THE KERNEL AND TRACE OPERATORS FOR IDEAL EXTENSIONS OF REGULAR SEMIGROUPS

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(Received 2 November, 1992)

Abstract. Let V be a regular semigroup and an ideal extension of a semigroup S by a semigroup Q. Congruences on V can be represented by triples of the form (σ, P, τ), here called admissible, where σ is a congruence on S, P is an ideal of Q and τ is a 0-restricted congruence on Q/P satisfying certain conditions. We characterize the trace relation T on V in terms of admissible triples. When the extension V of S is strict, for a congruence ν on V given in terms of an admissible triple, we characterize ν_K, ν^K, ν_T and ν^T again in terms of admissible triples.

1. Introduction and summary. Let S be a regular semigroup and $\mathscr{C}(S)$ be its congruence lattice. For $\rho \in \mathscr{C}(S)$, the kernel (respectively, trace) of ρ is the set of all elements of S ρ-related to idempotents (respectively, the restriction of ρ to idempotents of S). The relation on $\mathscr{C}(S)$ which identifies congruences on S with the same kernel (respectively, trace) is the kernel relation K (respectively, the trace relation T) for S. The classes of these (equivalence) relations are intervals and it is convenient to introduce the notation $\rho K = [\rho_K, \rho^K]$ and $\rho^T = [\rho_T, \rho^T]$ for the respective classes of $\rho \in \mathscr{C}(S)$. Then $\rho \to \rho_K$ and $\rho \to \rho^K$ are the kernel operators and $\rho \to \rho_T$ and $\rho \to \rho^T$ are the trace operators. The nature of these operators furnishes valuable information concerning both congruences on S and the congruence lattice $\mathscr{C}(S)$.

Now let V be a regular semigroup with an ideal S. Then V is an (ideal) extension of S by $Q = V/S$, where the latter is the Rees quotient semigroup. We may set $V = S \cup Q^*$ where $Q^* = Q\backslash\{0\}$. In such a case, both S and Q are regular semigroups so the above analysis can be applied to S, Q and V. The problem is to reduce this analysis for V to those for S and Q. As a first step, we must express the congruences on V in terms of S and Q, and if possible, in terms of $\mathscr{C}(S)$ and $\mathscr{C}(Q)$. In the present setting, this problem was solved in **[3]** as follows. Each congruence on V is expressed in terms of an (admissible) triple (σ, P, τ), where $\sigma \in \mathscr{C}(S)$, P is an ideal of Q and τ is a 0-restricted congruence on Q/P, satisfying certain conditions.

Representing the congruences on V in terms of triples as above, we may ask whether the relations K and T on $\mathscr{C}(V)$ can be expressed by means of the same relations on $\mathscr{C}(S)$ and $\mathscr{C}(Q)$. We may go one step further by asking for ν_K, ν^K, ν_T, ν^T for a congruence ν on V expressed in terms of a triple. The first task is easy: expressing the kernel and the trace. However, the problem of characterizing K on $\mathscr{C}(V)$ does not seem to admit a convenient solution, whereas T admits a simple expression. The problem with the kernel and the trace operators in this generality does not seem amenable to a successful treatment.

In order to make some progress in this context, we restrict our attention to the special case when V is a strict (or retract) extension of S and in various situations add further restrictions. For strict extensions, we are able to characterize ν_K, ν^K, ν_T and ν^T.

Section 2 contains some terminology, notation, background material and preliminary results. The relation T on $\mathscr{C}(V)$ is characterized in Section 3. In the remaining part of the paper, we assume that V is a strict extension of S. For a congruence ν on V given by means of an admissible triple, we calculate ν_T and ν^T in Section 4 and ν_K and ν^K in Section 5 again in terms of admissible triples.

2. Preliminaries. In addition to the standard terminology and notation, which can be found, for example, in [**1**], we state explicitly the following nomenclature and symbolism.

Let X be a set. The equality relation on X is denoted by ϵ_X or simply by ϵ. The universal relation on X is denoted by ω_X. The restriction of a function or a relation θ to X is denoted by $\theta|_X$. If θ is an equivalence relation on X and $x \in X$, then $x\theta$ denotes the θ-class containing x. If also $A \subseteq X$, then

$$A\theta = \{x \in X \mid x\theta a \text{ for some } a \in A\}$$

is the *saturation* of A by θ; if $A\theta = A$, then θ *saturates* A. If Y is also a set, then $X \backslash Y = \{x \in X \mid x \notin Y\}$.

Let R be a semigroup. If $A \subseteq R$, then $E(A)$ denotes the set of all idempotents in A. If R has an identity, then $R^1 = R$ otherwise R^1 stands for R with an identity adjoined. The congruence lattice of R is denoted by $\mathscr{C}(R)$. Assume that R has a zero. If $A \subseteq R$, then $A^* = A \backslash \{0\}$. An equivalence relation θ on R is 0-*restricted* if $\{0\}$ is a θ-class; the set of all 0-restricted congruences on R is denoted by $\mathscr{C}_0(R)$. Further, R is *categorical at zero* if for any $a, b, c \in R$, $ab \neq 0$ and $bc \neq 0$ imply $abc \neq 0$.

If θ is a relation on R, then θ^* denotes the congruence on R generated by θ. If θ is an equivalence relation on R, then θ^0 denotes the greatest congruence on R contained in θ; explicitly

$$a\theta^0 b \text{ if } xay\theta xby \text{ for all } x, y \in R^1.$$

If $A \subseteq Q$, let θ be the equivalence relation on R whose classes are A and $Q \backslash A$ (whichever one is nonempty), then $\pi_A = \theta^0$ is the *principal congruence* relative to A; explicitly

$$a\pi_A b \text{ if } (xay \in A \Leftrightarrow xby \in A \text{ for all } x, y \in R^1).$$

In fact, π_A is the greatest congruence ρ on R which saturates A. We shall sometimes write π_A^R for emphasis. If R has a zero, then $\zeta_R = \pi_{\{0\}}$ is the greatest 0-restricted congruence on R. If I is an ideal of R, then R/I denotes the *Rees quotient semigroup* of R relative to I; as a set $R/I = (R \backslash I) \cup \{0\}$.

Let R be a regular semigroup, that is for every $a \in R$ there exists $x \in R$ such that $a = axa$. Let $\rho \in \mathscr{C}(R)$. Then

$$\ker \rho = \{a \in R \mid a\rho e \text{ for some } e \in E(R)\}, \quad \operatorname{tr} \rho = \rho|_{E(R)}$$

are the *kernel* and the *trace* of ρ, respectively. They induce a complete $\wedge$-congruence K and a complete congruence T on $\mathscr{C}(R)$ by

$$\lambda K \rho \text{ if } \ker \lambda = \ker \rho, \qquad \lambda T \rho \text{ if } \operatorname{tr} \lambda = \operatorname{tr} \rho,$$

respectively. The K- and T-classes are intervals, so we use the notation

$$\rho K = [\rho_K, \rho^K], \qquad \rho T = [\rho_T, \rho^T].$$

LEMMA 2.1. ([**2**], Theorem 3.2). *Let R be a regular semigroup and $\rho \in \mathscr{C}(R)$. Then*

$$\rho_K = \{(x, x^2) \mid x \in \ker \rho\}^*, \qquad \rho^K = \pi_{\ker \rho}. \quad \square$$

We say that ρ is *idempotent pure* if $\ker \rho = E(S)$.

