# The Apolar Locus of Two Tetrads of Points.

By Dr William P. Milne.

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1. In the present paper I propose to investigate the fundamental geometrical properties of the Apolar Locus of two tetrads of points in a plane.

The Apolar Locus of two tetrads of points $K, L, M, N$ and $P, Q, R, S$ is defined in the locus of the point $X$, moving so that the pencils $X[K, L, M, N]$ and $X[P, Q, R, S]$ are apolar.

In particular, I wish to find in a convenient form the necessary and sufficient conditions which a quartic curve has to satisfy in order that it may be the Apolar Locus of two assigned tetrads.

2. We shall begin by proving some geometrical properties of a tetrad of points on a quartic curve, as these results will be required in the sequel.

Lemma I.—*The Principal Conics.*

Let $P, Q, R, S$ be a tetrad of points on the quartic curve

$$F \equiv ax^4 + by^4 + cz^4 + 6fy^2z^2 + 6gz^2x^2 + 6hx^2y^2 + \text{etc.} = 0, \quad \ldots\ldots (1)$$

no two of the four points being coincident.

Let the tetrad be regarded as the common points of the pencil of conics

$$S_1 + \lambda S_2 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Then it is plain that a unique pair of conics

$$pS_1^2 + 2qS_1S_2 + rS_2^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

of the system (2) can be so determined that the quartic

$$F + pS_1^2 + 2qS_1S_2 + rS_2^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

shall pass through the vertices of the self-polar triangle of the pencil $S_1 + \lambda S_2$. For if we refer to the self-polar triangle as the triangle of reference and take the equations of $S_1$ and $S_2$ in the form

$$\left.\begin{aligned} S_1 &\equiv \alpha x^2 + \beta y^2 + \gamma z^2 \\ S_2 &\equiv x^2 + y^2 + z^2, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

the conditions that the quartic (4) pass through the vertices of the triangle of reference are

$$\begin{aligned} a+p\alpha^2+2q\alpha+r&=0\\ b+p\beta^2+2q\beta+r&=0\\ c+p\gamma^2+2q\gamma+r&=0 \end{aligned} \qquad \text{.......................(6)}$$

which gives us unique values for $p$, $q$, $r$. The quartic we require is therefore (by (4) and (6))

$$\begin{vmatrix} F & S_1^2 & S_1 S_2 & S_2^2 \\ a & \alpha^2 & \alpha & 1 \\ b & \beta^2 & \beta & 1 \\ c & \gamma^2 & \gamma & 1 \end{vmatrix} = 0. \qquad \text{..................(7)}$$

We shall call (7) the "*Auxiliary Quartic*" of $F$ with respect to the tetrad $P$, $Q$, $R$, $S$. This Auxiliary Quartic possesses the property of passing through the four points of the tetrad and also the vertices of their self-polar triangle.

Furthermore, we shall call the conics obtained from (3) and (6), viz.

$$\begin{vmatrix} 0 & S_1^2 & S_1 S_2 & S_2^2 \\ a & \alpha^2 & \alpha & 1 \\ b & \beta^2 & \beta & 1 \\ c & \gamma^2 & \gamma & 1 \end{vmatrix} = 0 \qquad \text{...................(8)}$$

the "*Principal Conics*" of the tetrad $PQRS$ with respect to the quartic $F$.

LEMMA II.—*The Generating Conic.*

*If* $S_1+\lambda S_2=0$ *and* $U_1+\mu U_2=0$ *are two pencils of conics, and there is some member* $S_0$ *of the first pencil apolar (regarded as an envelope) to all members of the second, then the locus of the intersections of a member of the first (regarded as an envelope) with the member of the second (regarded as a locus) to which it is apolar, is a quartic through the base-points of the first pencil and the vertices of their common self-polar triangle.*

For, if we state the condition that $S_1+\lambda S_2$ (regarded as an envelope) is apolar to $U_1+\mu U_2$ (regarded as a locus), we obtain a (2, 1) correspondence between $\lambda$ and $\mu$.

