pi n' y)\\
&\quad = \sum_{(0,n)} \widehat{\mathcal{P}_s f_s}(0,n)\left[\sin(\pi n y) - \mathbf{2}(-1)^{n/2}\cos\left(\frac{2n\pi x}{2\sqrt{3}}\right)\sin\left(\frac{3\pi y}{2}\right)\right]\\
&\quad = \sum_{(0,n)} \widehat{\mathcal{P}_s f_s}(0,n)\left[\cos\left(\frac{\pi\cdot 0\cdot x}{\sqrt{3}}\right)\sin(\pi n y)\right.\\
&\qquad\qquad + (-1)^{(0+n)/2}\cos\left(\frac{\pi x}{2\sqrt{3}}(0+3n)\right)\sin\left(\frac{\pi m}{2}(0-n)\right)\\
&\qquad\qquad \left. -(-1)^{(0-n)/2}\cos\left(\frac{\pi x}{2\sqrt{3}}(3n-0)\right)\sin\left(\frac{\pi m}{2}(n-0)\right)\right]\\
&\quad = \sum_{(0,n)} \frac{\widehat{\mathcal{P}_s f_s}(0,n)}{2} v_{0,n}(x,y) \quad \text{by (5.4)}
\end{aligned}$$

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$$= \sum_{(0,n)} f_s^{\triangle}(0,n)\mathcal{P}_s v_{0,n}(x,y).$$

The bold factor of 2 in the second line arises from the fact that the normalization constant for $\widehat{\mathcal{P}_s f_s}(m,n)$, $m \neq 0$, is twice that of $\widehat{\mathcal{P}_s f_s}(0,n)$; see Equations (5.2) and (5.3). The last line follows from Equations (5.7) and (6.1). Once again, we have

$$\mathcal{P}_s\left[S_N^{T_1} f_s\right](x,y) = S_N^R \mathcal{P}_s f_s(x,y),$$

and we can conclude that $S_N^{T_1} f_s \to f_s$ in $L^p(T_1)$. □

7. Main results for the equilateral triangle

We now wrap up with the main results as they apply to the original equilateral triangle, T.

Theorem 4. [14] *The functions*

$$\begin{aligned} &u_{m,n}: \quad m,n = 1,2,\ldots, && m \equiv n \mod 2, \quad 0 < m < n; \\ &v_{m,n}: \quad m = 0,1,2,\ldots, n = 1,2,\ldots, && m \equiv n \mod 2, \quad 0 \leqslant m < n \end{aligned}$$

form a complete, orthogonal system on T, consisting of eigenfunctions of the Dirichlet Laplacian. Furthermore, each eigenfunction, $u_{m,n}$ or $v_{m,n}$, corresponds to the eigenvalue $\pi^2\left(\frac{m^2}{3} + n^2\right)$.

As for the L^p theory, we can combine our Theorems 2 and 3 to obtain the full result on $L^p(T)$. As before, we use the following notation to keep track of our indices in the asymmetric and symmetric parts of the sums:

$$\begin{aligned} \mathcal{U}_N &:= \{(m,n) : 0 < m < n \leqslant N \text{ and } m \equiv n \mod 2\}, \\ \mathcal{V}_N &:= \{(m,n) : 0 \leqslant m < n \text{ and } m \equiv n \mod 2\}. \end{aligned}$$

Theorem 5. *Let $f \in L^p(T)$ with $1 < p < \infty$. With the notation of the previous sections, let*

$$S_N^T f := \sum_{(m,n)\in\mathcal{U}_N} f_a^{\triangle}(m,n)u_{m,n} + \sum_{(m,n)\in\mathcal{V}_N} f_s^{\triangle}(m,n)v_{m,n}.$$

Then, $S_N^T f \to f$ in $L^p(T)$.

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Proof. For any function f on T, write its decomposition into a symmetric and antisymmetric part with respect to the y-axis:

$$f = f_a + f_s.$$

Then clearly

$$\|f_\nu\|^p_{L^p(T)} = 2\,\|f_\nu\|^p_{L^p(T_1)} \quad \text{for } \nu = a, s,$$

whence

$$\|f\|_{L^p(T)} \leqslant 2^{1/p}\left(\|f_a\|_{L^p(T_1)} + \|f_s\|_{L^p(T_1)}\right).$$

Hence, since

$$\left[S_N^T f\right]_\nu = S_N^{T_1} f_\nu \quad \text{for } \nu = a, s,$$

it follows that

$$\left\|S_N^T f - f\right\|_{L^p(T)} \leqslant 2^{1/p}\left(\left\|S_N^{T_1} f_a - f_a\right\|_{L^p(T_1)} + \left\|S_N^{T_1} f_s - f_s\right\|_{L^p(T_1)}\right) \to 0$$

by Theorems 2 and 3. □

8. Concluding remarks

Our argument made use of Práger's cunning triangle-to-rectangle transformation in order to reduce the convergence problem on the triangle to the well-known convergence on the rectangle. In a similar but more direct way, we were able to obtain these results for the 45-90-45 triangle.