LEMMA 2.2. ([**2**], Theorem 3.2). *Let R be a regular semigroup, $\rho \in \mathscr{C}(R)$, $\theta = \operatorname{tr} \rho$ and denote by juxtaposition the product of binary relations. Then*

$$\rho_T = \theta^*, \qquad \rho^T = (\mathscr{L}\theta\mathscr{L}\theta\mathscr{L} \cap \mathscr{R}\theta\mathscr{R}\theta\mathscr{R})^0. \quad \square$$

Throughout the paper we fix the following notation: V is a regular semigroup and an (ideal) extension of S by Q. Hence S is an ideal of V, the Rees quotient $V/S \cong Q$, where we set $V = S \cup Q^*$; in addition, both S and Q are regular semigroups.

From ([**3**, Corollary 1 to Theorem 1]) we deduce the following description of congruences on V. Let $\sigma \in \mathscr{C}(S)$, P be an ideal of Q and $\tau \in \mathscr{C}_0(Q)$ satisfy the following conditions:

(i) $a, b \in Q \backslash P$, $a\tau b$, $x\sigma y \Rightarrow ax\sigma by$, $xa\sigma yb$,

(ii) for every $a \in P^*$ there exists $a' \in S$ such that $x \in S \Rightarrow ax\sigma a'x$, $xa\sigma xa'$.

In such a case, we say that a and a' are *σ-linked*, call (σ, P, τ) an *admissible triple* and define a relation ν on V by

$$a\nu b \Leftrightarrow \begin{cases} a\tau b & \text{if} \quad a, b \in Q\backslash P, \\ a'\sigma b' & \text{if} \quad a, b \in P^*, \\ a'\sigma b & \text{if} \quad a \in P^*, b \in S, \\ a\sigma b' & \text{if} \quad a \in S, b \in P^* \\ a\sigma b & \text{if} \quad a, b \in S, \end{cases}$$

where a, a' and b, b' are σ-linked. Then ν is a congruence on V and, conversely, every congruence on V has this form. According to ([**4**, Corollary 3.2]), this representation is unique.

The notation $\nu = \mathscr{C}(\sigma, P, \tau)$ will always denote the above congruence implicitly implying that (σ, P, τ) is an admissible triple.

In fact, given $\nu \in \mathscr{C}(V)$, the admissible triple for ν is (σ, P, τ), where

$$\sigma = \nu|_S, \qquad P = \{a \in Q^* \mid a\nu b \text{ for some } b \in S\} \cup \{0\},$$

$$a\tau b \Leftrightarrow a, b \in Q\backslash P, \qquad a\nu b, \text{ and } 0\tau 0.$$

Note that if a, a' are σ-linked and $\sigma \subseteq \sigma'$ for $\sigma' \in \mathscr{C}(S)$, then a, a' are also σ'-linked. We shall need the following criterion for inclusion of congruences on V.

LEMMA 2.3. ([**4**, Lemma 3.1]). *Let $\nu_i = \mathscr{C}(\sigma_i, P_i, \tau_i)$ for $i = 1, 2$. Then $\nu_1 \subseteq \nu_2$ if and only if $\sigma_1 \subseteq \sigma_2$, $P_1 \subseteq P_2$, τ_1 saturates $P_2 \backslash P_1$ and $\tau_1|_{Q\backslash P_2} \subseteq \tau_2|_{Q\backslash P_2}$.* $\square$

LEMMA 2.4. *Let $\nu = \mathscr{C}(\sigma, P, \tau)$. Then*

$$\ker \nu = \ker \sigma \cup \{a \in P^* \mid a' \in \ker \sigma \textit{ for some } a' \in S \ \sigma\textit{-linked to } a\} \cup (\ker \tau)^*.$$

Proof. Let $a \in V$. For $a \in S$, clearly $a \in \ker \nu$ if and only if $a \in \ker \sigma$. Similarly, for $a \in Q\backslash P$, clearly $a \in \ker \nu$ if and only if $a \in \ker \tau$.

Next let $a \in P^*$. Assume first that $a \in \ker \nu$. Then $a\nu e$ for some $e \in E(V)$ and we must have $e \in E(S \cup P^*)$. Let a' be an element of S σ-linked to a. Then $a'\nu e$ and $a'\nu \cap S$ is an idempotent σ-class and thus must contain an idempotent, say f. Then $a'\sigma f$ so that $a' \in \ker \sigma$. Conversely, suppose that $a' \in \ker \sigma$ for some $a' \in S$ σ-linked to a. Then $a'\sigma e$ for some $e \in E(S)$ and thus $a\nu a'\nu e$ so that $a \in \ker \nu$. □

A mapping $\varphi : Q^* \to S$ is a *partial homomorphism* if for any $a, b \in Q^*$, $ab \neq 0$ in Q implies $(ab)\varphi = (a\varphi)(b\varphi)$. If in addition,

$$ab = \begin{cases} (a\varphi)b & \text{if } a \in Q^*, b \in S, \\ a(b\varphi) & \text{if } a \in S, b \in Q^*, \\ (a\varphi)(b\varphi) & \text{if } a, b \in Q^*, ab \in S, \end{cases}$$

then the multiplication in V is *determined by* φ and V is a *strict extension* of S.

Starting with Section 4, we assume that V is a strict extension of S, where the multiplication is determined by the partial homomorphism $\varphi : Q^* \to S$. The mapping $\psi = \varphi \cup \iota_S$ is a retraction of V onto S, where ι_S is the identity mapping on S. If $1 \in V^1$ and $1 \notin V$, we write $1\varphi = 1\psi = 1 \in S^1$.

In such a case, we have the following important simplification.

LEMMA 2.5. ([**3**, Proposition 2]). *Let V be a strict extension of S, where the multiplication is determined by a partial homomorphism $\varphi : Q^* \to S$. Let $\sigma \in \mathscr{C}(S)$, P be an ideal of Q and $\tau \in \mathscr{C}_0(Q/P)$. Then (σ, P, τ) is an admissible triple if and only if*

$$a, b \in Q \backslash P, a\tau b \Rightarrow a\varphi\sigma b\varphi. \quad □$$

LEMMA 2.6. *Let V be a strict extension of S, where the multiplication is determined by a partial homomorphism $\varphi : Q^* \to S$. Let $\nu = \mathscr{C}(\sigma, P, \tau)$. Then*

$$\ker \nu = \ker \sigma \cup \{a \in P^* \mid a\varphi \in \ker \sigma\} \cup (\ker \tau)^*.$$

Proof. This is a direct consequence of Lemma 2.4. □

3. The trace relation. A technical lemma is needed here in order to characterize the relation T on $\mathscr{C}(V)$ in terms of T on $\mathscr{C}(S)$ and $\mathscr{C}(Q/P)$, where P is an ideal of Q.

