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STRONG SYSTEM EQUIVALENCE (I)

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Abstract

Strong system equivalence is defined for polynomial realizations of a rational matrix. It is shown that any polynomial realization is strongly system equivalent to a generalized state-space realization, and two generalized state-space realizations are strongly system equivalent if and only if they are constant system equivalent.

1. Introduction

The important concept of (strict) *system equivalence* of polynomial realizations in linear systems theory was introduced by Rosenbrock [11] and has been further studied by Fuhrmann [6], Pernebo [10] and Rosenbrock [12]. Coppel [3] has shown that the theory of polynomial realizations may be extended to realizations over an arbitrary principal ideal domain and has pointed out in [4] that system equivalence may also be defined in this more general setting. The present work gives a significant application of this generalization within systems theory itself.

It is a commonplace in complex analysis that rational functions should be studied not only in the finite plane but also "at infinity". Since the transfer matrix of a (finite-dimensional, time-invariant) linear system is a matrix of rational functions, it is natural to study also the behaviour of the system at infinity. Indeed, it is essential if one is interested in impulsive, or distributional, solutions. This point of view has been most extensively pursued by Verghese [14]. (Less complete accounts have appeared in [15–19]. Some other works in this area are [1], [2], [13].) Verghese defines strong controllability to mean controllability in the ordinary sense together with "controllability at infinity", and likewise strong

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observability, and derives a number of basic properties. He defines also a concept of strong equivalence for generalized state-space systems, but leaves it as an open problem to define an appropriate concept of strong system equivalence for arbitrary polynomial realizations. This is the problem that will be considered here.

We define a notion of *system equivalence at infinity*, and in Propositions 1 and 2 show that it can be expressed in ways analogous to the definitions by Fuhrmann and Rosenbrock of ordinary system equivalence. We also define *strong system equivalence* to mean ordinary system equivalence together with system equivalence at infinity, and show that it preserves strong controllability and strong observability. In Theorem 1 we show that any polynomial realization is strongly system equivalent to a generalized state-space realization, and in Theorem 2 that two generalized state-space realizations are strongly system equivalent if and only if they are constant system equivalent. Here constant system equivalence is a convenient reformulation of Verghese's strong equivalence. The paper also contains several other results concerning the behaviour at infinity of linear systems.

A generalized state-space realization is a *linearisation* of a matrix of rational functions. Our theory of strong system equivalence makes it possible to replace an arbitrary polynomial realization by a linearisation which is intrinsically connected with it and shares its essential properties.

2. System equivalence at infinity

Let K be an arbitrary field and let $K(s)$ denote the field of rational functions with coefficients from K. For example, K may be the field of real numbers, the field of complex numbers, or the field with two elements: 0 and 1. A rational function $r(s)$ in $K(s)$ has the form $r(s) = p(s)/q(s)$, where $p(s)$ and $q(s)$ are coprime polynomials. The rational function $r(s)$ is said to be *causal* if the degree of $p(s)$ does not exceed the degree of $q(s)$ and *strictly causal* if the degree of $p(s)$ is actually less than the degree of $q(s)$. The set H of all causal rational functions is a principal ideal domain; in fact the proper ideals in H are $s^{-1}H$, $s^{-2}H$, $s^{-3}H, \ldots$. Since this principal ideal domain is the valuation ring of the field $K(s)$ associated with the degree valuation, or valuation at infinity, it is natural to suspect that it is the appropriate setting for studying the behaviour at infinity of a linear system. This suspicion is confirmed by the results we will establish. [The use of polynomial matrices to study system equivalence at infinity on the other hand raises the difficulty that the individual polynomial matrices in a polynomial realization all have a pole structure at infinity, and it is necessary to disentangle some of this structure if one wants to talk about the structure at infinity of the whole realization.]

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A $p \times m$ matrix of rational functions will be said to be causal if its entries are all causal, and strictly causal if its entries are all strictly causal. An $m \times m$ matrix of rational functions will be said to be *bicausal* if it is causal and has a causal inverse. Similarly an $m \times m$ matrix of polynomials will be said to be *bipolynomial* if it has a polynomial inverse.

A *causal realization* of a $p \times m$ rational matrix R is a representation of the form

$$R = \mathscr{W} + \mathscr{V}\mathscr{T}^{-1}\mathscr{U}, \tag{1}$$

where $\mathscr{T}$, $\mathscr{U}$, $\mathscr{V}$, $\mathscr{W}$ are $n \times n$, $n \times m$, $p \times n$, $p \times m$ causal matrices and $\mathscr{T}$ is non-singular. The positive integer n may vary from one realization to another. The realization is (causally) *controllable* if the causal matrices $\mathscr{T}$ and $\mathscr{U}$ are left coprime, and (causally) *observable* if the causal matrices $\mathscr{T}$ and $\mathscr{V}$ are right coprime. The realization is (causally) *irreducible* if it is both controllable and observable. Two causal realizations

$$R = \mathscr{W}_1 + \mathscr{V}_1\mathscr{T}_1^{-1}\mathscr{U}_1 = \mathscr{W}_2 + \mathscr{V}_2\mathscr{T}_2^{-1}\mathscr{U}_2 \tag{2}$$

of the same rational matrix are said to be (causally) *system equivalent* if there exist bicausal matrices $\bar{\mathscr{M}}$, $\bar{\mathscr{N}}$ and causal matrices $\bar{\mathscr{X}}$, $\bar{\mathscr{Y}}$ such that

$$\begin{bmatrix} \bar{\mathscr{M}} & 0 \\ \bar{\mathscr{X}} & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & -\mathscr{T}_1 & \mathscr{U}_1 \\ 0 & \mathscr{V}_1 & \mathscr{W}_1 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & -\mathscr{T}_2 & \mathscr{U}_2 \\ 0 & \mathscr{V}_2 & \mathscr{W}_2 \end{bmatrix} \begin{bmatrix} \bar{\mathscr{N}} & \bar{\mathscr{Y}} \\ 0 & I \end{bmatrix}, \tag{3}$$

where the adjoined unit matrices may be of any compatible dimensions. Otherwise stated, the causal realizations (2) are system equivalent if and only if there exist causal matrices $\mathscr{M}$, $\mathscr{N}$, $\mathscr{X}$, $\mathscr{Y}$ with $\mathscr{M}$ and $\mathscr{T}_2$ left coprime, and $\mathscr{N}$ and $\mathscr{T}_1$ right coprime, such that

$$\begin{bmatrix} \mathscr{M} & 0 \\ \mathscr{X} & I \end{bmatrix} \begin{bmatrix} -\mathscr{T}_1 & \mathscr{U}_1 \\ \mathscr{V}_1 & \mathscr{W}_1 \end{bmatrix} = \begin{bmatrix} -\mathscr{T}_2 & \mathscr{U}_2 \\ \mathscr{V}_2 & \mathscr{W}_2 \end{bmatrix} \begin{bmatrix} \mathscr{N} & \mathscr{Y} \\ 0 & I \end{bmatrix}. \tag{4}$$

By replacing "causal" by "polynomial" throughout, we recover the usual definitions of polynomial systems theory. The properties of causal realizations which are completely analogous to those of polynomial realizations will be used without special comment below.

To define system equivalence at infinity of polynomial realizations, we cannot simply operate with causal matrices in the above way on their Rosenbrock system matrices. Two preliminary steps are required. First the polynomial realization is replaced by a normalised one for which U and V are constant matrices and $W = 0$. (This procedure was also used by Verghese.) Secondly we associate with this normalized realization a causal realization. Two polynomial realizations will then be defined to be system equivalent at infinity, if their associated causal realizations are causally system equivalent. Subsequently we will show that this

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definition can be given forms similar to the definitions of ordinary system equivalence due to Fuhrmann and Rosenbrock.

Now let the $p \times m$ rational matrix R have the polynomial realization

$$R = W + VT^{-1}U. \tag{5}$$

With this realization we associate not only Rosenbrock's *system matrix*

$$P = \begin{bmatrix} -T & U \\ V & W \end{bmatrix} \tag{6}$$

but also the *extended system matrix*

$$Q = \begin{bmatrix} -T & U & 0 \\ V & W & -I_p \\ 0 & I_m & 0 \end{bmatrix}. \tag{7}$$

It is easily verified that Q is non-singular, with inverse

$$Q^{-1} = \begin{bmatrix} -T^{-1} & 0 & T^{-1}U \\ 0 & 0 & I_m \\ -VT^{-1} & -I_p & R \end{bmatrix}. \tag{8}$$

Hence

$$R = \mathscr{C}Q^{-1}\mathscr{B}, \tag{9}$$

where

$$\mathscr{B} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}, \qquad \mathscr{C} = \begin{bmatrix} 0 & 0 & I_p \end{bmatrix}. \tag{10}$$

Let Q^{-1} have the *irreducible* causal realization

$$Q^{-1} = \mathscr{W} + \mathscr{V}\mathscr{T}^{-1}\mathscr{U}.$$

Then R has the *induced* causal realization

$$R = \mathscr{C}\mathscr{W}\mathscr{B} + (\mathscr{C}\mathscr{V})\mathscr{T}^{-1}(\mathscr{U}\mathscr{B}).$$

The following definitions are basic for the present work.

DEFINITION 1. Two polynomial realizations

$$R = W_1 + V_1T_1^{-1}U_1 = W_2 + V_2T_2^{-1}U_2 \tag{11}$$

of a rational matrix R are said to be *system equivalent at infinity* if the corresponding induced causal realizations are causally system equivalent.

This definition does not depend on the choice of irreducible causal realization of Q^{-1}. For any two irreducible causal realizations of Q^{-1} are causally system

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equivalent, and a simple calculation then shows that the corresponding induced causal realizations of R are also causally system equivalent.

DEFINITION 2. Two polynomial realizations (11) of a rational matrix R are said to be *strongly system equivalent* if they are both (polynomially) system equivalent and system equivalent at infinity.

The object of this paper is to show that these definitions adequately solve the problem of Verghese, mentioned in the Introduction. We show first that the polynomial realizations (5) and (9) are strongly system equivalent.