There are, however, limitations to this approach, as a consequence of the following theorem due to Lamé [12] and reported as Theorem 3.1 in [13].

Theorem 6 (Lamé's Fundamental Theorem). *Suppose that $f(x, y)$ can be represented by the trigonometric series*

$$f(x, y) = \sum_i A_i \sin(\lambda_i x + \mu_i y + \alpha_i) + B_i \cos(\lambda_i x + \mu_i y + \beta_i)$$

with $\lambda_i^2 + \mu_i^2 = k^2$. Then f is antisymmetric about any line about which it vanishes.

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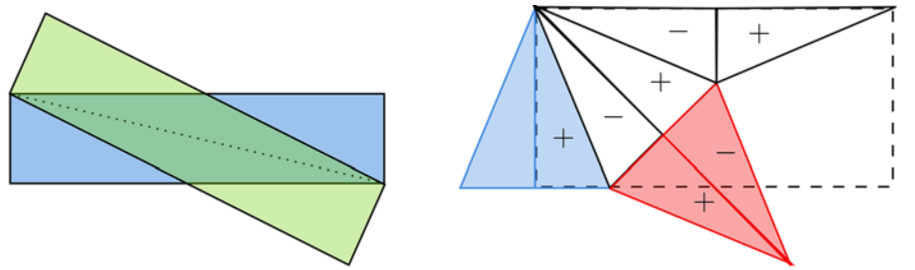

(a) In general, the diagonal is not a line of symmetry, so it cannot be a nodal line.

(b) Mimicking the Prâger construction with an angle of $\pi/2n$ ($n \neq 2, 3$) does not tile a rectangle.

Figure 5. Limitations of the triangle-to-rectangle constructions. (a) In general, the diagonal is not a line of symmetry, so it cannot be a nodal line. (b) Mimicking the Prâger construction with an angle of $\pi/2n$ ($n \neq 2, 3$) does not tile a rectangle.

The implication for our argument is that, assuming Dirichlet boundary conditions on the hypothenuse of T_1, we will generate a nodal (i.e. vanishing) line along the diagonal of the rectangle (see, e.g., Figure 2). Lamé's Theorem then requires the eigenfunctions to have a line of anti-symmetry along the diagonal. The only possibilities are symmetry along the diagonal line itself, as in the square (see Figure 5(a)) or an arrangement of smaller triangles inside the rectangle as in Prâger's construction with three hemiequilateral triangles. In this case, the upper right-angle of the rectangle is cut into three angles of $\pi/6$. Attempts to replace this decomposition by $n > 3$ triangles with angles $\pi/2n$ will not tile a rectangle (see Figure 5(b)).

These limitations are related to the Method of Images used to construct Green's functions for differential equations in Euclidean domains. The classification of admissible domains for the Method of Images goes back to [11], wherein our three 'special triangles', *viz.* equilateral, hemiequilateral and 45-90-45, are identified as the only triangular domains for which the method is applicable.

Acknowledgement. The author owes a special thanks to his doctoral advisor, Prof James C. Robinson, for many insightful discussions and his helpful criticism of early drafts. The author also wishes to thank an anonymous reviewer for pointing out the connection to the Method of Images.

Funding Statement. The author thanks the EPSRC studentship 2443915 under the project EP/V520226/1 for their support for conducting this work.

Competing interests. The author declares no conflict of interest.

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Appendix 1. Appendix: Non-convergence for circular cutoffs

Let us return to the Fourier series on the 45-90-45 triangle T from § 2. There we ordered the eigenfunctions $u_{m,n}$ by the condition $0 < m < n$ and summed in the range $n \leqslant N$. As we remarked in the introduction, a more natural choice would be to sum over all $u_{m,n}$ with eigenvalue $\lambda_{m,n} \leqslant N$. Since $\lambda_{m,n} = \pi(m^2 + n^2)$, this is essentially equivalent to summing within the quarter of the disk $m^2 + n^2 \leqslant N^2$ of radius N in the first quadrant. Proposition 1 immediately extends to show that these partial sums are related to the circular cut-offs of the traditional Fourier series on the square $[0, 1]^2$, which, as follows from Fefferman's classical result on the ball multiplier, do not in general converge in L^p.

Given a counterexample f on the square, we would need to find a function g on the triangle such that $\mathcal{P}g = f$ on $[0, 1]^2$. It is therefore clear that only a counterexample that is antisymmetric along the diagonal will serve this purpose. However, Fefferman's result is non-constructive, and we still lack an explicit example of a function on the torus whose circular partial sums do not converge in L^p. Thus, we cannot assume *a priori* that such counterexamples are available on the triangle.

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Nevertheless, we can carefully replicate Fefferman's argument, coupled with a transference result, to draw the conclusion that circular cut-offs of Fourier series on T will in general not converge in L^p. (The material in this section follows the presentation in [15]; there is a more detailed account in [8]. For a discussion of transference, see [4, 7].)

The proof requires three ingredients: (1) a transference theorem, (2) Meyer's Lemma and (3) a Besikovitch set.