LEMMA 3.1. *Let $\nu_i = \mathscr{C}(\sigma_i, Q, \epsilon)$ for $i = 1, 2$, $\sigma_1 T \sigma_2$, and $e \in E(Q^*)$. Then there exist $e_1, e_2 \in E(S)$ such that $e_1 \nu_1 e \nu_2 e_2$ and $e_1 \sigma_1 \wedge \sigma_2 e_2$.*

Proof. Let $i = 1, 2$. Since $e\nu_i \cap S$ is an idempotent σ_i-class, it contains an idempotent, e_i say. Clearly $e_1 \nu_1 e \nu_2 e_2$; we shall show that $e_1 \sigma_1 \wedge \sigma_2 e_2$. First let u_i be an inverse of ee_i and let $g_i = ee_i u_i e$. Then

$$g_i^2 = (ee_i u_i ee_i) u_i e = ee_i u_i e = g_i \in E(S),$$

$$g_i = ee_i u_i e \nu_i ee_i u_i ee_i = ee_i \nu_i e_i,$$

so that $g_i < e$ and $g_i \nu_i e$, $i = 1, 2$. Here $<$ is the natural partial order on the idempotents. Now $e\nu_2 g_2$ implies $g_1 = g_1 e \nu_2 g_1 g_2$ and $g_1 = eg_1 \nu_2 g_2 g_1$ so that

$$g_1 \sigma_2 g_1 g_2 \sigma_2 g_2 g_1. \tag{1}$$

Interchanging the roles of g_1, g_2 and σ_1, σ_2, we obtain

$$g_2 \sigma_1 g_2 g_1 \sigma_1 g_1 g_2. \tag{2}$$

Let v be an inverse of g_1g_2 and let $h = g_1g_2vg_1$. Then $h \in E(S)$ and we get

$$h = g_1g_2vg_1\sigma_2g_1g_2vg_1g_2 = g_1g_2\sigma_2g_1.$$

Hence $g_1\sigma_2h$ and the hypothesis implies that $g_1\sigma_1h$. Together with (2), this gives $g_1\sigma_1g_1g_2vg_1\sigma_1g_2vg_1$ whence

$$g_1\sigma_1g_2g_1. \tag{3}$$

Similarly, we let w be an inverse g_2g_1 and $t = g_1wg_2g_1$. Then $t \in E(S)$ and by (1), we obtain

$$t = g_1wg_2g_1\sigma_2g_2g_1wg_2g_1 = g_2g_1\sigma_2g_1.$$

Hence $g_1\sigma_2t$ and the hypothesis implies that $g_1\sigma_1t$. Together with (2), this gives $g_1\sigma_1g_1wg_2g_1\sigma_1g_1wg_2$ whence $g_1\sigma_1g_1g_2$. This together with (1) and (3) yields

$$g_1\sigma_1 \wedge \sigma_2g_1g_2\sigma_1 \wedge \sigma_2g_2g_1. \tag{4}$$

Now interchanging the roles of g_1, g_2 and σ_1, σ_2 we obtain

$$g_2\sigma_1 \wedge \sigma_2g_2g_1\sigma_1 \wedge \sigma_2g_1g_2$$

which together with (4) yields

$$g_1\sigma_1 \wedge \sigma_2g_2. \tag{5}$$

We have seen above that $e_iv_iev_ig_i$ so that $e_i\sigma_ig_i$ for $i = 1, 2$. The hypothesis implies that $e_1\sigma_1g_1$ gives $e_1\sigma_2g_1$ and $e_2\sigma_2g_2$ gives $e_2\sigma_1g_2$. This together with (5) finally yields

$$e_1\sigma_1 \wedge \sigma_2g_1\sigma_1 \wedge \sigma_2g_2\sigma_1 \wedge \sigma_2e_2$$

so that $e_1\sigma_1 \wedge \sigma_2e_2$, as required. □

We are now ready for the relation T.

THEOREM 3.2. *Let* $v_i = \mathscr{C}(\sigma_i, P_i, \tau_i)$ *for* $i = 1, 2$. *Then*

$$v_1Tv_2 \Leftrightarrow \sigma_1T\sigma_2, \qquad P_1 = P_2, \qquad \tau_1T\tau_2.$$

Proof. (a)$\Rightarrow$. If $e, f \in E(S)$ are such that $e\sigma_1f$, then ev_1f so by hypothesis ev_2f whence $e\sigma_2f$. Therefore $\operatorname{tr}\sigma_1 \subseteq \operatorname{tr}\sigma_2$ and by symmetry, $\sigma_1T\sigma_2$.

Let $e \in E(P_1 \backslash P_2)$. Then $ev_1 \cap S$ is an idempotent σ_1-class so it contains an idempotent, f say. Hence ev_1f which by hypothesis implies that ev_2f which is impossible since $e \notin P_2$ and v_2 saturates $S \cup P_2^*$. Therefore $E(P_1 \backslash P_2) = \varnothing$. If $a \in P_1 \backslash P_2$, then for any inverse a' of a, we have $aa' \in P_1 \backslash P_2$ which we have just seen to be impossible. Thus $P_1 \backslash P_2 = \varnothing$ that is $P_1 \subseteq P_2$. The equality $P_1 = P_2$ now follows by symmetry.

If $e, f \in E(T \backslash P)$ are such that $e\tau_1f$, then ev_1f so by hypothesis ev_2f whence $e\tau_2f$. Therefore $\operatorname{tr}\tau_1 \subseteq \operatorname{tr}\tau_2$ and by symmetry, $\tau_1T\tau_2$.

(b)$\Leftarrow$. Let $e, f \in E(V)$ be such that ev_1f. If $e, f \in S$, then $\sigma_1T\sigma_2$ implies that ev_2f, and if $e, f \in Q \backslash P$, then $\tau_1T\tau_2$ implies that ev_2f. Consider the case $e, f \in P_1^*$. By Lemma 3.1, there exist e_1, e_2, f_1, $f_2 \in E(S)$ such that

$$\left.\begin{array}{lll} e_1v_1ev_2e_2, & f_1v_1fv_2f_2, & ev_1f \\ e_1\sigma_1 \wedge \sigma_2e_2, & f_1\sigma_1 \wedge \sigma_2f_2. & \end{array}\right\} \tag{6}$$

https://doi.org/10.1017/S0017089500030767 Published online by Cambridge University Press

Consequently $e_1 \nu_1 e \nu_1 f \nu_1 f_1$ so that $e_1 \sigma_1 f_1$ which by hypothesis gives $e_1 \sigma_2 f$ whence

$$e \nu_2 e_2 \sigma_2 e_1 \sigma_2 f_1 \sigma_2 f_2 \nu_2 f$$

and thus $e \nu_2 f$.

By symmetry, it remains to consider the case $e \in P_1^*$, $f \in S$. With the above notation, we have $e_1 \nu_1 e \nu_1 f$ so that $e_1 \sigma_1 f$ which by hypothesis gives $e_1 \sigma_2 f$ where $e \nu_2 e_2 \sigma_2 e_1 f$ and thus $e \nu_2 f$.

Therefore in all cases $e \nu_2 f$ which proves that $\operatorname{tr} \nu_1 \subseteq \operatorname{tr} \nu_2$ and by symmetry equality prevails. Consequently $\nu_1 T \nu_2$. □

It would be natural to attempt to characterize ν_T and ν^T in terms of admissible triples when ν itself is given in this form. In this generality, this does not seem feasible. We limit ourselves to the following special case.