LEMMA 1. *Let R be a $p \times m$ rational matrix with the polynomial realization* (5). *Then the polynomial realization* (9), *where Q is the extended system matrix* (7) *and $\mathscr{B}$ and $\mathscr{C}$ are defined by* (10), *is strongly system equivalent to the given realization* (5).

PROOF. The realizations are system equivalent, since

$$\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & U & 0 \\ 0 & -I & W & 1 \end{bmatrix}\begin{bmatrix} T & -U & 0 & 0 \\ -V & -W & I & 0 \\ 0 & -I & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -T & U \\ 0 & 0 & V & W \end{bmatrix}\begin{bmatrix} 0 & -I & 0 & I \\ -V & -W & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{12}$$

Let

$$\bar{Q} = \begin{bmatrix} -Q & \mathscr{B} & 0 \\ \mathscr{C} & 0 & -I_p \\ 0 & I_m & 0 \end{bmatrix}$$

be the extended system matrix of the realization (9). If Q^{-1} has the irreducible causal realization $Q^{-1} = \mathscr{T}^{-1}\mathscr{U}$, then $\bar{Q}^{-1}$ has the irreducible causal realization

$$\bar{Q}^{-1} = \bar{\mathscr{T}}^{-1}\bar{\mathscr{U}} = \begin{bmatrix} -\mathscr{T} & \mathscr{U}\mathscr{B} & 0 \\ \mathscr{C} & 0 & -I_p \\ 0 & I_m & 0 \end{bmatrix}^{-1}\begin{bmatrix} \mathscr{U} & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

We wish to show that the induced causal realizations

$$R = \mathscr{C} \cdot \mathscr{T}^{-1} \cdot \mathscr{U}\mathscr{B} = \bar{\mathscr{C}} \cdot \bar{\mathscr{T}}^{-1} \cdot \bar{\mathscr{U}}\bar{\mathscr{B}},$$

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where $\bar{\mathscr{B}}$ and $\bar{\mathscr{C}}$ are defined analogously to $\mathscr{B}$ and $\mathscr{C}$, are causally system equivalent. But, since

$$\begin{bmatrix} -\bar{\mathscr{T}} & \overline{\mathscr{U}\mathscr{B}} \\ \bar{\mathscr{C}} & 0 \end{bmatrix} = \begin{bmatrix} \mathscr{T} & -\mathscr{U}\mathscr{B} & 0 & 0 \\ -\mathscr{C} & 0 & I_p & 0 \\ 0 & -I_m & 0 & I_m \\ 0 & 0 & I_p & 0 \end{bmatrix},$$

this follows by a similar computation to that in the first part of the proof.

The remaining results of this section characterize system equivalence at infinity directly in terms of the given polynomial realizations, without reference to induced causal realizations.

PROPOSITION 1. *Two polynomial realizations*

$$R = W_1 + V_1T_1^{-1}U_1 = W_2 + V_2T_2^{-1}U_2 \tag{11}$$

of a rational matrix R, *with extended system matrices* Q_1 *and* Q_2, *are system equivalent at infinity if and only if there exist causal rational matrices* $\mathscr{M}$, $\mathscr{N}$, $\mathscr{X}$, $\mathscr{Y}$ *such that* $[\mathscr{M}Q_2]$ *has a causal right inverse*, $[{\mathscr{N} \atop Q_1}]$ *has a causal left inverse, and*

$$\begin{bmatrix} \mathscr{M} & 0 \\ \mathscr{X} & I \end{bmatrix}\begin{bmatrix} -Q_1 & \mathscr{B}_1 \\ \mathscr{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} -Q_2 & \mathscr{B}_2 \\ \mathscr{C}_2 & 0 \end{bmatrix}\begin{bmatrix} \mathscr{N} & \mathscr{Y} \\ 0 & I \end{bmatrix}, \tag{13}$$

where the common value of both sides is causal.

PROOF. Let

$$Q_1^{-1} = \mathscr{T}_1^{-1}\mathscr{U}_1, \qquad Q_2^{-1} = \mathscr{V}_2\mathscr{T}_2^{-1}$$

be irreducible causal realizations. If the realizations (11) are system equivalent at infinity there exist causal matrices $\bar{\mathscr{M}}$, $\bar{\mathscr{N}}$, $\bar{\mathscr{X}}$, $\bar{\mathscr{Y}}$, such that $\bar{\mathscr{M}}$ and $\mathscr{T}_2$ are left coprime, $\bar{\mathscr{N}}$ and $\mathscr{T}_1$ are right coprime, and

$$\begin{bmatrix} \bar{\mathscr{M}} & 0 \\ \bar{\mathscr{X}} & I \end{bmatrix}\begin{bmatrix} -\mathscr{T}_1 & \mathscr{U}_1\mathscr{B}_1 \\ \mathscr{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} -\mathscr{T}_2 & \mathscr{B}_2 \\ \mathscr{C}_2\mathscr{V}_2 & 0 \end{bmatrix}\begin{bmatrix} \bar{\mathscr{N}} & \bar{\mathscr{Y}} \\ 0 & I \end{bmatrix}. \tag{14}$$

If we set

$$\begin{aligned} \mathscr{M} &= \bar{\mathscr{M}}\mathscr{U}_1, & \mathscr{N} &= \mathscr{V}_2\bar{\mathscr{N}}, \\ \mathscr{X} &= \bar{\mathscr{X}}\mathscr{U}_1, & \mathscr{Y} &= \mathscr{V}_2\bar{\mathscr{Y}}, \end{aligned} \tag{15}$$

then (13) holds. Since $\mathscr{T}_1$ and $\mathscr{U}_1$ are left coprime, there exist causal matrices $\mathscr{F}_1$, $\mathscr{G}_1$ such that

$$\mathscr{T}_1\mathscr{F}_1 + \mathscr{U}_1\mathscr{G}_1 = I.$$

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Then

$$\mathcal{M}\mathcal{G}_1 + Q_2\mathcal{N}\mathcal{F}_1 = \mathcal{M}\mathcal{U}_1^{-1} - \mathcal{M}Q_1\mathcal{F}_1 + Q_2\mathcal{N}\mathcal{F}_1$$
$$= \overline{\mathcal{M}}.$$

On the other hand, since $\overline{\mathcal{M}}$ and $\mathcal{T}_2$ are left coprime, there exist causal matrices $\mathcal{F}_2$, $\mathcal{G}_2$ such that

$$\overline{\mathcal{M}}\mathcal{F}_2 + \mathcal{T}_2\mathcal{G}_2 = I.$$

Then

$$\mathcal{M}\mathcal{F} + Q_2\mathcal{G} = I,$$

where

$$\mathcal{F} = \mathcal{G}_1\mathcal{F}_2, \qquad \mathcal{G} = \mathcal{N}\mathcal{F}_1\mathcal{F}_2 + \mathcal{V}_2\mathcal{G}_2.$$

Thus $[\mathcal{M}Q_2]$ has a causal right inverse, and similarly $[{}^{\mathcal{N}}_{Q_1}]$ has a causal left inverse.

Conversely, suppose there exist causal matrices $\mathcal{M}$, $\mathcal{N}$, $\mathcal{X}$, $\mathcal{Y}$ with the properties in the statement of the Proposition. If we define $\overline{\mathcal{M}}$, $\overline{\mathcal{N}}$, $\overline{\mathcal{X}}$, $\overline{\mathcal{Y}}$ by (15), then (14) holds. Moreover $\overline{\mathcal{M}} = \mathcal{M}\mathcal{U}_1^{-1}$ is causal, since $\mathcal{M}Q_1$ is causal and the realization $Q_1 = \mathcal{U}_1^{-1}\mathcal{T}_1$ is irreducible. Similarly $\overline{\mathcal{X}}$, $\overline{\mathcal{N}}$, and $\overline{\mathcal{Y}}$ are causal. There exist causal matrices $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{M}\mathcal{F} + Q_2\mathcal{G} = I$. Thus $Q_2\mathcal{G}$ is causal, and hence $\mathcal{G} = \mathcal{V}_2\overline{\mathcal{G}}$ for some causal $\overline{\mathcal{G}}$. Then

$$\overline{\mathcal{M}}\mathcal{U}_1\mathcal{F} + \mathcal{T}_2\overline{\mathcal{G}} = I,$$

which shows that $\overline{\mathcal{M}}$ and $\mathcal{T}_2$ are left coprime. Similarly $\overline{\mathcal{N}}$ and $\mathcal{T}_1$ are right coprime. Consequently, the realizations (11) are system equivalent at infinity.

Proposition 1 characterizes system equivalence at infinity in the manner of Fuhrman's definition of ordinary system equivalence. We will now characterize system equivalence at infinity in the manner of Rosenbrock's definition of system equivalence.