But, given $\mu$, one of the corresponding values of $\lambda$ thus obtained is the parameter of $S_0$. Hence, omitting this constant value in

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all cases, we obtain a (1, 1) correspondence between $\lambda$ and $\mu$. The required locus of the intersections of $S_1+\lambda S_2$ and $U_1+\mu U_2$ is therefore a quartic curve. To show that this quartic curve passes through the four points common to all members of the pencil $S_1+\lambda S_2$, we take one of these points $P$ and consider the member $U_1+\mu U_2$ which passes through $P$. A unique member of the pencil $S_1+\lambda S_2$ can be found apolar to this last-mentioned conic, and as $P$ is a point of intersection of both these corresponding conics, $P$ must lie on the quartic generated by such corresponding conics. Furthermore, the quartic passes through the vertices of the self-polar triangle. For, if we choose $\lambda$ so that $S_1+\lambda S_2$ is a pair of straight lines, then the reciprocal of $S_1+\lambda S_2$ will be the tangential equation to the corresponding vertex $A$ of the self-polar triangle taken twice over. Hence the corresponding conic $U_1+\mu U_2$ being apolar (as a locus) to $A^2$ (as an envelope) passes through $A$; the result of which is that $A$ lies on a pair of corresponding conics, and consequently on the quartic. We call $S_0$ the *Generating Conic* of the quartic passing through the four given points and their self-polar triangle.

Conversely, any quartic through four points and their self-polar triangle can be generated by the above method, and has a unique Generating Conic. For, any quartic through the points common to $S_1$ and $S_2$ can be expressed in the form $S_1T_2=S_2T_1$, which can be written in the more extended form

$$S_1(a_2x^2+b_2y^2+c_2z^2+2f_2yz+2g_2zx+2h_2xy)$$
$$=S_2(a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy). \quad \ldots(9)$$

Hence, using (5), the quartic (9) passes through the vertices of the self-polar triangle if

$$\begin{aligned}\alpha a_2&=a_1\\ \beta b_2&=b_1\\ \gamma c_2&=c_1.\end{aligned} \quad \ldots\ldots\ldots\ldots(10)$$

Now (9) can be written in the form

$$S_1(T_2+\kappa S_2)=S_2(T_1+\kappa S_1), \quad \ldots\ldots\ldots\ldots(11)$$

in which $S_1$ (envelope) is apolar to $T_1+\kappa S_1$ (locus), if

$$\frac{a_1+\kappa\alpha}{\alpha}+\frac{b_1+\kappa\beta}{\beta}+\frac{c_1+\kappa\gamma}{\gamma}=0\ ; \quad \ldots\ldots\ldots\ldots(12)$$

and also $S_2$ (envelope) is apolar to $T_2+\kappa S_2$ (locus), if

$$(a_2+\kappa)+(b_2+\kappa)+(c_2+\kappa)=0. \qquad \text{.................} (13)$$

Now (12) and (13) are consistent in virtue of (10), which proves that every quartic through the common points of $S_1$, $S_2$ and the vertices of the self-polar triangle can be reduced to the form

$$S_1 U_2 = S_2 U_1,$$

in which $S_1$ and $S_2$ (as envelopes) are respectively apolar to $U_1$ and $U_2$ (as loci).

We have still to show that $S_1+\lambda S_2$ (envelope) is apolar to its corresponding member of the $U$-pencil, viz. $U_1+\lambda U_2$.

Let us take

$$\left.\begin{array}{l} S_1 \equiv \alpha x^2+\beta y^2+\gamma z^2 \\ S_2 \equiv x^2+y^2+z^2 \\ U_1 \equiv a_1 x^2+b_1 y^2+c_1 z^2+2f_1 yz+2g_1 zx+2h_1 xy \\ U_2 \equiv a_2 x^2+b_2 y^2+c_2 z^2+2f_2 yz+2g_2 zx+2h_2 xy \end{array}\right\} \text{......} (14)$$

with the conditions

$$\left.\begin{array}{ll} \alpha\, a_2=a_1 & \\ \beta\, b_2=b_1 & \text{as in (10)} \\ \gamma\, c_2=c_1 & \end{array}\right\} \text{..............} (15)$$

and

$$\left.\begin{array}{l} \dfrac{a_1}{\alpha}+\dfrac{b_1}{\beta}+\dfrac{c_1}{\gamma}=0 \\ a_2+b_2+c_2=0 \end{array}\right\}. \text{......................} (16)$$

The two conditions (16) are equivalent in virtue of (15), and exist in virtue of the condition that $S_1$ and $S_2$ regarded as envelopes are to be apolar to $U_1$ and $U_2$ respectively, regarded as loci.