COROLLARY 3.3. *Let* $\nu = \mathscr{C}(\sigma, Q, \epsilon)$. *Then* $\nu^T = (\sigma^T, Q, \epsilon)$.

Proof. Since (σ, Q, ϵ) is an admissible triple, so is (σ^T, Q, ϵ). Let $\theta = \mathscr{C}(\sigma^T, Q, \epsilon)$. By Theorem 3.2, we have $\nu T \theta$. Next let $\nu' = \mathscr{C}(\sigma', P', \tau')$ be such that $\nu' T \nu$. By Theorem 3.2, we get $\sigma' T \sigma$, $P = Q$ and $\tau' = \epsilon$. Hence $\sigma' \subseteq \sigma^T$ which by Lemma 2.3 implies that $\nu' \subseteq \theta$. This proves the required maximality of θ. □

A similar analysis for the kernel relation is not possible because of the fact that a situation of the form $\mathscr{C}(\sigma, P, \tau) K \mathscr{C}(\sigma', P', \tau')$ with $P \neq P'$ is possible. We limit ourselves only to the analogue of Corollary 3.3 for the kernel.

PROPOSITION 3.4. *Let* $\nu = \mathscr{C}(\sigma, Q, \epsilon)$. *Then* $\nu^K = \mathscr{C}(\sigma^K, Q, \epsilon)$.

Proof. Since (σ, Q, ϵ) is an admissible triple so is (σ^K, Q, ϵ). Let $\theta = \mathscr{C}(\sigma^K, Q, \epsilon)$. By Lemma 2.4, we have

$$\ker \nu = \ker \sigma \cup \{a \in Q^* \mid a' \in \ker \sigma \text{ for some } a' \in S \ \sigma\text{-linked to } a\}, \tag{7}$$

$$\ker \theta = \ker \sigma^K \cup \{a \in Q^* \mid a'' \in \ker \sigma^K \text{ for some } a'' \in S \ \sigma^K\text{-linked to } a\}, \tag{8}$$

where $\ker \sigma = \ker \sigma^K$. If $a \in Q^*$ and $a' \in \ker \sigma$ are σ-linked, they are also σ^K-linked so that (7) is contained in (8). Conversely, let $a \in Q^*$ and $a'' \in \ker \sigma$ be σ^K-linked. Then for all $x \in S$, we have $ax \sigma^K a''x$ and $xa \sigma^K xa''$. Now let $a' \in S$ be such that a, a' are σ-linked. Then for all $x \in S$, we have $ax \sigma a'x$ and $xa \sigma xa'$ which implies that also $ax \sigma^K a'x$ and $xa \sigma^K xa'$. It follows that $a'x \sigma^K a''x$ and $xa' \sigma^K xa''$ for all $x \in S$. Recall that a semigroup T is weakly reductive if for any $a, b \in T$, $xa = xb$ and $ax = bx$ for all $x \in T$ implies that $a = b$. Weak reductivity of S/σ^K gives that $a' \sigma^K a''$. Since $a'' \in \ker \sigma^K$, we must have that $a' \in \ker \sigma^K = \ker \sigma$. Therefore (8) is contained in (7) and equality prevails so that $\nu K \theta$.

Next let $\nu' = \mathscr{C}(\sigma', P', \tau')$ be such that $\nu' K \nu$. By Lemma 2.4 we have an expression for $\ker \nu'$ analogous to that for $\ker \nu$ in (7) which implies that $\ker \sigma' = \ker \sigma$. It follows that $\sigma' K \sigma$ and thus $\sigma' \subseteq \sigma^K$. The remaining three conditions in Lemma 2.3 are trivially satisfied which gives that $\nu' \subseteq \theta$ which establishes the required maximality of θ. □

4. The lower and upper ends of trace classes. We reiterate first that henceforth V is a strict extension of S, where the multiplication is determined by a partial homomorphism $\varphi : Q^* \to S$. This will not be stated explicitly. Besides characterizing the ends, we include several consequences of these results.

THEOREM 4.1. *Let* $\nu = \mathscr{C}(\sigma, P, \tau)$. *Then* $\nu_T = \mathscr{C}(\sigma_T, P, \tau_T)$.

Proof. In view of Theorem 3.2, it suffices to show that (σ_T, P, τ_T) is an admissible triple.

Let $\eta = \operatorname{tr} \sigma$ and $\theta = \operatorname{tr} \tau$. By Lemma 2.2, we have $\sigma_T = \eta^*$ and $\tau_T = \theta^*$, the first of these taken within S and the second one within Q/P. Let $a\tau_T b$ for $a, b \in Q \backslash P$. Then there is a sequence

$$a = x_1 e_1 y_1, \qquad x_1 f_1 y_1 = x_2 e_2 y_2, \ldots, x_n f_n y_n = b \tag{9}$$

for some $x_i, y_i \in (Q/P)^1$ and $e_i \theta f_i$, $i = 1, 2, \ldots, n$. Since τ is 0-restricted and $\tau_T \subseteq \tau$, we have that τ_T is 0-restricted. In the above sequence, we have

$$a\tau_T x_1 f_1 y_1 \tau_T x_2 f_2 y_2 \ldots \tau_T b$$

which together with $a \neq 0$ in Q/P implies that all these elements are nonzero in Q/P. We may thus apply φ to the sequence (9) thereby obtaining

$$a\varphi = (x_1\varphi)(e_1\varphi)(y_1\varphi), (x_1\varphi)(f_1\varphi)(y_1\varphi) = (x_2\varphi)(e_2\varphi)(y_2\varphi) \ldots (x_n\varphi)(f_n\varphi)(y_n\varphi) = b\varphi$$

with $x_i\varphi, y_i\varphi \in S^1$. In addition $e_i \tau f_i$ implies that $e_i\varphi\sigma f_i\varphi$ by Lemma 2.5 so that $e_i\varphi\eta f_i\varphi$. Since $\sigma_T = \eta^*$ by Lemma 2.2, we get $a\varphi\sigma_T b\varphi$. By Lemma 2.5, (σ_T, P, τ_T) is an admissible triple and therefore $\nu_T = \mathscr{C}(\sigma_T, P, \tau_T)$. $\square$

THEOREM 4.2. *Let* $\nu = \mathscr{C}(\sigma, P, \tau)$. *Then* $\nu^T = \mathscr{C}(\sigma^T, P, \tau^T \cap \bar{\sigma})$ *where*

$$a\bar{\sigma}b \quad \text{if} \quad a, b \in Q \backslash P, \qquad a\varphi\sigma^T b\varphi, \qquad 0\bar{\sigma}0.$$

Proof. Let $\nu^T = \mathscr{C}(\sigma', \ , \ )$ where the blanks stand for entries to be determined. By Theorem 3.2, $\sigma T \sigma'$ and thus $\sigma' \subseteq \sigma^T$. Let $a\sigma^T b$. Then for $\theta = \operatorname{tr} \sigma$, by Lemma 2.2, we have

$$a(\mathscr{L}\theta\mathscr{L}\theta\mathscr{L} \cap \mathscr{R}\theta\mathscr{R}\theta\mathscr{R})^0 b$$

within S and thus for every $x, y \in S^1$,

$$xay\mathscr{L}\theta\mathscr{L}\theta\mathscr{L} \cap \mathscr{R}\theta\mathscr{R}\theta\mathscr{R}xby.$$