PROPOSITION 2. *Two polynomial realizations* (11) *of a rational matrix R are system equivalent at infinity if and only if there exist causal matrices $\mathcal{M}$, $\mathcal{N}$, $\mathcal{X}$, $\mathcal{Y}$ with $\mathcal{M}$ and $\mathcal{N}$ non-singular such that*

(i)

$$\begin{bmatrix} \mathcal{M} & 0 \\ \mathcal{X}\mathcal{M} & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & -Q_1 & \mathcal{B}_1 \\ 0 & \mathcal{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & -Q_2 & \mathcal{B}_2 \\ 0 & \mathcal{C}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{N} & \mathcal{N}\mathcal{Y} \\ 0 & I \end{bmatrix}$$

where Q_i, $\mathcal{B}_i$ and $\mathcal{C}_i$ are defined as in (7) *and* (10),

(ii)

$$\mathcal{G}\begin{bmatrix} I & 0 \\ 0 & -Q_1 \end{bmatrix} \text{ is causal for some causal } \mathcal{G} \text{ if and only if } \mathcal{G} = \overline{\mathcal{G}}\mathcal{M} \text{ for some causal } \overline{\mathcal{G}},$$

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(iii)

$$\begin{bmatrix} I & 0 \\ 0 & -Q_2 \end{bmatrix} \mathscr{K} \quad \textit{is causal for some causal } \mathscr{K} \textit{ if and only if } \mathscr{K} = \mathscr{N}\bar{\mathscr{K}} \textit{ for some causal } \bar{\mathscr{K}}.$$

PROOF. Again, let

$$Q_1^{-1} = \mathscr{T}_1^{-1}\mathscr{U}_1, \qquad Q_2^{-1} = \mathscr{V}_2\mathscr{T}_2^{-1}$$

be irreducible causal realizations. If the given polynomial realizations (11) are system equivalent at infinity there exist bicausal matrices $\bar{\mathscr{M}}$, $\bar{\mathscr{N}}$ and causal matrices $\bar{\mathscr{X}}$, $\bar{\mathscr{Y}}$ such that

$$\begin{bmatrix} \bar{\mathscr{M}} & 0 \\ \bar{\mathscr{X}} & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & -\mathscr{T}_1 & \mathscr{U}_1\mathscr{B}_1 \\ 0 & \mathscr{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & -\mathscr{T}_2 & \mathscr{B}_2 \\ 0 & \mathscr{C}_2\mathscr{V}_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathscr{N}} & \bar{\mathscr{Y}} \\ 0 & I \end{bmatrix}. \tag{16}$$

If we set

$$\begin{aligned} &\mathscr{M} = \bar{\mathscr{M}} \begin{bmatrix} I & 0 \\ 0 & \mathscr{U}_1 \end{bmatrix}, \qquad \mathscr{N} = \begin{bmatrix} I & 0 \\ 0 & \mathscr{V}_2 \end{bmatrix} \bar{\mathscr{N}}, \\ &\mathscr{X} = \bar{\mathscr{X}}\bar{\mathscr{M}}^{-1}, \qquad \mathscr{Y} = \bar{\mathscr{N}}^{-1}\bar{\mathscr{Y}}, \end{aligned} \tag{17}$$

then $\mathscr{M}$, $\mathscr{N}$, $\mathscr{X}$, $\mathscr{Y}$ are causal, with $\mathscr{M}$ and $\mathscr{N}$ non-singular, and (i) holds. Moreover

$$\mathscr{M} \begin{bmatrix} I & 0 \\ 0 & -Q_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 \\ 0 & -Q_2 \end{bmatrix} \mathscr{N}$$

are causal. Since the causal realization

$$\begin{bmatrix} I & 0 \\ 0 & -Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathscr{U}_1 \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & -\mathscr{T}_1 \end{bmatrix} \tag{18}$$

is irreducible, $\mathscr{G}\left[\begin{smallmatrix} I & 0 \\ 0 & -Q_1 \end{smallmatrix}\right]$ causal for some causal $\mathscr{G}$ implies $\mathscr{G} = \mathscr{G}_1\left[\begin{smallmatrix} I & 0 \\ 0 & -\mathscr{U}_1 \end{smallmatrix}\right]$ for some causal $\mathscr{G}_1$ and hence $\mathscr{G} = \bar{\mathscr{G}}\mathscr{M}$, where $\bar{\mathscr{G}} = \mathscr{G}_1\bar{\mathscr{M}}^{-1}$ is causal. This proves (ii), and the proof of (iii) is analogous.

Conversely, suppose that there exist causal $\mathscr{M}$, $\mathscr{N}$, $\mathscr{X}$, $\mathscr{Y}$ with $\mathscr{M}$ and $\mathscr{N}$ non-singular such that (ii)–(iii) hold. If we define $\bar{\mathscr{M}}$, $\bar{\mathscr{N}}$, $\bar{\mathscr{X}}$, $\bar{\mathscr{Y}}$ by (17), then (16) holds. Furthermore, since the realization (18) is irreducible, (ii) implies that $\bar{\mathscr{M}}$ is bicausal. Similarly, (iii) implies that $\bar{\mathscr{N}}$ is bicausal. Then $\bar{\mathscr{X}}$ and $\bar{\mathscr{Y}}$ are causal, and (16) shows that the polynomial realizations (11) are system equivalent at infinity.

It is readily seen that if (i) holds, condition (ii) in the statement of Proposition 2 is equivalent to requiring the causal matrix $\mathscr{M}\left[\begin{smallmatrix} I & 0 \\ 0 & -Q_1 \end{smallmatrix}\right]$ to have the same zero structure at infinity as the polynomial matrix Q_1, and condition (iii) is equivalent to requiring $\left[\begin{smallmatrix} I & 0 \\ 0 & -Q_2 \end{smallmatrix}\right]\mathscr{N}$ to have the same zero structure at infinity as Q_2. Our original derivation of these results was in precisely the reverse order to that adopted here. By mapping the point at infinity to a finite point at which Q_1 and Q_2 were non-singular, and by imposing natural requirements at this finite point

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for a suitable definition of system equivalence at infinity, we were led to the conditions of Proposition 2 in the form just stated.

A simpler result in the style of Rosenbrock is the following

PROPOSITION 3. *Two polynomial realizations* (11) *of a rational matrix R are system equivalent at infinity if there exist bicausal matrices $\mathscr{M}$, $\mathscr{N}$ and causal matrices $\mathscr{X}$, $\mathscr{Y}$ such that*

$$\begin{bmatrix} \mathscr{M} & 0 \\ \mathscr{X} & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & -T_1 & U_1 \\ 0 & V_1 & W_1 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & -T_2 & U_2 \\ 0 & V_2 & W_2 \end{bmatrix} \begin{bmatrix} \mathscr{N} & \mathscr{Y} \\ 0 & I \end{bmatrix}.$$

PROOF. We have

$$\overline{\mathscr{M}} \begin{bmatrix} I & 0 \\ 0 & Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Q_2 \end{bmatrix} \overline{\mathscr{N}},$$

where

$$\overline{\mathscr{M}} = \begin{bmatrix} \mathscr{M} & 0 & 0 \\ \mathscr{X} & I & 0 \\ 0 & 0 & I_m \end{bmatrix}, \qquad \overline{\mathscr{N}} = \begin{bmatrix} \mathscr{N} & \mathscr{Y} & 0 \\ 0 & I & 0 \\ 0 & 0 & I_p \end{bmatrix}$$

are bicausal rational matrices. Let Q_1^{-1} and Q_2^{-1} have the irreducible causal realizations

$$Q_1^{-1} = \mathscr{T}_1^{-1}\mathscr{U}_1, \qquad Q_2^{-1} = \mathscr{T}_2^{-1}\mathscr{U}_2.$$

The induced causal realizations

$$R = \mathscr{C}_1 \cdot \mathscr{T}_1^{-1} \cdot \mathscr{U}_1\mathscr{B}_1, \qquad R = \mathscr{C}_2 \cdot \mathscr{T}_2^{-1} \cdot \mathscr{U}_2\mathscr{B}_2$$

are respectively causally system equivalent to the realizations

$$R = [0 \quad \mathscr{C}_1] \cdot \begin{bmatrix} I & 0 \\ 0 & \mathscr{T}_1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} I & 0 \\ 0 & \mathscr{U}_1 \end{bmatrix} \begin{bmatrix} 0 \\ \mathscr{B}_1 \end{bmatrix},$$

$$R = [0 \quad \mathscr{C}_2] \cdot \begin{bmatrix} I & 0 \\ 0 & \mathscr{T}_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} I & 0 \\ 0 & \mathscr{U}_2 \end{bmatrix} \begin{bmatrix} 0 \\ \mathscr{B}_2 \end{bmatrix}.$$

Moreover, since

$$[0 \quad \mathscr{C}_1]\overline{\mathscr{N}} = [0 \quad \mathscr{C}_1],$$

$$\overline{\mathscr{M}} \begin{bmatrix} 0 \\ \mathscr{B}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathscr{B}_1 \end{bmatrix},$$

the first is causally system equivalent to the realization

$$R = [0 \quad \mathscr{C}_1] \cdot \left\{ \overline{\mathscr{M}} \begin{bmatrix} I & 0 \\ 0 & \mathscr{T}_1 \end{bmatrix} \overline{\mathscr{N}}^{-1} \right\}^{-1} \cdot \overline{\mathscr{M}} \begin{bmatrix} I & 0 \\ 0 & \mathscr{U}_1 \end{bmatrix} \overline{\mathscr{M}}^{-1} \begin{bmatrix} 0 \\ \mathscr{B}_1 \end{bmatrix}.$$

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But the causal realizations

$$\begin{bmatrix} I & 0 \\ 0 & Q_2^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathscr{T}_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} I & 0 \\ 0 & \mathscr{U}_2 \end{bmatrix}$$

$$= \left\{ \bar{\mathscr{M}} \begin{bmatrix} I & 0 \\ 0 & \mathscr{T}_1 \end{bmatrix} \bar{\mathscr{N}}^{-1} \right\}^{-1} \cdot \bar{\mathscr{M}} \begin{bmatrix} I & 0 \\ 0 & \mathscr{U}_1 \end{bmatrix} \bar{\mathscr{M}}^{-1}$$

are both irreducible and hence causally system equivalent. It follows that the original two induced causal realizations of R are also causally system equivalent.

Proposition 3 provides a sufficient condition for system equivalence at infinity. However, this condition is not also necessary. For example, the zero matrix has the polynomial realization

$$0 = 0 + 0 \cdot (sI_n)^{-1} \cdot 0$$

for any positive integer n. The corresponding extended system matrix Q has a causal inverse, and the induced causal realizations $R = \mathscr{C}Q^{-1}\mathscr{B}$ for two different values $n = n_1$, n_2 are easily seen to be causally system equivalent. But the condition of Proposition 3 is not satisfied, since there do not exist bicausal matrices $\mathscr{M}$, $\mathscr{N}$ such that

$$\mathscr{M} \begin{bmatrix} I & 0 \\ 0 & -sI_{n_1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -sI_{n_2} \end{bmatrix} \mathscr{N}.$$

This example illustrates the fact that, although system equivalence at infinity preserves the *zero* structure at infinity of Q, it need not preserve its *pole* structure at infinity.

Finally we relate realizations of a non-singular matrix to realizations of its inverse. The following result extends a well-known property of ordinary system equivalence to system equivalence at infinity.