The transference theorem exploits the intuition that a Fourier transform is the limit of a Fourier series with an increasingly large period. It is then argued that, since L^p convergence of Fourier series is equivalent to the uniform boundedness of the partial sum operators, this uniform bound is equivalent to the boundedness of a Fourier multiplier. For the spherical partial sums, this means the boundedness of the operator

$$T_B f(x) := \int_{|\xi|<1} \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\, \mathrm{d}\xi.$$

The 'increasing period' of the Fourier series can be seen as rescaling the square so that it 'converges' to the entire space $\mathbb{R}^2$. But we need to keep the antisymmetry constraint along the diagonal. Thus, we will be concerned with the space $L^p_-(\mathbb{R}^2)$: the L^p closure of the smooth functions f of compact support that are antisymmetric about the line $y = -x$.

Theorem 7. *(Antisymmetric Transference) Suppose the circular cut-off operators S_N^T are uniformly bounded in $N \geqslant 1$ from $L^p(T)$ to $L^p(T)$. Then, there is a constant $C_p > 0$ such that*

$$\|T_B f\|_{L^p(\mathbb{R}^2)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^2)} \quad \textit{for all } f \in L^p_-(\mathbb{R}^2). \tag{A.1}$$

This inequality is used to prove a vector-valued inequality known as Meyer's Lemma. The key idea here is that, on small enough scales, the disk at $u \in S^1$ 'looks like' the half plane $\{\xi : \xi \cdot u > 0\}$. Thus, denoting by S^u the multiplier operator of this half plane

$$S^u f(x) = \int_{\xi\cdot u>0} \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\, \mathrm{d}\xi,$$

we have our second ingredient.

Theorem 8 (Meyer's Lemma for L^p_-). *Assume Equation (A.1) holds. Then, for all $u_1, \ldots, u_k \in S^1$ and all $f_1, \ldots, f_k \in L^p_-$,*

$$\left\| \left(\sum_j |S^{u_j} f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)} \leqslant C_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)}. \tag{A.2}$$

The proof is identical to that of the original Meyer's Lemma, since it only involves rescaling the ball in Fourier space and multiplying each f_j by a complex scalar, neither of which operations alter the antisymmetry of the f_j.

https://doi.org/10.1017/S0013091523000226 Published online by Cambridge University Press

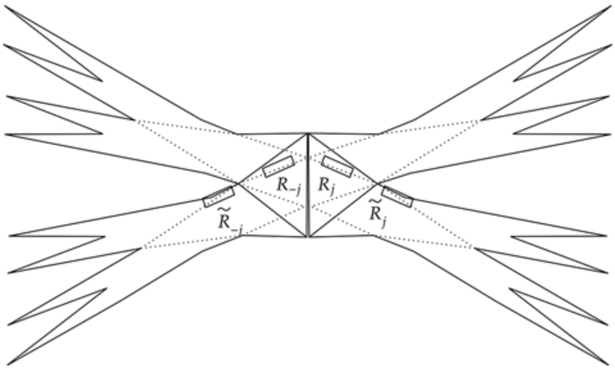

Figure A1. Duplication of the rectangles in the Besicovitch set.

The crucial next step is finding a set of functions that violate the bound (A.2). This is where the third ingredient, the Besikovitch set, comes in: a collection of congruent rectangles R_j that overlap to form a small total area, but whose translates $\widetilde{R}_j$ along the longest side remain disjoint. (We use '$\sqcup$' to denote disjoint unions.)

Theorem 9 (Besikovitch Set). *For any ε, there is a finite collection of congruent rectangles R_j such that $\left|\bigcup_j R_j\right| < \varepsilon$ but $\left|\bigsqcup_j \widetilde{R}_j\right| = 1$.*

A simple pointwise calculation shows that there is an absolute constant $c > 0$ such that

$$\left|S_{u_j}\chi_{R_j}\right| \geqslant c\chi_{\widetilde{R}_j} \quad \text{for all } j,$$

where $u_j \in S^1$ is the unit direction of the longest side of R_j, and thus the collection of 'rectangles' (more precisely: their indicator functions) violates Equation (A.2), as required.

The construction of the Besikovitch set proceeds by dissecting a triangle and shifting the resulting triangles parallel to their base so as to create overlap. Thus, all the mass lies on one side of the diagonal and we can duplicate the whole thing to create a *disjoint* reflection with respect to this line (see Figure A1, where we rotated the picture). This new figure has double the area but introduces no additional overlap. Hence, if we denote the reflection of R_j by R_{-j}, we may take

$$f_j = \chi_{R_j} - \chi_{R_{-j}} \in L^p_-, \quad \text{for all } j \geqslant 1$$

and similarly conclude that Equation (A.2), and therefore Equation (A.1), fails.

The situation for the hemi- and equilateral triangles is much more complicated, since we cannot immediately scale the rectangles in Figures 2 and 4 to $\mathbb{R}^2$. The inner rhomboid $T_2 \cup T_3 \cup T_4 \cup T_5$ can be scaled, while preserving the two diagonals as symmetry lines, but the 'fringe' triangles T_1 and T_6 would be, so to speak, 'lost at infinity'.