Let $\theta' = \operatorname{tr} \nu$. Then $\theta = \theta'|_S$ and for any $x, y \in V^1$,

$$(x\psi)a(y\psi)\mathscr{L}\theta'\mathscr{L}\theta'\mathscr{L} \cap \mathscr{R}\theta'\mathscr{R}\theta'\mathscr{R}(x\psi)b(y\psi)$$

and thus, for any $x, y \in V^1$,

$$xay\mathscr{L}\theta'\mathscr{L}\theta'\mathscr{L} \cap \mathscr{R}\theta'\mathscr{R}\theta'\mathscr{R}xby$$

so that $a(\mathscr{L}\theta'\mathscr{L}\theta'\mathscr{L} \cap \mathscr{R}\theta'\mathscr{R}\theta'\mathscr{R})^0 b$ within V. Therefore $a\nu^T b$ whence $a\sigma' b$. Consequently $\sigma^T \subseteq \sigma'$ and equality prevails. From Theorem 3.2 we get that $\nu^T = \mathscr{C}(\sigma^T, P, \ )$ where the blank stands for the entry to be determined.

Clearly $\bar{\sigma}$ is an equivalence relation on Q/P. The set $0\tau^T$ is an ideal of Q. If it contains a nonzero element, it also contains a nonzero idempotent, say e. But then $e\tau^T0$ so that $e\tau 0$, which contradicts the hypothesis that τ is 0-restricted.

Hence also τ^T is 0-restricted. Set $\eta = \tau^T \cap \bar{\sigma}$. Next let $a, b \in Q\backslash P$ be such that $a\eta b$ and let $c \in Q\backslash P$. If $ac \neq 0$ in Q/P, then $ac\tau^T bc$ implies $bc \neq 0$ and $a\bar{\sigma}b$ gives $(ac)\varphi = (a\varphi)(c\varphi)\sigma^T(b\varphi)(c\varphi) = (bc)\varphi$ so that $ac\bar{\sigma}bc$. If $ac = 0$ in Q/P, then $ac\tau^T bc$ implies $bc = 0$. Therefore $ac\eta bc$ in all cases; similarly $ca\eta cb$ which proves that η is a congruence on Q/P. Trivially η is 0-restricted. By Lemma 2.5, we conclude that (σ^T, P, η) is an admissible triple; let $\xi = \mathscr{C}(\sigma^T, P, \eta)$.

If $e, f \in E(Q\backslash P)$ are such that $e\tau f$, then by Lemma 2.5 we have $e\varphi\sigma f\varphi$ whence $e\varphi\sigma^T f\varphi$. It follows that $\operatorname{tr}\tau \subseteq \operatorname{tr}(\tau^T \cap \bar{\sigma}) = \operatorname{tr}\eta$. Conversely, trivially $\operatorname{tr}\eta \subseteq \operatorname{tr}\tau^T = \operatorname{tr}\tau$ and thus $\operatorname{tr}\tau = \operatorname{tr}\eta$. Therefore $\tau T\eta$ which by Theorem 3.2 implies that $\nu T\xi$.

Finally, let $\nu' = \mathscr{C}(\sigma', P', \tau')$ be such that $\nu' T\nu$. By Theorem 3.2, we get $\sigma' T\sigma$, $P' = P$ and $\tau' T\tau$. Hence $\sigma' \subseteq \sigma^T$ and $\tau' \subseteq \tau^T$. If $a, b \in Q\backslash P$ are such that $a\tau' b$, then by Lemma 2.5 we have $a\varphi\sigma' b\varphi$ whence $a\varphi\sigma^T b\varphi$ so that $a\bar{\sigma}b$. Therefore $\tau' \subseteq \bar{\sigma}$ which implies that $\tau' \subseteq \eta$. Now Lemma 2.3 implies that $\nu' \subseteq \xi$ which establishes the desired maximality of ξ. □

In the next consequence of the above theorem we have a case in which $\bar{\sigma}$ in the theorem may be omitted.

Corollary 4.3. *Let $\nu = \mathscr{C}(\sigma, P, \tau)$ and assume that φ maps $Q\backslash P$ onto S. Then $\nu^T = \mathscr{C}(\sigma^T, P, \tau^T)$.*

Proof. In view of Theorem 4.2, it suffices to prove that $\tau^T \subseteq \bar{\sigma}$. Hence let $a\tau^T b$, $\theta = \operatorname{tr}\tau$ and $\eta = \operatorname{tr}\sigma$. By Lemma 2.2, for any $x, y \in (Q/P)^1$, we have

$$xay\mathscr{L}\theta\mathscr{L}\theta\mathscr{L} \cap \mathscr{R}\theta\mathscr{R}\theta\mathscr{R}xby$$

so that

$$xay\mathscr{L}e\theta f\mathscr{L}g\theta h\mathscr{L}xby, \qquad xay\mathscr{R}e'\theta f'\mathscr{R}g'\theta h'\mathscr{R}xby \tag{10}$$

for some $e, f, g, h, e', f', g', h' \in E(Q/P)$. Since $\mathscr{L}$, $\mathscr{R}$ and θ are 0-restricted, we have $xay \neq 0$ if and only if $xby \neq 0$. Now assuming that $xay \neq 0$, we may apply φ to the sequences in (10) so that by Lemma 2.5, writing $1\varphi = 1$, we obtain

$$(x\varphi)(a\varphi)(y\varphi)\mathscr{L}e\varphi\eta f\varphi\mathscr{L}g\varphi\eta h\varphi\mathscr{L}(x\varphi)(b\varphi)(y\varphi),$$

$$(x\varphi)(a\varphi)(y\varphi)\mathscr{R}e'\varphi\eta f'\varphi\mathscr{R}g'\varphi\eta h'\varphi\mathscr{R}(x\varphi)(b\varphi)(y\varphi)$$

and thus

$$(x\varphi)(a\varphi)(y\varphi)\mathscr{L}\eta\mathscr{L}\eta\mathscr{L} \cap \mathscr{R}\eta\mathscr{R}\eta\mathscr{R}(x\varphi)(b\varphi)(y\varphi).$$

Since this holds for all $x, y \in (Q\backslash P) \cup \{1\}$ and φ maps $Q\backslash P$ onto S, we conclude that $a\varphi(\mathscr{L}\eta\mathscr{L}\eta\mathscr{L} \cap \mathscr{R}\eta\mathscr{R}\eta\mathscr{R})^0 b\varphi$ so that by Lemma 2.2 $a\varphi\sigma^T b\varphi$ and thus $a\bar{\sigma}b$. □

The next result provides a copy of the trace class of a congruence on V expressed by means of an admissible triple.