Lemma 2. *Let R be a non-singular $m \times m$ rational matrix. Then the polynomial realizations*

$$R = W_1 + V_1 T_1^{-1} U_1 = W_2 + V_2 T_2^{-1} U_2 \tag{11}$$

are system equivalent at infinity if and only if the polynomial realizations

$$R^{-1} = [0 \quad I] \begin{bmatrix} -T_1 & U_1 \\ V_1 & W_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = [0 \quad I] \begin{bmatrix} -T_2 & U_2 \\ V_2 & W_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

are system equivalent at infinity.

Proof. It is sufficient to establish the lemma for the modified realizations

$$R^{-1} = [0 \quad I] \begin{bmatrix} T_i & -U_i \\ -V_i & -W_i \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I \end{bmatrix} \qquad (i = 1, 2),$$

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whose extended system matrices are

$$\bar{Q}_i = \begin{bmatrix} Q_i & -\mathscr{B}_i \\ \mathscr{C}_i & 0 \end{bmatrix} \qquad (i = 1, 2).$$

Suppose first that the realizations (11) are system equivalent at infinity. If Q_1^{-1} has the irreducible causal realization $Q_1^{-1} = \mathscr{T}_1^{-1}\mathscr{U}_1$ then $\bar{Q}_1^{-1}$ has the irreducible causal realization

$$\bar{Q}_1^{-1} = \begin{bmatrix} \mathscr{T}_1 & -\mathscr{U}_1\mathscr{B}_1 \\ \mathscr{C}_1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathscr{U}_1 & 0 \\ 0 & I \end{bmatrix}.$$

Similarly for $Q_2^{-1} = \mathscr{V}_2\mathscr{T}_2^{-1}$ and $\bar{Q}_2^{-1}$. By hypothesis there exist causal matrices $\mathscr{M}, \mathscr{N}, \mathscr{X}, \mathscr{Y}$ with $\mathscr{M}, \mathscr{T}_2$ left coprime and $\mathscr{N}, \mathscr{T}_1$ right coprime such that

$$\begin{bmatrix} \mathscr{M} & 0 \\ \mathscr{X} & I \end{bmatrix} \begin{bmatrix} -\mathscr{T}_1 & \mathscr{U}_1\mathscr{B}_1 \\ \mathscr{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} -\mathscr{T}_2 & \mathscr{B}_2 \\ \mathscr{C}_2\mathscr{V}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{N} & \mathscr{Y} \\ 0 & I \end{bmatrix}.$$

Then the induced causal realizations of R^{-1} are causally system equivalent, since

$$\begin{bmatrix} \mathscr{M} & 0 & 0 \\ -\mathscr{X} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} -\mathscr{T}_1 & \mathscr{U}_1\mathscr{B}_1 & 0 \\ -\mathscr{C}_1 & 0 & I \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} -\mathscr{T}_2 & \mathscr{B}_2 & 0 \\ -\mathscr{C}_2\mathscr{V}_2 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \mathscr{N} & \mathscr{Y} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Conversely, suppose the realizations of R^{-1} are system equivalent at infinity. Applying what we have already proved to R^{-1}, we see that the realizations

$$R = \mathscr{C}_1 Q_1^{-1}\mathscr{B}_1 = \mathscr{C}_2 Q_2^{-1}\mathscr{B}_2$$

are system equivalent at infinity. Hence, by Lemma 1, the realizations (11) are also system equivalent at infinity.

Lemma 2 continues to hold if "system equivalence at infinity" is replaced by "strong system equivalence", since it also holds for ordinary system equivalence.

3. Controllability and observability at infinity

A polynomial realization

$$R = W + VT^{-1}U \tag{5}$$

will be said to be *controllable at infinity* (*observable at infinity*) if the corresponding induced causal realization is controllable (observable). It will be said to be *irreducible at infinity* if it is both controllable at infinity and observable at infinity. These definitions are independent of the choice of irreducible causal realization of Q^{-1}. Also, if two polynomial realizations are system equivalent at infinity and one is controllable (observable) at infinity, then so is the other. It follows at once from

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the corresponding result for causal realizations that if two polynomial realizations of the same rational matrix are irreducible at infinity then they are system equivalent at infinity.

A polynomial realization (5) will be said to be *strongly controllable* if it is controllable, i.e., the polynomial matrices T and U are left coprime, and also controllable at infinity. Similarly it will be said to be *strongly observable* if it is observable, i.e., the polynomial matrices T and V are right coprime, and also observable at infinity. It will be said to be *strongly irreducible* if it is both irreducible and irreducible at infinity, i.e., if it is both strongly controllable and strongly observable. If two polynomial realizations are strongly system equivalent and one is strongly controllable (observable), then so is the other. Conversely, two strongly irreducible polynomial realizations of the same rational matrix are necessarily strongly system equivalent.

The next result shows that these definitions are equivalent to those of Verghese [14].

PROPOSITION 4. *The polynomial realization* (5) *is controllable at infinity if and only if the polynomial matrix*

$$\begin{bmatrix} -T & U & 0 \\ V & W & -I \end{bmatrix} \tag{19}$$

has a causal right inverse, and it is observable at infinity if and only if the polynomial matrix

$$\begin{bmatrix} -T & U \\ V & W \\ 0 & I \end{bmatrix} \tag{20}$$

has a causal left inverse.

PROOF. It is sufficient to establish the controllability criterion, since the observability criterion follows by taking transposes. Let Q^{-1} have the irreducible causal realization $Q^{-1} = \mathscr{V}\mathscr{T}^{-1}$ and suppose first that the matrix (19) has a causal right inverse $\mathscr{F}_1$. If $\mathscr{F} = [\mathscr{F}_1 \quad 0]$ then $\mathscr{G} = Q\mathscr{F}$ is a causal matrix of the form

$$\mathscr{G} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathscr{G}_1 & \mathscr{G}_2 & 0 \end{bmatrix}.$$

Since $\mathscr{F} = Q^{-1}\mathscr{G}$ is causal we can write $\mathscr{G} = \mathscr{T}\bar{\mathscr{G}}$ for some causal matrix $\bar{\mathscr{G}}$. Since

$$\mathscr{T}\bar{\mathscr{G}} - \mathscr{B}[\mathscr{G}_1 \quad \mathscr{G}_2 \quad -I] = I,$$

it follows that the causal matrices $\mathscr{T}$ and $\mathscr{B}$ are left coprime. Thus the induced causal realization $R = \mathscr{C}\mathscr{V}\mathscr{T}^{-1} \cdot \mathscr{B}$ is causally controllable and the given polynomial realization is controllable at infinity.

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Suppose on the other hand that the given polynomial realization is controllable at infinity. Then there exist causal matrices $\mathscr{H}, \mathscr{K}$ such that

$$\mathscr{T}\mathscr{H} + \mathscr{B}\mathscr{K} = I.$$

Since $\mathscr{T}\mathscr{H} = Q\mathscr{V}\mathscr{H}$, this shows that the matrix $[Q \quad \mathscr{B}]$ has a causal right inverse. It follows at once that the matrix (19) has a causal right inverse.

Proposition 4 is the key result of this section. There are however further insights which can be derived, and these are contained in the next three propositions.

The following simple result appears to have been overlooked by Verghese.

PROPOSITION 5. *If the rational matrix R is causal then the polynomial realization* (5) *is irreducible at infinity if and only if T^{-1}, $T^{-1}U$ and VT^{-1} are causal.*

PROOF. The sufficiency of the condition follows immediately from Proposition 4 and the expression (8) for Q^{-1}. Conversely, suppose R is causal and the realization (5) is irreducible at infinity. Since the realization is controllable at infinity, the matrix (19) has a causal right inverse

$$\begin{bmatrix} \mathscr{F}_1 & \mathscr{F}_2 \\ \mathscr{G}_1 & \mathscr{G}_2 \\ \mathscr{H}_1 & \mathscr{H}_2 \end{bmatrix}.$$

It follows that

$$\mathscr{F}_1 = T^{-1}U\mathscr{G}_1 - \mathscr{T}^{-1}$$

and hence that

$$R\mathscr{G}_1 - VT^{-1} - \mathscr{H}_1 = 0.$$

Thus VT^{-1} is causal. Similarly, since the matrix (20) has a causal left inverse, $T^{-1}U$ is causal. It now follows from the equation displayed above that T^{-1} is causal.

Another immediate consequence of Proposition 4 is

PROPOSITION 6. *Let R be a non-singular $m \times m$ rational matrix with the polynomial realization* (5). *Then the polynomial realization*

$$R^{-1} = [0 \quad I]\begin{bmatrix} -T & U \\ V & W \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

of R^{-1} is controllable (observable) at infinity if and only if the given realization of R is controllable (observable) at infinity.

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If the $p \times m$ rational matrix R has rank r, then there exist bicausal rational matrices $\mathcal{U}$, $\mathcal{V}$ such that

$$\mathcal{U} R \mathcal{V} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where the diagonal matrix $D = [s^{d_1}, \ldots, s^{d_r}]$ and $d_1 \geqslant \cdots \geqslant d_r$. This is the Smith-McMillan form of R at infinity. We can write

$$D = [R_+, R_0, R_-],$$

where the diagonal matrices R_+, R_0 and R_- contain respectively the positive, zero and negative powers of s in D. It follows directly from the general theory of realizations over a principal ideal domain [3, Theorems 12 and 13], that if the polynomial realization (5) is irreducible at infinity, and if P and Q are the corresponding system matrix and extended system matrix, then

$$R_+ = Q_-^{-1}, \qquad R_- = P_-. \tag{21}$$

These results were originally derived by Verghese [14, Theorem 3.9] by a more special argument.

The McMillan degree $\delta(R)$ of a rational matrix R is defined, over the complex field, to be the total polar degree of R. Over an arbitrary field it may be defined in the following way. Let $\nu(R)$ denote the degree of the least common denominator of all minors of R, and let $\nu_\infty(R)$ denote the maximum non-negative degree of any minor of R. (We include the "empty" minor, which has the value 1.) Then

$$\delta(R) = \nu(R) + \nu_\infty(R).$$

If R has Smith-McMillan form $\{\varepsilon_1/\psi_1, \ldots, \varepsilon_r/\psi_r\}$, then $\nu(R)$ is the degree of the polynomial $\psi_1 \cdots \psi_r$. If R has Smith-McMillan form at infinity $\{s^{d_1}, \ldots, s^{d_r}\}$, then $\nu_\infty(R)$ is the sum of all positive exponents d_k.

