9

The $\overline{MS}$ renormalization scheme

9.1 Renormalizability and power counting rules

The notion of a superficial degree of divergences, based on the power counting rule of a given Feynman diagram, is often used for studying the renormalizability of the interactions in the Lagrangian. For instance, if we consider the previous two-point correlator $\Psi_5(q^2)$, we can see, for n-dimensions space–time, that, to lowest order, it behaves for large p^2 as:

$$\Psi_5(q^2) \sim \lim_{p\to\infty} p^{n-2} \ , \tag{9.1}$$

and its degree of divergence is:

$$d = n - 2 \ . \tag{9.2}$$

More generally, for an arbitrary Green's function $\mathcal{G}$, the superficial degree of divergence reads:

$$d = nl + \sum_v \delta_v - 2n_B - n_F \ , \tag{9.3}$$

where:

$$n = \text{space–time dimensions},$$
$$l = \text{number of loops (independent integrals)},$$
$$\delta_v = \text{number of momentum factors at the vertex } v, \tag{9.4}$$
$$n_B = \text{number of internal boson lines (we consider a theory with massless bosons)}, \tag{9.5}$$
$$n_F = \text{number of internal fermion lines}. \tag{9.6}$$

For a given interaction Lagrangian term $\mathcal{L}_\text{I}$, which one can write symbolically as:

$$\mathcal{L}_\text{I} \sim g(\partial)^\delta(\phi)^b(\psi)^f \ , \tag{9.7}$$

where ϕ and ψ are the bosonic and fermion fields, one can define the *index of divergence* of the interaction Lagrangian as:

$$r = \left(\frac{n-2}{2}\right) b + \left(\frac{n-1}{2}\right) f + \delta - n \ , \tag{9.8}$$

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where:

$$\delta = \text{number of space–time derivatives in } \mathcal{L}_{\rm I},$$
$$b = \text{number of boson fields in } \mathcal{L}_{\rm I},$$
$$f = \text{number of fermion lines in } \mathcal{L}_{\rm I}. \tag{9.9}$$

Actually, using the fact that the action:

$$\mathcal{S} = \int \mathcal{L}_{\rm I}\, d^n x \tag{9.10}$$

is dimensionless, one can deduce from a dimensional analysis that:

$$n = \dim[g] + \left(\frac{n-2}{2}\right) b + \left(\frac{n-1}{2}\right) f + \delta\ , \tag{9.11}$$

such that:

$$r = -\dim[g]\ . \tag{9.12}$$

One can define respectively by:

$$v = \text{number of vertices corresponding to } \mathcal{L}_{\rm I}^1 \text{ in the Green's function } \mathcal{G},$$
$$N_B = \text{number of external boson lines in } \mathcal{G},$$
$$N_F = \text{number of external fermion lines in } \mathcal{G}, \tag{9.13}$$

which obey the relations:

$$2n_B + N_B = vb; \qquad 2n_F + N_F = vf\ ,$$
$$l = n_B + n_F - v + 1\ , \quad \sum_v \delta_v = v\delta\ . \tag{9.14}$$

Eliminating for instance the internal fields through Eq. (9.14), one can rewrite Eq. (9.3) as:

$$d = rv - \left(\frac{n-2}{2}\right) N_B - \left(\frac{n-1}{2}\right) N_F + n\ , \tag{9.15}$$

where r is the index divergence given above. This result can be generalized to any numbers of interaction Lagrangians by the substitution:

$$rv \to \sum_i r_i v_i \tag{9.16}$$

From these definitions, one can classify the different theories as:

- If one of the r_i *is positive*, the divergences cannot be removed by any finite numbers of renormalization constants and interaction parameters. Then the theory is *not renormalizable*.
- If all $r_i \le 0$, then there is a possibility to remove the divergences by finite numbers of renormalization constants and interaction parameters. The theory is *a candidate for a renormalizable theory*.
- If $r_i < 0$ for all i, then the theory is *super renormalizable* since the number of types of divergent diagrams, and the number of diagrams are finite.

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- If $r_i = 0$, the theory is *renormalizable in a narrow sense*, which is the case of QCD. As QCD has a dimensionless coupling, then comes the conclusion from Eq. (9.12).