Corollary 4.4. *Denote by $\mathscr{AT}$ the set of all admissible triples and, for $(\sigma, P, \tau) \in \mathscr{AT}$, let*

$$(\sigma, P, \tau)T = \{(\sigma', P', \tau') \in \mathscr{AT} \mid \mathscr{C}(\sigma', P', \tau')T\mathscr{C}(\sigma, P, \tau)\}.$$

https://doi.org/10.1017/S0017089500030767 Published online by Cambridge University Press

Then, for any $(\sigma, P, \tau) \in \mathscr{AT}$, *we have*

$$(\sigma, P, \tau)T = ([\sigma_T, \sigma^T] \times \{P\} \times [\tau_T, \tau^T \cap \bar{\sigma}]) \cap \mathscr{AT}. \tag{11}$$

Proof. Let $(\sigma', P', \tau') \in (\sigma, P, \tau)T$. Then

$$(\mathscr{C}(\sigma, P, \tau))_T \subseteq \mathscr{C}(\sigma', P', \tau') \subseteq (\mathscr{C}(\sigma, P, \tau))^T$$

which by Theorems 4.1 and 4.2 gives

$$\mathscr{C}(\sigma_T, P, \tau_T) \subseteq \mathscr{C}(\sigma', P', \tau') \subseteq \mathscr{C}(\sigma^T, P, \tau^T \cap \bar{\sigma})$$

which in turn implies that $\sigma_T \subseteq \sigma' \subseteq \sigma^T$, $P = P'$ and $\tau_T \subseteq \tau' \subseteq \tau^T \cap \bar{\sigma}$. It follows that (σ', P', τ') is contained in the right hand side of (11). The proof of the converse follows essentially by reversing the steps above. □

5. The lower and upper ends of kernel classes. We continue with the hypothesis that V is a strict extension of S and characterize these ends including some special cases.

THEOREM 5.1. *Let* $\nu = \mathscr{C}(\sigma, P, \tau)$. *Then* $\nu_K = \mathscr{C}(\sigma_K, P', \tau')$ *where*

$$A = \{a \in P^* \mid a\varphi \in \ker \sigma\} \cup (\ker \tau)^* \tag{12}$$

and P' *is the set of all* a *in* Q^* *for which there exists a sequence*

$$a = x_1 u_1 y_1, x_1 v_1 y_1 = x_2 u_2 y_2, \ldots, x_{n-1} v_{n-1} y_{n-1} = x_n u_n y_n \tag{13}$$

in Q^*, $x_n v_n y_n \in S$ *with* $x_i, y_i \in (Q^1)^*$ *and*

$$\{u_i, v_i\} = \{z_i, z_i^2\}, \qquad z_i \in A \quad \text{for} \quad i = 1, 2, \ldots, n-1,$$

and for $i = n$, *either the same condition or* $v_n = u_n^2$, $u_n \in A$, $B = [A \cap (Q \backslash P')] \cup \{0\}$ *and* $\tau' = (\pi_B^{Q/P'})_K$.

Proof. Let $\nu_K = \mathscr{C}(\xi, R, \eta)$. By Lemma 2.6, we have

$$\begin{aligned} \ker \nu &= \ker \sigma \cup \{a \in P^* \mid a\varphi \in \ker \sigma\} \cup (\ker \tau)^*, \\ \ker \nu_K &= \ker \xi \cup \{a \in R^* \mid a\varphi \in \ker \xi\} \cup (\ker \eta)^* \end{aligned} \tag{14}$$

and $\nu K \nu_K$ implies that $\ker \sigma = \ker \xi$ and

$$A = \{a \in R^* \mid a\varphi \in \ker \sigma\} \cup (\ker \eta)^* = \ker \nu_K \cap Q^*. \tag{15}$$

The former implies that $\sigma K \xi$ whence $\sigma_K \subseteq \xi$.

Next let $a\xi b$. Then $a\nu_K b$ and thus there exists a sequence of the form

$$\left.\begin{aligned} &a = x_1 u_1 y_1, x_1 v_1 y_1 = x_2 u_2 v_2, \ldots, x_n v_n y_n = b \text{ for some} \\ &x_i, y_i \in V^1, \{u_i, v_i\} = \{z_i, z_i^2\}, z_i \in \ker \nu \text{ for } i = 1, 2, \ldots, n. \end{aligned}\right\} \tag{16}$$

Then (16) implies that

$$a = (x_1\psi)(u_1\psi)(y_1\psi), (x_1\psi)(v_1\psi)(y_1\psi) = (x_2\psi)(u_2\psi)(v_2\psi), \ldots, (x_n\psi)(v_n\psi)(y_n k) = b$$

with $\{u_i\psi, v_i\psi\} = \{z_i\psi, (z_i\psi)^2\}$, and in view of (14), $z_i\psi \in \ker \sigma$. Therefore $a\sigma_K b$ which implies that $\xi \subseteq \sigma_K$ and equality prevails.

Note that

$$R = \{a \in Q^* \mid a\nu_K b \text{ for some } b \in S\} \cup \{0\}. \tag{17}$$

First let $a \in P'$ with the notation as in the statement of the theorem. For $i = 1, 2, \ldots, n$, in view of (12) and (14), $z_i \in A$ implies that $z_i \in \ker \nu$ and $u_n \in A$ implies that $u_n \in \ker \nu$. Hence $a\nu_K x_n v_n y_n \in S$ and thus (17) implies that $a \in R$. Therefore $P' \subseteq R$.

Conversely, let $a \in R^*$. In view of (17), there exists a sequence of the form (16) with $b \in S$. Since $a = x_1 u_1 y_1 \notin S$, there exists a least positive integer j such that $x_i u_i y_i \notin S$ for all $i \leq j$. Without loss of generality, we may assume that $j = n - 1$. We thus have arrived at a sequence in Q^* of the form (13). It follows that $x_i, y_i \in (Q^1)^*$ for $i = 1, 2, \ldots, n$. In view of (15), we also have $z_i \in A$ for $i = 1, 2, \ldots, n-1$. For $i = n$, we have $\{u_n, v_n\} = \{z_n, z_n^2\}$ with $z_n \in \ker v$ and also $x_n u_n y_n \notin S$, $x_n v_n y_n \in S$. If $z_n^2 \in Q^*$, then $z_n \in Q^*$ and thus $z_n \in A$. Otherwise $z_n^2 \in S$ and we must have $v_n = z_n^2$ so that $u_n = z_n$ whence $v_n = u_n^2$ and $u_n \in A$. Therefore $a \in P'$. Consequently $R \subseteq P'$ and equality prevails.

Now let $a, b \in Q \backslash P'$. Assume first that $a\eta b$. Then $a\nu b$ and we have a sequence of the form (16). Here $z_i \in \ker \nu$ and since $a, b \in Q\backslash P'$, we must have, in view of (15), that

$$z_i \in \ker \nu \cap (Q\backslash P') = A \cap (Q\backslash P') = B^*.$$

In particular, $B = \ker \eta$ which in view of ([**2**, Theorem 2.13]) yields that $B = \ker \pi_B^{Q/P'}$. Now the sequence (16) where $z_i \in \ker \pi_B^{Q/P'}$ gives that $a(\pi_B^{Q/P'})_K b$, that is $a\tau' b$. Since η is 0-restricted, this shows that $\eta \subseteq \tau'$. Furthermore, $\ker \eta = B = \ker \pi_B^{Q/P'} = \ker \tau'$ so that $\tau' = \tau'_K \subseteq \eta$ and equality prevails. □

Corollary 5.2. *Let* $\nu = \mathscr{C}(\sigma, \{0\}, \tau)$. *Then* $\nu_K = \mathscr{C}(\sigma_K, \{0\}, \tau_K)$.