Now $S_1+\lambda S_2$ (envelope) will be apolar to $U_1+\lambda U_2$ (locus) if

$$\frac{a_1+\lambda a_2}{\alpha+\lambda}+\frac{b_1+\lambda b_2}{\beta+\lambda}+\frac{c_1+\lambda c_2}{\gamma+\lambda}=0,$$

*i.e.* if, by using (15),

$$\frac{a_2(\alpha+\lambda)}{\alpha+\lambda}+\frac{b_2(\beta+\lambda)}{\beta+\lambda}+\frac{c_2(\gamma+\lambda)}{\gamma+\lambda}=0, \qquad \text{.............} (17)$$

which is true in virtue of (16).

We have thus shewn that every such Seven-Point Quartic passing through the four points of a quadrangle and the vertices of the self-polar triangle can be generated by the intersections of the corresponding members of two pencils of conics $S_1 + \lambda S_2$ and $U_1 + \lambda U_2$, in which every member of the $S$-pencil regarded as an envelope is apolar to every member of the $U$-pencil regarded as a locus.

It still remains to demonstrate the existence of the unique "*generating-conic.*" Using the notation of (14), we have to show that a unique member of the $S$-pencil exists, which, regarded as an envelope, is apolar to every member of the $U$-pencil regarded as a locus. Let the required generating-conic be $S_1 + \rho S_2$, and let it, if possible, be apolar to $U_1 + \lambda U_2$ for all values of $\lambda$. This requires the condition

$$\frac{a_1 + \lambda a_2}{\alpha + \rho} + \frac{b_1 + \lambda b_2}{\beta + \rho} + \frac{c_1 + \lambda c_2}{\gamma + \rho} = 0,$$

*i.e.* using (15),

$$\frac{a_2 (\alpha + \lambda)}{\alpha + \rho} + \frac{b_2 (\beta + \lambda)}{\beta + \rho} + \frac{c_2 (\gamma + \lambda)}{\gamma + \rho} = 0,$$

which becomes

$$(\rho - \lambda) \{(a_2 \alpha + b_2 \beta + c_2 \gamma) \rho - (a_2 \beta\gamma + b_2 \gamma\alpha + c_2 \alpha\beta)\} = 0,$$

after some reduction with the help of (15).

Hence, neglecting the factor $\rho - \lambda$, we obtain the required value for $\rho$, and the equation of the *generating-conic* becomes

$$(a_2 \alpha + b_2 \beta + c_2 \gamma) S_1 + (a_2 \beta\gamma + b_2 \gamma\alpha + c_2 \alpha\beta) S_2 = 0. \quad \ldots\ldots(18)$$

Hence we have the following result :—

*Any quartic through four points and the vertices of their self-polar triangle can be generated by the method just described. For every such quartic the method of generation is unique, and to it corresponds a single member of the pencil, its "generating conic."*

3. We shall next require some of the geometrical properties existing between a pencil of conics $S_1 + \lambda S_2$ and a Class-Quartic,

$$\psi_t^4 \equiv Al^4 + Bm^4 + Cn^4 + 6Fm^2 n^2 + 6Gn^2 l^2 + 6Hl^2 m^2 + \text{etc.}$$

LEMMA III.—*The Self-Conjugate Conics.*

*Given any conic $S_1 + \lambda S_2$ of the S-pencil, there exists a unique conic $S_1 + \mu S_2$ belonging to the pencil, such that $(S_1 + \lambda S_2)(S_1 + \mu S_2)$ regarded as a quartic-locus is apolar to $\psi_t^4$ regarded as a quartic envelope.*

For if
$$S_1 \equiv \alpha x^2 + \beta y^2 + \gamma z^2 \\ S_2 \equiv \ x^2 + \ y^2 + \ z^2 \ , \qquad \text{.....................(19)}$$
then $(S_1 + \lambda S_2)(S_1 + \mu S_2)$ is apolar to $\psi_l{}^4 = 0$ if

$$A(\alpha + \lambda)(\alpha + \mu) + B(\beta + \lambda)(\beta + \mu) + C(\gamma + \lambda)(\gamma + \mu)$$
$$+ F\{(\beta + \lambda)(\gamma + \mu) + (\beta + \mu)(\gamma + \lambda)\}$$
$$+ G\{(\gamma + \lambda)(\alpha + \mu) + (\gamma + \mu)(\alpha + \lambda)\}$$
$$+ H\{(\alpha + \lambda)(\beta + \mu) + (\alpha + \mu)(\beta + \lambda)\} = 0.$$

This is a (1, 1) algebraic correspondence between $\lambda$ and $\mu$, having in general two self-corresponding members. We shall call these the two "*self-conjugate*" conics of the pencil.