9.2 The QCD Lagrangian counterterms

As we have seen before, one can remove the UV divergences of a renormalizable theory by finite numbers of counterterms to any orders of perturbation theory. In QCD, the counterterms of the Langrangian are:

$$\begin{aligned}\Delta\,\mathcal{L}_{\rm QCD} = &\Delta_{3YM}\,\frac{1}{4}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\,(\partial^\mu A^\nu - \partial^\mu A^\nu)\\ &+\Delta_{1YM}\,\frac{1}{2}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\,g\vec{A}^\nu \times \vec{A}^\mu\\ &+\Delta_5\,\frac{1}{4}\,g^2(\vec{A}_\mu \times \vec{A}_\nu)\,(\vec{A}^\mu \times \vec{A}^\nu)\\ &-\Delta_{2F}\,i\sum_j \bar{\psi}_j\,\gamma^\mu\,\partial_\mu\,\psi_j + \Delta_4\sum_j m_j\,\bar{\psi}_j\,\psi_j\\ &-\Delta_{1F}\,g\,\bar{\psi}\,\frac{\lambda}{2}\,\gamma^\mu\,\psi\,\vec{A}_\mu\\ &+\Delta_6\,\frac{1}{2\alpha_G}\,(\partial_\mu\,\vec{A}^\mu)^2 + \tilde{\Delta}_3\,(\partial_\mu\,\bar{\varphi})^2 + \tilde{\Delta}_1\,g\,\partial_\mu\,\vec{\varphi}\,A^\mu \times \psi\,,\end{aligned} \tag{9.17}$$

which are all we need for removing the UV divergences of the theory. We have used the notation:

$$\vec{A}_\mu \times \vec{A}_\nu \equiv f_{abc}\,A^b_\mu\,A^c_\nu\,. \tag{9.18}$$

It is possible to rescale the fields in such a way that $\mathcal{L}_{\rm QCD}$ has the form in Eq. (5.12) but in terms of 'bare' quantities. This manipulation is correlated to the introduction of renormalization constants and then to the choices of renormalization schemes.

9.3 Dimensional renormalization

In QED, it is natural to use the *on-shell renormalization scheme*:

$$\Psi_5(q^2)_R = \Psi_5(q^2) - \Psi_5(q^2 = 0)\,, \tag{9.19}$$

for defining a renormalized Green's function, as the photon and electron are observed, and then are on their mass-shells (for a electron self-energy diagram, on can, for example, do the subtraction at $p^2 = m_e^2$), which is not the case of QCD, as quarks are off-shell due to confinement. Therefore, there is a freedom to choose the renormalizaton schemes. We shall discuss these different renormalization schemes and their relations in the following sections. t'Hooft [123] has introduced the MS (renormalization) scheme, which is specific for dimensional regularization. In this scheme, one only has to eliminate the $1/\epsilon$ poles [or in the $\overline{MS}$ scheme, the $1/\hat{\epsilon}$ poles defined in Eq. (8.44)] of the Green's functions. The renormalization constants are mass-independent and will appear as counterterms in the initial Lagrangian constrained by the Slavnov–Taylor identities [104].

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Table 9.1. *Dimensions of the couplings and fields in n dimensions*

Name	Notation	Dimension
gauge coupling	g	$\frac{1}{2}(4-n)$
quark mass	m_i	1
covariant gauge parameter	α_G	0
fermion field	$\psi_j(x)$	$\frac{1}{2}(n-1)$
gluon field	$A^a_\mu(x)$	$\frac{1}{2}(n-2)$
Faddeev–Popov field	$\varphi^a(x)$	$\frac{1}{2}(n-2)$

9.4 Renormalization constants

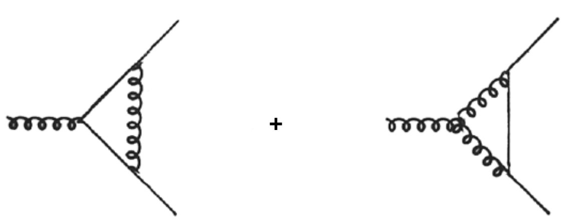
Taking into account the dimension obtained in the $4-\epsilon$ world (see Table 9.1) via the mass scale ν, one has relations between renormalized and bare parameters:

$$\begin{aligned}
g^R &= \nu^{-\epsilon/2} g^B Z_\alpha^{-1/2} \\
g^2/4\pi &\equiv \alpha_s , \\
m_j^R &= m_j^B Z_m^{-1} , \\
\alpha_G^R &= \alpha_G^B Z_G^{-1} , \\
\left(\psi_j^\alpha\right)^R &= \nu^{\epsilon/2} \left(\psi_j^\alpha\right)^B (Z_{2F})^{-1/2} , \\
\left(A_\mu^a\right)_R &= \nu^{\epsilon/2} \left(A_\mu^a\right)_B (Z_{3YM})^{-1/2} , \\
(\varphi^a)_R &= \nu^{\epsilon/2} (\varphi^a)_B (\tilde{Z}_3)^{-1/2} ,
\end{aligned} \tag{9.20}$$

where $Z_i \equiv 1-\Delta_i$. One can introduce the renormalization constant for the quark-gluon-quark vertex as:

$$(g\bar{\psi} A \psi)_R = (g_B \bar{\psi}_B A_B \psi_B) \nu^\epsilon Z_{1F}^{-1} , \tag{9.21}$$

which corresponds to the Feynman diagrams (Fig. 9.1).



Analogously, one can introduce the three-gluon renormalization constant (Z_{1YM}) corresponding to the vertex (Fig. 9.2).

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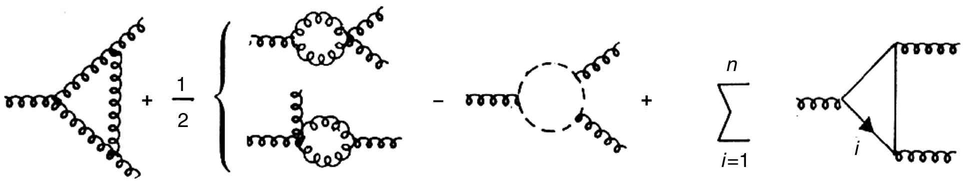
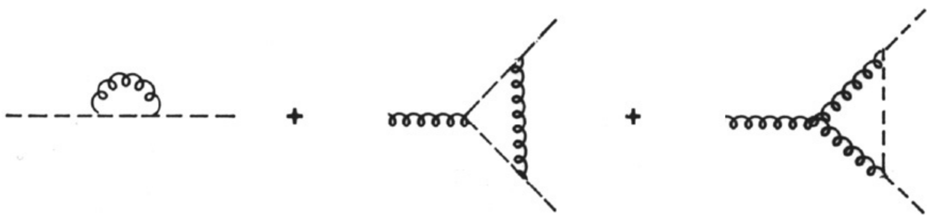


and, ($\tilde{Z}_1$) ghost self-energy and ($\tilde{Z}_3$) ghost-gluon-ghost vertex one (Fig. 9.3).



and (Z_5) four-gluon vertices one. Then, one can deduce:

$$\begin{aligned} g_B^{YM} &= Z_{1YM}\, Z_{3YM}^{-3/2}\, g_R\ , \\ \tilde{g}_B &= \tilde{Z}_1^{-1}\, Z_{3YM}^{-1/2}\, g_R\ , \\ g_B^F &= Z_{1F}\, Z_{3YM}^{-1/2}\, Z_{2F}^{-1}\, g_R\ , \\ \left(g_B^{(5)}\right)^2 &= Z_5\, Z_{3YM}^{-2}\, g_R^2\ , \end{aligned} \tag{9.22}$$

which are related to each other by BRS [103] invariance:

$$g_B^{YM} = \ldots\ldots = g_{(5)}^B\ , \tag{9.23}$$

leading to the Slavnov–Taylor [104] identities:

$$\begin{aligned} Z_{3YM}/Z_{1YM} &= \tilde{Z}_3/\tilde{Z}_1 = Z_{2F}/Z_{1F}\ , \\ Z_5 &= Z_{1YM}^2/Z_{3YM}\ . \end{aligned} \tag{9.24}$$

This is the analogue of the QED relation:

$$Z_{1F} = Z_2\ . \tag{9.25}$$

The mass renormalization constant is:

$$m_B = \left(Z_m \equiv Z_4\, Z_{2F}^{-1}\right) m_R\ , \tag{9.26}$$

and the gauge one is:

$$\alpha_G^B = \alpha_G^R\, Z_G^{-1}\, Z_{3YM}\ . \tag{9.27}$$

Z_{3YM} comes from the evaluation of the gluon propagator (Fig. 9.4).