The following result is essentially contained in Verghese [14], but it is not explicitly formulated there.

Proposition 7. *Let the rational matrix R have the polynomial realization* (5) *and let Q be the corresponding extended system matrix. Then*

$$\delta(R) \leqslant \delta(Q),$$

with equality if and only if the realization is strongly irreducible.

Proof. By ordinary realization theory [3], $\nu(R)$ is at most equal to the degree of the polynomial $\det Q$, with equality if and only if the given realization is irreducible. Similarly it follows from (21) that $\nu_\infty(R)$ is at most equal to the degree of the polynomial $\det Q_-^{-1}$, with equality if and only if the given realization

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is irreducible at infinity. Hence

$$\delta(R) \leqslant \delta(\det Q) - \delta(\det Q_-)$$
$$= \delta(\det Q_+) = \delta(Q),$$

with equality if and only if the given realization is strongly irreducible.

4. Matrix fraction realizations

Matrix fraction realizations, i.e., polynomial realizations of the form

$$R = \overline{VT}^{-1}, \tag{22}$$

have received particular attention in the literature. Such a realization is necessarily strongly controllable, and it is tempting to conjecture that any strongly controllable realization (5) is strongly system equivalent to some realization (22). Unfortunately this is false. For example, the strongly controllable scalar realizations

$$s + (-1) \cdot s^{-1} \cdot (s^2 - 1) = 0 + 1 \cdot s^{-1} \cdot 1$$

are system equivalent. However, they are not strongly system equivalent, since the Smith-McMillan forms at infinity of the corresponding extended system matrices are $\{s^2, 1, s^{-1}\}$ and $\{s, 1, 1\}$, which do not have the same zero structure. Nevertheless it is possible to say when two polynomial realizations of the form (22) are strongly system equivalent. The main result of this section now follows:

PROPOSITION 8. *Two polynomial realizations*

$$R = V_1 T_1^{-1} = V_2 T_2^{-1} \tag{23}$$

of a rational matrix R are system equivalent at infinity if and only if the matrices

$$\begin{bmatrix} Z \\ T_2 \\ V_2 \end{bmatrix}, \quad \begin{bmatrix} Z^{-1} \\ T_1 \\ V_1 \end{bmatrix}, \tag{24}$$

where $Z = T_1^{-1} T_2$, have causal left inverses.

PROOF. Suppose first that the realizations (23) are system equivalent at infinity. Then there exist causal matrices $\mathscr{M}, \mathscr{N}, \mathscr{X}, \mathscr{Y}$ such that

$$\begin{bmatrix} \mathscr{M} & 0 \\ \mathscr{X} & I \end{bmatrix} \begin{bmatrix} -Q_1 & \mathscr{B}_1 \\ \mathscr{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} -Q_2 & \mathscr{B}_2 \\ \mathscr{C}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{N} & \mathscr{Y} \\ 0 & I \end{bmatrix}.$$

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Corresponding to the partitions (7) of Q_1 and Q_2, write

$$\begin{aligned} \mathscr{M} &= \left[\mathscr{M}_{jk}\right], & \mathscr{N} &= \left[\mathscr{N}_{jk}\right], \\ \mathscr{X} &= \left[\mathscr{X}_j\right], & \mathscr{Y} &= \left[\mathscr{Y}_k\right], \end{aligned} \qquad (j, k = 1, 2, 3).$$

Then

$$\mathscr{F} = \left[\mathscr{N}_{11} \quad \mathscr{N}_{12} + \mathscr{Y}_1 \quad \mathscr{N}_{13}\right]$$

is causal and a straightforward calculation shows that $\mathscr{F}$ is a left inverse of the first matrix (24). Since system equivalence at infinity is a symmetric relation, it follows that also the second matrix (24) has a causal left inverse.

Suppose next that the two matrices (24) have causal left inverses

$$\mathscr{F} = \left[\mathscr{F}_1 \quad \mathscr{F}_2 \quad \mathscr{F}_3\right] \quad \text{and} \quad \mathscr{G} = \left[\mathscr{G}_1 \quad \mathscr{G}_2 \quad \mathscr{G}_3\right]$$

respectively. We may assume that

$$\begin{bmatrix} Z \\ T_2 \\ V_2 \end{bmatrix} \mathscr{F}, \qquad \begin{bmatrix} Z^{-1} \\ T_1 \\ V_1 \end{bmatrix} \mathscr{G}$$

are also causal. In fact, by transforming to Smith-McMillan form at infinity, it is readily seen that if a rational matrix $\bar{R}$ has a causal left inverse then it has a causal left inverse $\mathscr{H}$ such that $\bar{R}\mathscr{H}$ is also causal. If we set

$$\mathscr{M} = \begin{bmatrix} I - T_2\mathscr{F}_2 & T_2\mathscr{F}_3 & 0 \\ V_2\mathscr{F}_2 & I - V_2\mathscr{F}_3 & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad \mathscr{N} = \begin{bmatrix} \mathscr{F}_1 & \mathscr{F}_2 & \mathscr{F}_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

then

$$\begin{bmatrix} \mathscr{M} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -Q_1 & \mathscr{B}_1 \\ \mathscr{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} -Q_2 & \mathscr{B}_2 \\ \mathscr{C}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{N} & 0 \\ 0 & I \end{bmatrix}$$

and the common value of both sides is causal. Moreover $[\mathscr{M} \quad Q_2]$ has a causal right inverse, since

$$\mathscr{M} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} + Q_2 \begin{bmatrix} -\mathscr{F}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} = I,$$

and $\left[\begin{smallmatrix} \mathscr{N} \\ Q_1 \end{smallmatrix}\right]$ has a causal left inverse, since

$$\begin{bmatrix} \mathscr{G}_1 & \mathscr{G}_2 & \mathscr{G}_3 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \mathscr{N} + \begin{bmatrix} -\mathscr{G}_1\mathscr{F}_2 - \mathscr{G}_2 & \mathscr{G}_1\mathscr{F}_3 + \mathscr{G}_3 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} Q_1 = I.$$

Therefore, by Proposition 1, the realizations (23) are system equivalent at infinity.

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COROLLARY. *For a causal matrix R, the polynomial realizations* (23) *are system equivalent at infinity if and only if the matrices*

$$\begin{bmatrix} Z \\ T_2 \end{bmatrix}, \qquad \begin{bmatrix} Z^{-1} \\ T_1 \end{bmatrix},$$

where $Z = T_1^{-1}T_2$, have causal left inverses.

It is easily seen that Proposition 8 can be reformulated in the following way: *the polynomial realizations* (23) *are system equivalent at infinity if and only if, for every causal matrix $\mathscr{X}$ such that $R\mathscr{X}$ is causal, T_1^{-1} is causal if and only if $T_2^{-1}\mathscr{X}$ is causal.*

In fact, suppose the realizations (23) are system equivalent at infinity. Then there exist causal matrices $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ such that

$$\mathscr{F}_1 Z + \mathscr{F}_2 T_2 + \mathscr{F}_3 V_2 = I.$$

Multiplying on the right by $T_2^{-1}\mathscr{X}$, it follows at once that $\mathscr{X}$, $R\mathscr{X}$ and $T_1^{-1}\mathscr{X}$ causal imply $T_2^{-1}\mathscr{X}$ causal. Similarly $\mathscr{X}$, $R\mathscr{X}$ and $T_2^{-1}\mathscr{X}$ causal imply $T_1^{-1}\mathscr{X}$ causal. Thus the condition is necessary.

Conversely, suppose the condition is satisfied. If

$$\begin{bmatrix} T_k^{-1} \\ V_k T_k^{-1} \end{bmatrix} = \mathscr{V}_k \mathscr{T}_k^{-1} = \begin{bmatrix} \mathscr{U}_k \\ \mathscr{W}_k \end{bmatrix} \mathscr{T}_k^{-1} \qquad (k = 1, 2)$$

are irreducible causal realizations, it follows that $\mathscr{Z} = \mathscr{T}_2^{-1}\mathscr{T}_1$ must be bicausal. On the other hand, since the realizations are irreducible, there exist causal matrices $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ such that

$$\mathscr{F}\mathscr{U}_1 + \mathscr{G}\mathscr{W}_1 + \mathscr{H}\mathscr{T}_1 = I.$$

Then

$$\mathscr{U}_2\mathscr{Z}(\mathscr{F}Z + \mathscr{G}V_2 + \mathscr{H}T_2) = \mathscr{U}_2\mathscr{Z}(\mathscr{F}\mathscr{U}_1 + \mathscr{G}\mathscr{W}_1 + \mathscr{H}\mathscr{T}_1)\mathscr{Z}^{-1}\mathscr{U}_2^{-1} = I.$$

Thus the first matrix (24) has a causal left inverse, and similarly also the second matrix (24).

System equivalence at infinity of the realizations (23) does not imply that $Z = T_1^{-1}T_2$ is a bicausal matrix. For example, the realizations of zero with system matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} s & 1 \\ 0 & 0 \end{bmatrix}$$

are system equivalent at infinity.

On the other hand it is well-known, see [9, page 564], that the realizations (23) are system equivalent if and only if $Z = T_1^{-1}T_2$ is a bipolynomial matrix.

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Otherwise expressed, the realizations (23) are system equivalent if and only if, for every polynomial matrix X, $T_1^{-1}X$ is polynomial if and only if $T_2^{-1}X$ is polynomial. Combining these conditions with the conditions for system equivalence at infinity, we obtain necessary and sufficient conditions for the strong system equivalence of the realizations (23).