Proof. For $P = \{0\}$ in Theorem 5.1, we have $A = (\ker \tau)^*$ and $P' = \{0\}$ so that $B = \ker \tau$ and thus $\tau' = (\pi_{\ker \tau})_K = (\tau^K)_K = \tau_K$. □

Corollary 5.3. *Assume that Q is categorical at zero. Let $\nu = \mathscr{C}(\sigma, P, \tau)$ and suppose that $a \in P^*$ and $a\varphi \in \ker \sigma$ imply that $a^2 \in Q^*$. Then in the notation of Theorem* 5.1, $P' = \{0\}$ *and* $B = A \cup \{0\}$.

Proof. We adopt the notation of Theorem 5.1. Let $a \in P'^*$ and write x_n, u_n, v_n, y_n, z_n without subscripts. Then $xuy \neq 0$ in Q. Also $\{u, v\} = \{z, z^2\}$ with $z \in A \subseteq Q^*$. If $z \in (\ker \tau)^*$, then $z^2 \tau z$ so that $z^2 \in Q\backslash P$. Assume that $z \in P^*$. Then $z\varphi \in \ker \sigma$ and the hypothesis implis that $z^2 \in Q^*$. If $u = z$, then $v = u^2$ and thus $xu \neq 0$, $uu \neq 0$ and $uy \neq 0$ which by categoricity at zero yields $xu^2y \neq 0$, that is $xvy \neq 0$. Otherwise $u = z^2$ which gives $v^2 = u$ whence $xv^2y \neq 0$ which by categoricity at zero yields $xvy \neq 0$. Therefore $P' = \{0\}$. Hence $B = (A \cap Q^*) \cup \{0\} = A \cup \{0\}$. □

For a characterization of ν^K we need a preliminary result. Recall that $J(a)$ denotes the principal ideal generated by a.

Lemma 5.4. *Let* $R \subseteq Q^*$. *Then*

$$P = \{a \in Q^* \mid J(a) \cap R = \varnothing\} \cup \{0\}$$

is the union of all ideals I of Q such that $I \cap R = \varnothing$.

Proof. Since $R \subseteq Q^*$, we have $\{0\} \cap R = \varnothing$ and hence there exists at least one ideal of Q disjoint from R. Let U be the union of all such ideals.

If $a \in P$, then $J(a) \cap R = \varnothing$ and thus $J(a) \subseteq U$ so that $a \in U$. Therefore $P \subseteq U$. Conversely, let $a \in U$. Then there exists an ideal J of V such that $a \in J$ and $J \cap R = \varnothing$. Since $J(a) \subseteq J$, it follows that $J(a) \cap R = \varnothing$ and thus $a \in P$. Therefore $U \subseteq P$ and equality prevails. □

THEOREM 5.5. *Let* $v = \mathscr{C}(\sigma, P, \tau)$. *Then* $v^K = \mathscr{C}(\sigma^K, P', \tau')$ *where*

$$R = \{a \in Q\backslash \ker \tau \mid a\varphi \in \ker \sigma\},$$
$$P' = \{a \in Q^* \mid J(a) \cap R = \varnothing\} \cup P,$$
$$a\hat{\sigma}b \text{ if } a, b \in Q\backslash P',\ a\varphi\sigma^K b\varphi,\ 0\hat{\sigma}0,$$
$$\eta = (\tau|_{Q\backslash P'}) \cup \{(0,0)\}, \qquad \tau' = \eta^K \cap \hat{\sigma} \cap \zeta_{Q/P'}.$$

Proof. 1. *P' is an ideal of Q* by Lemma 5.4.

2. *τ saturates $P'\backslash P$.* Indeed, let $a\tau b$ with $a \in Q\backslash P$ and $b \in P'\backslash P$. Hence $J(b) \cap R = \varnothing$ and thus, for every $x, y \in Q^1$, $xby \notin R$. We wish to show that $xay \notin R$. If $xby \in \ker \tau$, then $xay\tau xby$ implies that $xay \in \ker \tau$ and thus $xay \notin R$. Otherwise, $xby \notin \ker \tau$. Since $xby \notin R$, we must have $(xby)\varphi \notin \ker \tau$. The hypothesis $a\tau b$ implies that $a\varphi\sigma b\varphi$ by Lemma 2.5. Hence $(xay)\varphi\sigma(xby)\varphi$ and thus $(xay)\varphi \notin \ker \sigma$. Also $xby \notin \ker \tau$ implies that $xay \notin \ker \tau$. Therefore again $xay \notin R$. Consequently $a \in P'$ and thus τ saturates $P'\backslash P$.

3. *τ saturates $Q\backslash P'$ and $\eta \in \mathscr{C}_0(Q/P')$.* The first assertion follows from part 2 and the fact that τ saturates $Q\backslash P$. The second assertion is a consequence of the first.

4. *$\tau' \in \mathscr{C}_0(Q/P')$.* Let $a, b, c \in Q\backslash P'$ be such that $a\tau' b$ and $ac \in Q\backslash P'$. Then $a\zeta_{Q/P'}b$ implies that $ac\zeta_{Q/P'}bc$ and thus $bc \neq 0$ in Q/P', that is $bc \in Q\backslash P'$, since $\zeta_{Q/P'}$ is 0-restricted. In addition $a\eta^K b$ implies that $ac\eta^K bc$. Finally, $a\varphi\sigma^K b\varphi$ implies $(ac)\varphi = (a\varphi)(c\varphi)\sigma^K(b\varphi)(c\varphi) = (bc)\varphi$. Therefore $ac\tau' bc$. By symmetry, $bc \neq 0$ in Q/P' implies $ac \neq 0$ and the same conclusion is reached. Hence τ' is a right congruence and, by duality, it is a congruence. Since $\tau' \subseteq \zeta_{Q/P'}$ (or $\tau \subseteq \hat{\sigma}$), τ' is 0-restricted.

5. *(σ^K, P', τ') is an admissible triple.* Indeed, let $a, b \in Q\backslash P'$ be such that $a\tau' b$. Then $a\hat{\sigma}b$ which yields $a\varphi\sigma^K b\varphi$. The assertion now follows by Lemma 2.5. Let $\theta = \mathscr{C}(\sigma^K, P', \tau')$.

6. *$vK\theta$.* By Lemma 2.6, we have

$$\begin{aligned} \ker v &= \ker \sigma \cup \{a \in P^* \mid a\varphi \in \ker \sigma\} \cup (\ker \tau)^*, \\ \ker \theta &= \ker \sigma^K \cup \{a \in P'^* \mid a\varphi \in \ker \sigma^K\} \cup (\ker \tau')^*. \end{aligned} \tag{18}$$

In order to prove that $\ker v = \ker \theta$, we let $a \in V$ and consider the following cases.

(i) $a \in S: a \in \ker v \Leftrightarrow a \in \ker \sigma \Leftrightarrow a \in \ker \sigma^K \Leftrightarrow a \in \ker \theta$.

(ii) $a \in P\backslash S: a \in \ker v \Leftrightarrow a\varphi \in \ker \sigma \Leftrightarrow a\varphi \in \ker \sigma^K \Leftrightarrow a \in \ker \theta$.