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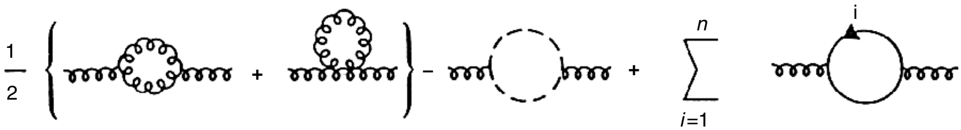

Z_{2F} and Z_m come from the quark self-energy diagram, which can be parametrized as:

$$\Sigma = m_B \Sigma_1 + (\hat{p} - m_B)\Sigma_2 \ , \tag{9.28}$$

and leads to:

$$Z_{2F} \equiv \frac{1}{1 - \Sigma_2|_{\text{pole}}} \ , \qquad Z_m = 1 - \Sigma_1|_{\text{pole}} \ , \tag{9.29}$$

More generally, for a Green's function with N_G, N_{FP} and N_F external gluons, ghost and fermion fields, one can associate the renormalization constants:

$$Z_\Gamma = \left(Z_{3YM}^{1/2}\right)^{-N_G} \left(Z_3^{1/2}\right)^{-N_{FP}} \left(Z_{2F}^{1/2}\right)^{-N_F} . \tag{9.30}$$

Expressions of these renormalization constants are known from standard diagram techniques (see Table 11.1).

9.5 Check of the renormalizability of QCD

We are now in a position to check the renormalizability of QCD. We want to see if the counterterms presented in Eq. (9.17) are sufficient for removing all divergences in Feynman integrals to all orders.

If one looks at the superficial degree of divergences for the Feynman diagrams given in Eq. (9.15), and using the fact in Eq. (9.12), we can see for QCD in four dimensions:

$$d = 4 - N_B - \frac{3}{2} N_F , \tag{9.31}$$

for N_B and N_F external lines of bosons and fermions. Here, N_B includes gluons N_G and Faddeev–Popov N_{FP} ghosts. Remarking that the coupling in the ghost-gluon-ghost vertex behaves like k_μ (see Appendix E), the number of boson fields become:

$$N_B = N_G + N_{FP} + \frac{1}{2} N_{FP} \ . \tag{9.32}$$

It is easy to see that the condition $d \geq 0$ for a superficially divergent integral is obtained for seven different cases of the set $(N_F,\ N_G,\ N_{FP})$ discarding the case (0, 0, 0) (vacuum)

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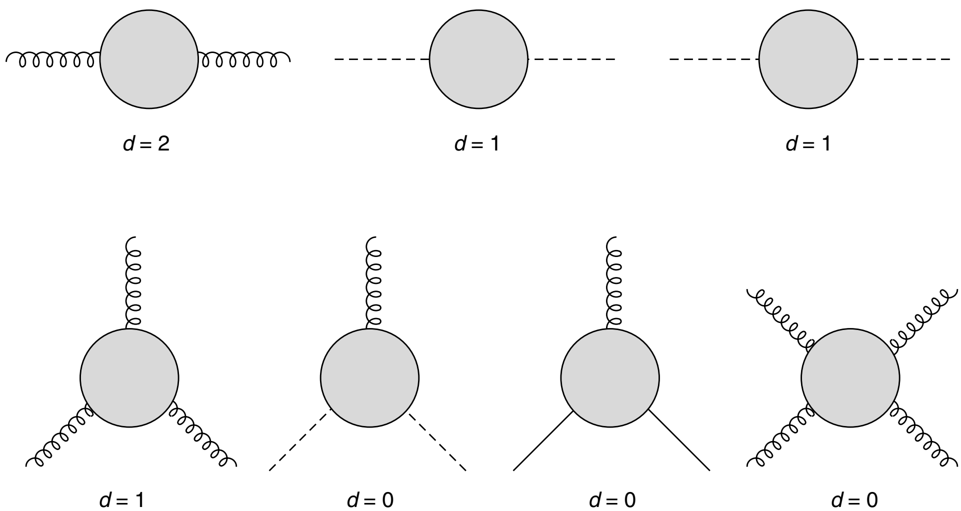
and the one (0, 1, 0) because of Lorentz invariance. These seven diagrams are displayed in Fig. (9.5):



and have the same structure as the counterterms. It is an easy exercise to show that these divergences can all be absorbed by the counterterms. One should also notice that owing to gauge and Lorentz invariances, the apparent degree of divergence 2, 1, 1, 1 of the self-energies of gluons, ghost, fermions, and of the three-gluon vertex become logarithmic. These features have explicitly shown the renormalizability of QCD, which is maintained to all orders of perturbative QCD [113,108,125,104,103].

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