5. Generalized state-space realizations

A generalized state-space realization of a rational matrix R is defined to be a polynomial realization of the special form

$$R = D + C(sE - A)^{-1}B,$$

where $A,\ldots,E$ are constant matrices. In this section we show that generalized state-space realizations play an analogous role with respect to strong system equivalence to that played by ordinary state-space realizations with respect to ordinary system equivalence.

There are two main results to obtain. First, any polynomial realization is strongly system equivalent to a generalized state-space realization and second, any two generalized state-space realizations are strongly system equivalent if and only if they are constant system equivalent (in a sense made precise later). Both results are of course extensions to strong system equivalence of important, long-standing results of linear systems theory.

The preliminary work for both theorems requires us to consider polynomial transfer matrices, which are excluded in the conventional theory. These results can then be combined with known results on proper (that is causal) rational matrices to give the main results for arbitrary, proper or improper, rational transfer matrices. We begin with:

PROPOSITION 9. *Any $p \times m$ polynomial matrix $\overline{P}(s)$ has a strongly irreducible polynomial realization of the form*

$$\overline{P}(s) = C(I - sJ)^{-1}B, \tag{25}$$

where J, B, C are constant matrices and J is nilpotent.

PROOF. Since $s^{-1}\overline{P}(s^{-1})$ is a strictly causal rational matrix, it admits a minimal state-space realization

$$s^{-1}\overline{P}(s^{-1}) = C(sI - A)^{-1}B,$$

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where A, B, C are constant matrices. There exists an invertible constant matrix S such that

$$S^{-1}AS = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 is nilpotent and A_2 is non-singular. Then

$$\begin{aligned}\bar{P}(s) &= C(I - sA)^{-1}B \\ &= CS\begin{bmatrix} I - sA_1 & 0 \\ 0 & I - sA_2 \end{bmatrix}^{-1} S^{-1}B \\ &= C_1(I - sA_1)^{-1}B_1 + C_2(I - sA_2)^{-1}B_2,\end{aligned}$$

where B_1, B_2, C_1, C_2 are constant matrices. The first term on the right is a polynomial matrix, since if $A_1^n = 0$ then

$$(I - sA_1)^{-1} = I + sA_1 + \cdots + s^{n-1}A_1^{n-1}.$$

The second term on the right is a strictly causal rational matrix, since

$$(I - sA_2)^{-1} = -A_2^{-1}(sI - A_2^{-1})^{-1}.$$

Since $\bar{P}(s)$ is a polynomial matrix, it follows that

$$C_2(I - sA_2)^{-1}B_2 = 0.$$

Since the original realization of $s^{-1}\bar{P}(s^{-1})$ was assumed minimal, this is possible only if the second term does not in fact appear. That is, $A = J$ is itself nilpotent and (25) holds. The realization (25) is certainly irreducible, since $I - sJ$ is bipolynomial. On the other hand, $[sI - J \quad B]$ has a polynomial right inverse, since (J, B) is controllable. Then $[s^{-1}I - J \quad B]$ has a causal right inverse and, *a fortiori*, $[I - sJ \quad B]$ has a causal right inverse. Similarly $\left[\begin{smallmatrix} I - sJ \\ C \end{smallmatrix}\right]$ has a causal left inverse. It follows from Proposition 4 that the realization (25) is irreducible at infinity. This completes the proof.

Proposition 9 is stated without proof by Verghese [14, page 184]. In the proof of Theorem 1 we will also use the following

LEMMA 3. *Let the rational matrix R have a polynomial realization*

$$R = D + CT^{-1}B$$

with B, C, D constant matrices, which is irreducible at infinity. If

$$T^{-1} = \mathscr{V}\mathscr{T}^{-1}$$

is a irreducible causal realization, then

$$R = D + C\mathscr{V}\cdot\mathscr{T}^{-1}\cdot B$$

is also an irreducible causal realization.

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Proof. By Proposition 4 there exist causal matrices $\mathscr{F}_1$, $\mathscr{G}_1$ such that

$$T\mathscr{F}_1 + B\mathscr{G}_1 = I.$$

Since $T\mathscr{F}_1$ is causal and the realization $T = \mathscr{T}\mathscr{V}^{-1}$ is irreducible, we can write $\mathscr{F}_1 = \mathscr{V}\mathscr{H}_1$ for some causal $\mathscr{H}_1$. Then

$$\mathscr{T}\mathscr{H}_1 + B\mathscr{G}_1 = I,$$

which shows that the causal realization of R is controllable. Similarly there exist causal matrices $\mathscr{F}_2$, $\mathscr{G}_2$ such that

$$\mathscr{F}_2 T + \mathscr{G}_2 C = I$$

and hence

$$\mathscr{G}_2 C\mathscr{V} = \mathscr{V} - \mathscr{F}_2\mathscr{T}.$$

But there exist causal matrices $\mathscr{F}_3$, $\mathscr{G}_3$ such that

$$\mathscr{F}_3\mathscr{V} + \mathscr{G}_3\mathscr{T} = I.$$

It follows that

$$\mathscr{F}_3\mathscr{G}_2 C\mathscr{V} + (\mathscr{G}_3 + \mathscr{F}_3\mathscr{F}_2)\mathscr{T} = I,$$

which shows that the causal realization of R is observable.

Theorem 1. *Any polynomial realization*

$$R = W + VT^{-1}U \tag{5}$$

of a rational matrix R is strongly system equivalent to a generalized state-space realization.

Proof. We suppose again that R is a $p \times m$ matrix and T an $n \times n$ matrix. By Proposition 9 the extended system matrix Q, defined by (7), admits a strongly irreducible polynomial realization

$$Q(s) = C(sJ - I_q)^{-1}B,$$

where J, B, C are constant matrices and J is nilpotent. Therefore, by Proposition 6, the polynomial realization

$$Q^{-1}(s) = \begin{bmatrix} 0_{l,q} & I_l \end{bmatrix} L^{-1}(s) \begin{bmatrix} 0_{q,l} \\ I_l \end{bmatrix},$$

where $l = n + p + m$ and

$$L(s) = \begin{bmatrix} I_q - sJ & B \\ C & 0 \end{bmatrix},$$

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is irreducible at infinity (and even strongly irreducible). Thus $R = \mathscr{C}Q^{-1}\mathscr{B}$ has the generalized state-space realization

$$R = \begin{bmatrix} 0_{p,q} & \mathscr{C} \end{bmatrix} L^{-1}(s) \begin{bmatrix} 0_{q,m} \\ \mathscr{B} \end{bmatrix}. \tag{26}$$

We will show that this realization is strongly system equivalent to the given realization (5).

Since $sJ - I$ is bipolynomial and

$$\begin{bmatrix} I & 0 & 0 \\ C(sJ-I)^{-1} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} sJ-I & -B & 0 \\ -C & 0 & \mathscr{B} \\ 0 & \mathscr{C} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 & 0 \\ 0 & -Q & \mathscr{B} \\ 0 & \mathscr{C} & 0 \end{bmatrix} \begin{bmatrix} sJ-I & -B & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

the realization (26) is system equivalent to the realization $R = \mathscr{C}Q^{-1}\mathscr{B}$ and hence also to the realization (5). It remains to show that the two realizations are system equivalent at infinity.

Let L^{-1} have the irreducible causal realization $L^{-1} = \mathscr{V}\mathscr{T}^{-1}$. Then, by Lemma 3, Q^{-1} has the irreducible causal realization

$$Q^{-1} = \begin{bmatrix} 0 & I \end{bmatrix} \mathscr{V} \cdot \mathscr{T}^{-1} \cdot \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The extended system matrix

$$\bar{Q} = \begin{bmatrix} sJ-I & -B & 0 & 0 \\ -C & 0 & \mathscr{B} & 0 \\ 0 & \mathscr{C} & 0 & -I_p \\ 0 & 0 & I_m & 0 \end{bmatrix}$$

of the generalized state-space realization (26) has the irreducible causal realization

$$\bar{Q} = \begin{bmatrix} -\mathscr{T} & \bar{\mathscr{B}} & 0 \\ \bar{\mathscr{C}}\mathscr{V} & 0 & -I_p \\ 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} \mathscr{V} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_p \end{bmatrix}^{-1},$$

where

$$\bar{\mathscr{C}} = \begin{bmatrix} 0_{p,n+m+q} & I_p \end{bmatrix}, \qquad \bar{\mathscr{B}} = \begin{bmatrix} 0_{n+p+q,m} \\ I_m \end{bmatrix}.$$

The corresponding induced causal realizations

$$R = \bar{\mathscr{C}}\mathscr{V} \cdot \mathscr{T}^{-1} \cdot \bar{\mathscr{B}}$$

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and

$$R = \hat{\mathscr{C}}\begin{bmatrix} \mathscr{V} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_p \end{bmatrix} \cdot \begin{bmatrix} -\mathscr{T} & \overline{\mathscr{B}} & 0 \\ \overline{\mathscr{C}}\mathscr{V} & 0 & -I_p \\ 0 & I_m & 0 \end{bmatrix}^{-1} \cdot \hat{\mathscr{B}},$$

where

$$\hat{\mathscr{C}} = \begin{bmatrix} 0_{p,l+m+q} & I_p \end{bmatrix}, \quad \hat{\mathscr{B}} = \begin{bmatrix} 0_{l+p+q,m} \\ I_m \end{bmatrix},$$

are causally system equivalent, since

$$\begin{bmatrix} 0 & 0 & I_m & 0 \\ 0 & I_p & 0 & 0 \\ I_{l+q} & 0 & -\overline{\mathscr{B}} & 0 \\ 0 & I_p & 0 & -I_p \end{bmatrix}\begin{bmatrix} \mathscr{T} & -\overline{\mathscr{B}} & 0 & 0 \\ -\overline{\mathscr{C}}\mathscr{V} & 0 & I_p & 0 \\ 0 & -I_m & 0 & I_m \\ 0 & 0 & I_p & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & -\mathscr{T} & \overline{\mathscr{B}} \\ 0 & 0 & \overline{\mathscr{C}}\mathscr{V} & 0 \end{bmatrix}\begin{bmatrix} 0 & -I_m & 0 & I_m \\ -\overline{\mathscr{C}}\mathscr{V} & 0 & I_p & 0 \\ -I_{l+q} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \end{bmatrix}.$$

Thus the original polynomial realization (5) and the generalized state-space realization (26) are system equivalent at infinity.