(iii) $a \in P'\backslash P$: If $a \in (\ker \tau)^*$, then $a\tau e$ from $e \in E(Q\backslash P)$ which by Lemma 2.5 yields $a\varphi\sigma e\varphi$ and thus $a\varphi \in \ker \sigma$. Hence

$$a \in \ker v \Rightarrow a \in \ker \tau \Rightarrow a\varphi \in \ker \sigma \Rightarrow a\varphi \in \ker \sigma^K \Rightarrow a \in \ker \theta.$$

For the converse, we first note that $a \in P'$ implies $a \notin R$ and thus either $a \in \ker \tau$ or $a\varphi \notin \ker \sigma$. Hence the above sequence of implications can be reversed.

(iv) $a \in Q\backslash P'$: First note that

$$\begin{aligned} a \in \ker \theta &\Leftrightarrow a \in \ker \tau' \\ &\Leftrightarrow a \in \ker \eta^K, a\hat{\sigma}a^2, a \in \ker \zeta_{Q/P'} \\ &\Leftrightarrow a \in \ker \eta, a\varphi\sigma^K a^2\varphi, a \in \ker \zeta_{Q/P'} \\ &\Leftrightarrow a \in \ker \tau, a\varphi\sigma a^2\varphi, a \in \ker \zeta_{Q/P'} \\ &\Leftrightarrow a\tau a^2, a\varphi\sigma a^2\varphi, a\zeta_{Q/P'}a^2. \end{aligned} \tag{19}$$

Assume that $a\tau a^2$. By Lemma 2.5, we have $a\varphi\sigma a^2\varphi$. For any $x, y \in (Q/P')^1$, we have $xay\tau xa^2y$. By part 3, τ saturates $Q\backslash P'$ which then implies that $xay \neq 0$ if and only if $xa^2y \neq 0$ in Q/P'. Therefore $a\zeta_{Q/P'}a^2$. Now (19) implies that

$$a \in \ker \nu \Leftrightarrow a \in \ker \tau \Leftrightarrow a \in \ker \tau' \Leftrightarrow a \in \ker \theta.$$

Therefore $\ker \nu = \ker \theta$.

7. *If $\nu_1 K \nu$, then $\nu_1 \subseteq \theta$. Let* $\nu_1 = \mathscr{C}(\sigma_1, P_1, \tau_1)$ and assume that $\nu_1 K \nu$. By Lemma 2.6, we have

$$\ker \nu_1 = \ker \sigma_1 \cup \{a \in P_1^* \mid a\varphi \in \ker \sigma_1\} \cup (\ker \tau_1)^*$$

which, together with (18), by hypothesis gives $\ker \sigma_1 = \ker \sigma$ and

$$\{a \in P_1^* \mid a\varphi \in \ker \sigma_1\} \cup (\ker \tau_1)^* = \{a \in P^* \mid a\varphi \in \ker \sigma\} \cup (\ker \tau)^*. \tag{20}$$

It follows that $\sigma_1 K \sigma$ and hence $\sigma_1 \subseteq \sigma^K$. In order to prove that $\nu_1 \subseteq \theta$, by Lemma 2.3, it remains to show that

$$P_1 \subseteq P', \qquad \tau_1 \text{ saturates } P'\backslash P_1, \qquad \tau_1|_{Q\backslash P'} \subseteq \tau'|_{Q\backslash P'}. \tag{21}$$

Let $a \in R \cap P_1$. Then $a \notin \ker \tau$, $a\varphi \in \ker \sigma$ and $a \in P_1^*$. The last two conditions imply that a is in the left hand side of (20). But the first two conditions imply that a is not in the right hand side of (20). This being impossible, we conclude that $R \cap P_1 = \varnothing$. By Lemma 5.4, P' is the greatest ideal of Q which is disjoint from R and thus $P_1 \subseteq P'$. This establishes the first condition in (21).

In order to prove the second condition in (21), we let

$$A = \{a \in Q\backslash P_1 \mid a\tau_1 b \text{ for some } b \in P'\backslash P_1\} \cup P_1.$$

We show next that A is an ideal of Q. Indeed, let $a \in A^*$ and $c \in Q\backslash P_1$ be such that $ac \notin P_1$. There exists $b \in P'\backslash P_1$ such that $a\tau_1 b$ by the definition of A. Hence $ac\tau_1 bc$ whence $bc \notin P_1$ since τ_1 is 0-restricted. Thus $bc \in P'\backslash P_1$ which yields $ac \in A^*$. By duality and since P_1 is an ideal of Q, we conclude that A is an ideal of Q.

Now assume that $a \in A \cap R$. We have seen above that $P_1 \cap R = \varnothing$. Hence $a \in A\backslash P_1$ and there exists $b \in P'\backslash P_1$ such that $a\tau_1 b$. Further, $a \in R$ implies that $a \notin \ker \tau$ and $a\varphi \in \ker \sigma$. Since $a\tau_1 b$, by Lemma 2.5 we get $a\varphi\sigma_1 b\varphi$; also $a\varphi \in \ker \sigma$ implies $a\varphi \in \ker \sigma_1$ whence $b\varphi \in \ker \sigma_1$. Hence $b\varphi \in \ker \sigma$ and since $b \in P'$, we also have that $b \notin R$. But then $b \in \ker \tau$ by the definition of R. By (20), $a \notin \ker \tau$ implies $a \notin \ker \tau_1$, and $b \in \ker \tau$ implies that $b \in \ker \tau_1$. This is incompatible with $a\tau_1 b$. Therefore $A \cap R = \varnothing$.

We have proved that A is an ideal of Q disjoint from R which by Lemma 5.4 gives that $A \subseteq P'$. It follows that τ_1 saturates $P'\backslash P_1$.

It remains to establish the last condition in (21). Since τ_1 saturates both $Q\backslash P_1$ and $P'\backslash P_1$, it also saturates $Q\backslash P'$. Letting $\tau_2 = (\tau_1|_{Q\backslash P'}) \cup \{(0,0)\}$, we get $\tau_2 \in \mathscr{C}_0(Q/P')$. Now condition (20) yields that $\tau_2 K \tau$ which implies that $\tau_2 \subseteq \tau^K$. Let $a, b \in Q\backslash P'$ be such that $a\tau_1 b$. Then $a\tau_2 b$ and thus $a\tau^K b$. Also $a\tau_1 b$ implies that $a\varphi\sigma_1 b\varphi$ whence $a\varphi\sigma^K b\varphi$ since $\sigma_1 \subseteq \sigma^K$, and thus $a\hat{\sigma}b$. Since $\tau_2 \in \mathscr{C}_0(Q/P')$ and $\zeta_{Q/P'}$ is the greatest 0-restricted congruence on Q/P', we get $\tau_2 \subseteq \zeta_{Q/P'}$. In particular, $a\zeta_{Q/P'}b$. We have proved that $a\tau' b$ which shows that $\tau_1|_{Q\backslash P'} \subseteq \tau'|_{Q\backslash P'}$.

This completes the verification of the requirements in Lemma 2.3 for the inclusion $\nu_1 \subseteq \theta$. Therefore θ has the required maximality so that $\theta = \nu^K$. □

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