Second Meeting, 11th December 1896.

This meeting was postponed as the funeral of the President, Rev. John Wilson, M.A., F.R.S.E., took place that day.

Third Meeting, January 8th, 1897.

J. B. Clark, Esq., M.A., F.R.S.E., Vice-President, in the Chair.

Theorems on Normals of an Ellipse.

By Professor A. H. Anglin.

The condition that the normals at the points whose eccentric angles are α, β, γ *shall be concurrent is*

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$$

The following method of establishing this result, as compared with those given in works on *Concis*, is direct, and also has the advantage of simplicity, by first proving the Trigonometrical identity

$$\sin 2\alpha \sin(\beta-\gamma)+\ \ldots\ = 4\sin\frac{\beta-\gamma}{2}\ \ldots\ \{\sin(\beta+\gamma)+\ \ldots\ \}, \quad \text{(A)}$$

which may be simply done by multiplying both sides of the well-known identity

$$\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta) = -4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}$$

by $$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta),$$

when it will be easily found that the product in the first member reduces to that in (A) with opposite sign.

Now, writing the equation to the normal at ϕ in the form

$$ax\sin\phi - by\cos\phi = \frac{c^2}{2}\sin 2\phi$$

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the normals at a, β, γ are concurrent if (eliminating x, y as usual)

$$\begin{vmatrix} \sin 2a, & \sin a, & \cos a \\ . & . & . \\ . & . & . \end{vmatrix} = 0$$

that is, if $\sin 2a \sin(\beta - \gamma) + \ . \ . \ . \ = 0,$

which, as shown above, is equivalent to

$$\sin(\beta+\gamma) + \sin(\gamma + a) + \sin(a+\beta) = 0.$$

2. *The normals at the angular points of a maximum triangle in (and so at the points of contact of a minimum triangle about) an ellipse are concurrent.*

This may be shown without reference to the above *reduced* condition for concurrent normals.

For the condition

$$\sin 2a \sin(\beta - \gamma) + \ . \ . \ = 0$$

for concurrent normals follows *at once* from their equations; and at the angular points of a maximum inscribed triangle this condition is satisfied, for at these points we have

$$\beta - a = \gamma - \beta = \frac{2\pi}{3};$$

$$\therefore \quad \gamma - a = \frac{4\pi}{3}, \quad \text{and} \quad \gamma + a = 2\beta.$$

Hence

$$\sin 2a \sin(\beta - \gamma) + \ . \ .$$

$$= -\sin\frac{\pi}{3}(\sin 2a + \sin 2\beta + \sin 2\gamma)$$

$$= -\sin\frac{\pi}{3}(\sin 2\beta - \sin\overline{\gamma + a}) = 0,$$

and thus the normals are concurrent.

3. *To find the area of the triangle formed by the three normals to an ellipse.*

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In the case of the triangle formed by three lines whose equations are of the form $ax+by+c=0$,

$$2\Delta=(a_1b_2c_3)^2/\Pi(a_1b_2)$$
$$=[c_1(a_2b_3-a_3b_2)+\ .\ .\ ]^2/(a_2b_3-a_3b_2)\ .\ .$$

Hence, for the normals whose equations are of the form

$$ax\sin\phi-by\cos\phi=\frac{c^2}{2}\sin2\phi,$$

we have

$$2\Delta=\left[\frac{c^2ab}{2}\{\sin2\alpha\sin(\beta-\gamma)+\ .\ .\ \}\right]^2\Big/a^3b^3\sin(\beta-\gamma)\ .\ .$$
$$=\frac{c^4}{4ab}\{\sin2\alpha\sin(\beta-\gamma)+\ .\ .\ \}^2/\sin(\beta-\gamma)\ .\ .$$

Now use the identity (A), and we get

$$2\Delta=\frac{c^4}{2ab}\tan\frac{\beta-\gamma}{2}\ .\ .\ \{\sin(\beta+\gamma)+\ .\ .\ \}^2,$$

the required expression.

4. *If the normal at the points whose eccentric angles are* α, β, λ, δ *be concurrent, then*

$$\alpha+\beta+\gamma+\delta=(2n+1)\pi.$$

This is merely a question in Plane Trigonometry. For the equation to the normal at ϕ is

$$ax\sec\phi-by\operatorname{cosec}\phi=c^2,$$

which is of the *form* $a\sec\phi-b\operatorname{cosec}\phi=c.$

Denoting $\tan\phi$ by t, we have

$$a\sqrt{1+t^2}-\frac{b}{t}\sqrt{1+t^2}=c$$
$$\therefore\quad(1+t^2)(at-b)^2=c^2t^2,$$

an equation of the fourth degree in t, the roots of which are the tans. of the angles α, β, γ, δ. And since the coefficients of t and t^3 are obviously the same, we have, with the usual notation,

$$\tan(\alpha+\rho+\gamma+\delta)\equiv(s_1-s_3)/(1-s_2+s_4)=0$$
$$\therefore\quad\alpha+\beta+\gamma+\delta=n\pi,\text{ and not }(2n+1)\pi\textit{ necessarily}.$$

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Let us now denote $\tan\frac{\phi}{2}$ by t, when we get

$$a.\frac{1+t^2}{1-t^2} - b.\frac{1+t^2}{2t} = c$$

$$\therefore \quad b(t^4-1) + 2(a+c)t^3 + 2(a-c)t = 0,$$

where t^2 is absent, and $s_4 = -1$, so that $1 - s_2 + s_4 = 0$ while $s_1 - s_3$ is not $=0$. Thus

$$\tan\tfrac{1}{2}(\alpha+\beta+\gamma+\delta) = \infty$$

$$\therefore \quad \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) = n\pi + \frac{\pi}{2}$$

and $$\alpha+\beta+\gamma+\delta = (2n+1)\pi,$$

which is the correct result.

The reason why the first biquadratic does not give the exact result being that, to obtain it, we performed the operation of *squaring*; and when we square we must expect, as usual, a result of greater generality than, but not contradictory of, the actual result.