The procedure for constructing the generalized state-space realization (26) was given by Verghese [14, p. 209]. He recognised that (26) ought to be strongly system equivalent to the given realization (5), even though he lacked a general definition of strong system equivalence.

Our next objective is to obtain necessary and sufficient conditions for the strong system equivalence of two generalized state-space realizations. We first prove a division rule for the ring of causal rational functions.

LEMMA 4. *Let $\mathscr{A}(s)$ be a causal $p \times m$ matrix and J a constant nilpotent $p \times p$ matrix. Then there exists a unique causal matrix $\mathscr{B}(s)$ and constant matrix C such that*

$$\mathscr{A}(s) = (s^{-1}I - J)\mathscr{B}(s) + C. \tag{27}$$

PROOF. Let $\mathscr{A}(s)$ have the formal power series expansion

$$\mathscr{A}(s) = A_0 + A_1 s^{-1} + A_2 s^{-2} + \cdots.$$

If $\mathscr{B}(s) = B_0 + B_1 s^{-1} + B_2 s^{-2} + \cdots$ and C satisfy (27), then

$$\mathscr{B}(s) = (sI + s^2 J + \cdots + s^p J^{p-1})(-C + A_0 + A_1 s^{-1} + \cdots).$$

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Equating coefficients of s, we get

$$C = A_0 + JA_1 + \cdots + J^{p-1}A_{p-1}.$$

Equating coefficients of $1, s^{-1}, \ldots$ we now obtain in succession $B_0, B_1, \ldots$. Thus $\mathscr{B}(s)$ and C are uniquely determined.

Conversely, if we define C in this way then

$$\begin{aligned} JC &= JA_0 + J^2A_1 + \cdots + J^{p-1}A_{p-2} \\ &\cdots\cdots \\ J^{p-2}C &= J^{p-2}A_0 + J^{p-1}A_1 \\ J^{p-1}C &= J^{p-1}A_0. \end{aligned}$$

It follows that the rational matrix

$$\mathscr{B}(s) = (s^{-1}I - J)^{-1}(\mathscr{A}(s) - C)$$

is causal.

Clearly, there is an analogous result to Lemma 4 with right division, instead of left division, by $s^{-1}I - J$. These results will now be used to study strong system equivalence for generalized state-space realizations of a polynomial matrix.

PROPOSITION 10. *Let $\bar{P}(s)$ be a polynomial matrix with the generalized state-space realizations*

$$\bar{P}(s) = D_1 + C_1(sJ_1 - I)^{-1}B_1 = D_2 + C_2(sJ_2 - I)^{-1}B_2 \tag{28}$$

where J_1 and J_2 are nilpotent. If there exist causal matrices $\mathscr{M}, \mathscr{N}, \mathscr{X}, \mathscr{Y}$ such that $[\mathscr{M} \quad I - sJ_2]$ has a causal right inverse, $\left[\begin{smallmatrix}\mathscr{N}\\ I - sJ_1\end{smallmatrix}\right]$ has a causal left inverse and

$$\begin{bmatrix} \mathscr{M} & 0 \\ \mathscr{X} & I \end{bmatrix}\begin{bmatrix} I - sJ_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} I - sJ_2 & B_2 \\ C_2 & D_2 \end{bmatrix}\begin{bmatrix} \mathscr{N} & \mathscr{Y} \\ 0 & I \end{bmatrix},$$

then there exists an invertible constant matrix M_0 and constant matrices X_0, Y_0 such that

$$\begin{bmatrix} M_0 & 0 \\ X_0 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I - sJ_1 & B_1 \\ 0 & C_1 & D_1 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I - sJ_2 & B_2 \\ 0 & C_2 & D_2 \end{bmatrix}\begin{bmatrix} M_0 & Y_0 \\ 0 & I \end{bmatrix}.$$

PROOF. By Lemma 4 we can write

$$\mathscr{M} = (I - sJ_2)\mathscr{M}_1 + M_1,$$

where $\mathscr{M}_1$ is strictly causal and M_1 is a constant matrix. Then

$$\mathscr{N} = \mathscr{M}_1(I - sJ_1) + N_1,$$

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where $N_1 = (I - sJ_2)^{-1}M_1(I - sJ_1)$ is both polynomial and causal, and hence a constant matrix. Moreover $(I - sJ_2)N_1 = M_1(I - sJ_1)$ gives

$$N_1 = M_1, \quad J_2N_1 = M_1J_1.$$

Similarly from $\mathscr{M}B_1 = (I - sJ_2)\mathscr{Y} + B_2$ we obtain

$$M_1B_1 = (I - sJ_2)Y_1 + B_2,$$

where $Y_1 = \mathscr{Y} - \mathscr{M}_1B_1$ is a constant matrix, and from $\mathscr{X}(I - sJ_1) + C_1 = C_2\mathscr{N}$ we obtain

$$C_2N_1 = X_1(I - sJ_1) + C_1,$$

where $X_1 = \mathscr{X} - C_2\mathscr{M}_1$ is a constant matrix. Finally, equating constant terms in $\mathscr{X}B_1 + D_1 = C_2\mathscr{Y} + D_2$ we obtain

$$X_1B_1 + D_1 = C_2Y_1 + D_2.$$

By hypothesis there exist causal matrices $\mathscr{U}$, $\mathscr{V}$ such that

$$\mathscr{M}\mathscr{U} + (I - sJ_2)\mathscr{V} = I$$

and hence

$$M_1\mathscr{U} + (I - sJ_2)\mathscr{W} = I,$$

where $\mathscr{W} = \mathscr{V} + \mathscr{M}_1\mathscr{U}$. We can write

$$\mathscr{U} = (I - sJ_1)\mathscr{U}_1 + U_1,$$

where $\mathscr{U}_1$ is strictly causal and U_1 is a constant matrix. It follows that

$$M_1U_1 + (I - sJ_2)W_1 = I,$$

where $W_1 = \mathscr{W} + M_1\mathscr{U}_1$ is a constant matrix. Similarly there exist constant matrices $\overline{U}_1$, $\overline{W}_1$ such that

$$\overline{U}_1N_1 + \overline{W}_1(I - sJ_1) = I.$$

Equating coefficients of s, we obtain

$$J_2W_1 = 0, \quad \overline{W}_1J_1 = 0.$$

It follows that, for any constant matrix L_1,

$$\begin{bmatrix} L_1 & \overline{W}_1 & 0 \\ W_1 & M_1 & 0 \\ C_2W_1 & X_1 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I - sJ_1 & B_1 \\ 0 & C_1 & D_1 \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 & 0 \\ 0 & I - sJ_2 & B_2 \\ 0 & C_2 & D_2 \end{bmatrix}\begin{bmatrix} L_1 & \overline{W}_1 & \overline{W}_1B_1 \\ W_1 & M_1 & Y_1 \\ 0 & 0 & I \end{bmatrix}.$$

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Since $[W_1 \quad M_1]$ has full row rank and $\left[\begin{smallmatrix}\overline{W}_1\\ M_1\end{smallmatrix}\right]$ has full column rank we can choose L_1 so that the constant matrix

$$M_0 = \begin{bmatrix} L_1 & \overline{W}_1 \\ W_1 & M_1 \end{bmatrix}$$

is invertible. The result follows.

We will say that two generalized state-space realizations

$$R = D_1 + C_1(sE_1 - A_1)^{-1}B_1 = D_2 + C_2(sE_2 - A_2)^{-1}B_2 \tag{29}$$

of an arbitrary rational matrix R are *constant system equivalent* if there exist invertible constant matrices M_0, N_0 and constant matrices X_0, Y_0 such that

$$\begin{bmatrix} M_0 & 0 \\ X_0 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & A_1 - sE_1 & B_1 \\ 0 & C_1 & D_1 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & A_2 - sE_2 & B_2 \\ 0 & C_2 & D_2 \end{bmatrix}\begin{bmatrix} N_0 & Y_0 \\ 0 & I \end{bmatrix}. \tag{30}$$

It is not difficult to see that this is the same as the concept of strong equivalence for generalized state-space realizations which Verghese [14, p. 161] defined in a less direct way.

THEOREM 2. *Two generalized state-space realizations* (29) *of a rational matrix R are strongly system equivalent if and only if they are constant system equivalent.*

PROOF. Suppose first that (30) holds. Then the realizations (29) are certainly system equivalent and, by Proposition 3, they are also system equivalent at infinity.

Conversely, suppose that the realizations (29) are strongly system equivalent. It is well-known, see, for example, Gantmacher [7, Vol. II, p. 28], that for any non-singular pencil $sE - A$ there exist invertible constant matrices L, K such that

$$L(sE - A)K = \begin{bmatrix} sJ - I & 0 \\ 0 & sI - \overline{A} \end{bmatrix},$$

where J is nilpotent. Consequently, we may assume from the outset that the two given realizations have system matrices of the form

$$\begin{bmatrix} I - sJ_i & 0 & \hat{B}_i \\ 0 & \overline{A}_i - sI & \overline{B}_i \\ \hat{C}_i & \overline{C}_i & D_i \end{bmatrix},$$

where J_i is nilpotent $(i = 1, 2)$. Then

$$\overline{C}_1(sI - \overline{A}_1)^{-1}\overline{B}_1 = \overline{C}_2(sI - \overline{A}_2)^{-1}\overline{B}_2, \tag{31}$$

since each side is the strictly causal part of R.

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Since the realizations (29) are system equivalent, there exist polynomial matrices M, N, X, Y such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I - sJ_1 & 0 & \hat{B}_1 \\ 0 & \bar{A}_1 - sI & \bar{B}_1 \\ \hat{C}_1 & \bar{C}_1 & D_1 \end{bmatrix} = \begin{bmatrix} I - sJ_2 & 0 & \hat{B}_2 \\ 0 & \bar{A}_2 - sI & \bar{B}_2 \\ \hat{C}_2 & \bar{C}_2 & D_2 \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix},$$

where M and

$$A_2 - sE_2 = \begin{bmatrix} I - sJ_2 & 0 \\ 0 & \bar{A}_2 - sI \end{bmatrix}$$

are left coprime, and N and

$$A_1 - sE_1 = \begin{bmatrix} I - sJ_1 & 0 \\ 0 & \bar{A}_1 - sI \end{bmatrix}$$

are right coprime. Let

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

$$X = [X_1 \quad X_2], \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

be corresponding partitions. Then it follows that

$$\begin{bmatrix} \bar{M} & 0 \\ \bar{X} & I \end{bmatrix} \begin{bmatrix} \bar{A}_1 - sI & \bar{B}_1 \\ \bar{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_2 - sI & \bar{B}_2 \\ \bar{C}_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{N} & \bar{Y} \\ 0 & I \end{bmatrix},$$

where $\bar{M} = M_{22}$, $\bar{N} = N_{22}$, and

$$\bar{X} = X_2 - \hat{C}_2(I - sJ_2)^{-1}M_{12},$$

$$\bar{Y} = Y_2 - N_{21}(I - sJ_1)^{-1}\hat{B}_1.$$

Moreover $\bar{M} = M_{22}$ and $\bar{A}_2 - sI$ are left coprime polynomial matrices, since M and $A_2 - sE_2$ are left coprime and

$$M_{21} = (\bar{A}_2 - sI)N_{21}(I - sJ_1)^{-1}.$$

Similarly, $\bar{N} = N_{22}$ and $\bar{A}_1 - sI$ are right coprime polynomial matrices, since N and $A_1 - sE_1$ are right coprime and

$$N_{12} = (I - sJ_2)^{-1}M_{12}(\bar{A}_1 - sI).$$

Thus the state-space realizations (31) are system equivalent. By a basic property of state-space realizations, see for example [9, p. 562], this implies that there is an

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invertible constant matrix S such that

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_1 - sI & \bar{B}_1 \\ \bar{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_2 - sI & \bar{B}_2 \\ \bar{C}_2 & 0 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}. \tag{32}$$

On the other hand, since the realizations (29) are system equivalent at infinity, there exist by Proposition 1 causal matrices $\mathscr{M}, \mathscr{N}, \mathscr{X}, \mathscr{Y}$ such that $[\mathscr{M} \quad Q_2]$ has a causal right inverse, $[{}^{\mathscr{N}}_{Q_1}]$ has a causal left inverse, and

$$\begin{bmatrix} \mathscr{M} & 0 \\ \mathscr{X} & I \end{bmatrix} \begin{bmatrix} -Q_1 & \mathscr{B}_1 \\ \mathscr{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} -Q_2 & \mathscr{B}_2 \\ \mathscr{C}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{N} & \mathscr{Y} \\ 0 & I \end{bmatrix}. \tag{33}$$

Corresponding to the partitions

$$Q_i = \begin{bmatrix} I - sJ_i & 0 & \hat{B}_i & 0 \\ 0 & \bar{A}_i - sI & \bar{B}_i & 0 \\ \hat{C}_i & \bar{C}_i & D_i & -I \\ 0 & 0 & I & 0 \end{bmatrix} \qquad (i = 1, 2),$$

put

$$\mathscr{M} = \begin{bmatrix} \mathscr{M}_{11} & \mathscr{M}_{12} & \mathscr{M}_{13} & \mathscr{M}_{14} \\ \mathscr{M}_{21} & \mathscr{M}_{22} & \mathscr{M}_{23} & \mathscr{M}_{24} \\ \mathscr{M}_{31} & \mathscr{M}_{32} & \mathscr{M}_{33} & \mathscr{M}_{34} \\ \mathscr{M}_{41} & \mathscr{M}_{42} & \mathscr{M}_{43} & \mathscr{M}_{44} \end{bmatrix}, \qquad \mathscr{N} = \begin{bmatrix} \mathscr{N}_{11} & \mathscr{N}_{12} & \mathscr{N}_{13} & \mathscr{N}_{14} \\ \mathscr{N}_{21} & \mathscr{N}_{22} & \mathscr{N}_{23} & \mathscr{N}_{24} \\ \mathscr{N}_{31} & \mathscr{N}_{32} & \mathscr{N}_{33} & \mathscr{N}_{34} \\ \mathscr{N}_{41} & \mathscr{N}_{42} & \mathscr{N}_{43} & \mathscr{N}_{44} \end{bmatrix},$$

$$\mathscr{X} = [\mathscr{X}_1 \quad \mathscr{X}_2 \quad \mathscr{X}_3 \quad \mathscr{X}_4], \qquad \mathscr{Y} = \begin{bmatrix} \mathscr{Y}_1 \\ \mathscr{Y}_2 \\ \mathscr{Y}_3 \\ \mathscr{Y}_4 \end{bmatrix}.$$

Then it follows from (33) by a straightforward calculation that

$$\begin{bmatrix} \bar{\mathscr{M}} & 0 \\ \bar{\mathscr{X}} & I \end{bmatrix} \begin{bmatrix} I - sJ_1 & \hat{B}_1 \\ \hat{C}_1 & D_1 \end{bmatrix} = \begin{bmatrix} I - sJ_2 & \hat{B}_2 \\ \hat{C}_2 & D_2 \end{bmatrix} \begin{bmatrix} \bar{\mathscr{N}} & \bar{\mathscr{Y}} \\ 0 & I \end{bmatrix},$$

where

$$\bar{\mathscr{M}} = \mathscr{M}_{11} - \hat{B}_2 \mathscr{M}_{41}, \qquad \bar{\mathscr{N}} = \mathscr{N}_{11} + \mathscr{N}_{14} \hat{C}_1,$$

$$\bar{\mathscr{X}} = -\mathscr{X}_1 + \mathscr{M}_{31} - D_2 \mathscr{M}_{41} + \bar{C}_2 (sI - \bar{A}_2)^{-1} (\mathscr{M}_{21} - \bar{B}_2 \mathscr{M}_{41}),$$

$$\bar{\mathscr{Y}} = \mathscr{Y}_1 + \mathscr{N}_{13} + \mathscr{N}_{14} D_1 + (\mathscr{N}_{12} + \mathscr{N}_{14} \bar{C}_1)(sI - \bar{A}_1)^{-1} \bar{B}_1.$$

Moreover $[\bar{\mathscr{M}} \quad I - sJ_2]$ has a causal right inverse, since

$$[I \quad 0 \quad 0 \quad -\hat{B}_2] Q_2 = [I - sJ_2 \quad 0 \quad 0 \quad 0]$$

and

$$[I \quad 0 \quad 0 \quad -\hat{B}_2] \mathscr{M} = \bar{\mathscr{M}} [I \quad 0 \quad 0 \quad -\hat{B}_1] - (I - sJ_2)[0 \quad \mathscr{L}_2 \quad \mathscr{L}_3 \quad \mathscr{L}_4],$$

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where

$$\mathscr{L}_2 = (\mathscr{N}_{12} + \mathscr{N}_{14}\bar{C}_1)(sI - \bar{A}_1)^{-1},$$
$$\mathscr{L}_3 = \mathscr{N}_{14},$$
$$\mathscr{L}_4 = \mathscr{Y}_1 - \bar{\mathscr{Y}}$$

are causal. Similarly $[{}_{I-sJ_1}^{\mathscr{N}}]$ has a causal left inverse. It follows from Proposition 10 that there exist constant matrices M_0, X_0, Y_0 with M_0 invertible such that

$$\begin{bmatrix} M_0 & 0 \\ X_0 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I - sJ_1 & \hat{B}_1 \\ 0 & \hat{C}_1 & D_1 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I - sJ_2 & \hat{B}_2 \\ 0 & \hat{C}_2 & D_2 \end{bmatrix}\begin{bmatrix} M_0 & Y_0 \\ 0 & I \end{bmatrix}. \tag{34}$$

Combining (32) and (34), we obtain the theorem.

REMARKS. The proof of Theorem 2 shows that two generalized state-space realizations

$$\begin{aligned} R(s) &= \bar{C}_1(sI - \bar{A}_1)^{-1}\bar{B}_1 + \hat{C}_1(sJ_1 - I)^{-1}\hat{B}_1 + D_1 \\ &= \bar{C}_2(sI - \bar{A}_2)^{-1}\bar{B}_2 + \hat{C}_2(sJ_2 - I)^{-1}\hat{B}_2 + D_2 \end{aligned}$$

of a rational matrix $R(s)$ are strongly system equivalent (i.e. constant system equivalent) if and only if the realizations

$$\bar{C}_1(sI - \bar{A}_1)^{-1}\bar{B}_1 = \bar{C}_2(sI - \bar{A}_2)^{-1}\bar{B}_2$$

of its strictly causal part are system equivalent (i.e. similar) and the realizations

$$\hat{C}_1(sJ_1 - I)^{-1}\hat{B}_1 + D_1 = \hat{C}_2(sJ_2 - I)^{-1}\hat{B}_2 + D_2$$

of its polynomial part are system equivalent at infinity (i.e. constant system equivalent). In conjunction with Proposition 3, this proves that the hypothesis of Proposition 10 is satisfied if and only if the realizations (28) are system equivalent at infinity. This also follows more directly from Proposition 1 of [5]. Finally we note that in Proposition 10 the matrices X_0 and Y_0 cannot always be taken to be zero. A simple example is provided by the scalar realizations of the zero matrix

$$0 = 0 + 1 \cdot (-1)^{-1} \cdot 0 + 0 \cdot (-1)^{-1} \cdot 1.$$

6. Conclusion

The results which have been established provide an adequate theory of strong system equivalence and linearisation of polynomial realizations, in the sense of the Introduction. The only drawback is aesthetic. There are some massive

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matrices and tedious calculations. It has been shown in [8] and [4] that ordinary system equivalence can be abstractly characterised as an isomorphism of modules. A module-theoretic approach to strong system equivalence has also been developed and will be given in the continuation of this paper by Coppel and Cullen [5]. Although the treatment there in part supersedes the present one, we have chosen not to discard altogether our own order of discovery, since some may still prefer matrices to modules.

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