LEMMA IV.—*The Polo-Reciprocal Conic.*

If $l$ typify the tangential coordinates of a given line, then $\psi_{l'}{}^2 \psi_l{}^2 = 0$ is the polar conic of the line $l'$ with respect to the class-quartic $\psi_l{}^4$. The conic-envelope $\psi_{l'}{}^2 \psi_l{}^2$ is apolar to the conic locus $a_x{}^2 = 0$ if $\psi_{l'}{}^2 \psi_a{}^2 = 0$. Hence $\psi_l{}^2 \psi_a{}^2$ is the locus of lines whose polar conics are apolar to $a_x{}^2$, in virtue of which property $\psi_l{}^2 \psi_a{}^2$ is usually called the polar conic of $a_x{}^2$ with respect to $\psi_l{}^4$. Now the polar conics of all the members of the pencil $S_1 + \lambda S_2$ of (19) plainly form a pencil of conic envelopes

$$\Pi_1 + \lambda \Pi_2 = 0. \qquad \text{........................ (20)}$$

Let

$$\Pi_1 \equiv A_1 l^2 + B_1 m^2 + C_1 n^2 + 2F_1 mn + 2G_1 nl + 2H_1 lm = 0,$$
and
$$\Pi_2 \equiv A_2 l^2 + B_2 m^2 + C_2 n^2 + 2F_2 mn + 2G_2 nl + 2H_2 lm = 0. \qquad (21)$$

It has to be noted that the conics $S_1 \equiv a_x{}^2$ and $S_2 \equiv a_x'{}^2$ are together apolar to $\psi_l{}^4 = 0$ if $\psi_a{}^2 \psi_{a'}{}^2 = 0$, *i.e.* if $S_1$ be apolar to $\Pi_2$ or $S_2$ apolar to $\Pi_1$ (these being equivalent conditions, each requiring the vanishing of $\psi_a{}^2 \psi_{a'}{}^2$).

Hence, if $I_{12}$ denote the invariant expressing the condition that $S_1 S_2$ be apolar to $\psi_l{}^4$, we have

$$I_{11} \equiv \alpha A_1 + \beta B_1 + \gamma C_1 \qquad \text{.......................(22)}$$
$$I_{12} \equiv \alpha A_2 + \beta B_2 + \gamma C_2 \equiv A_2 + B_2 + C_2 \qquad \text{......(23)}$$
$$I_{22} \equiv A_2 + B_2 + C_2 . \qquad \text{.......................(24)}$$

It will also be convenient to use the following abbreviation :—

$$P \equiv \alpha^2 A_2 + \beta^2 B_2 + \gamma^2 C_2 \qquad \text{..................(25)}$$
$$Q \equiv \beta\gamma A_1 + \gamma\alpha B_1 + \alpha\beta C_1 . \qquad \text{....................(26)}$$

We therefore have, on expressing the condition that $S_1+\lambda S_2$ be apolar to $\Pi_1+\lambda\Pi_2$, the equation

$$\lambda^2 I_{22}+2\lambda I_{12}+I_{11}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(27)$$

giving the parameters of the two "self-conjugate conics" with respect to $\psi_\iota^4$ of the $S_1$, $S_2$ pencil.

Let us now form the reciprocal of the conic $\Pi_1+\mu\Pi_2$ with respect to the conic $S_1+\lambda S_2$. We obtain

$$(A_1+\mu A_2)(\alpha+\lambda)^2x^2+(B_1+\mu B_2)(\beta+\lambda)^2y^2+(C_1+\mu C_2)(\gamma+\lambda)^2z^2 + \text{etc}=0. \quad (28)$$

The conic (28) will be apolar to the conic $S_1+\mu S_2$ (regarded as an envelope) if

$$\Sigma\,(\beta+\mu)\,(\gamma+\mu)\,(\alpha+\lambda)^2\,(A_1+\mu A_2)=0,$$

that is, if

$$(\mu+\alpha)\,(\mu+\beta)\,(\mu+\gamma)\,(I_{22}\,\lambda^2+2I_{12}\lambda+I_{11}) + (\lambda-\mu)^2\,(\mu\overline{P-I_{11}}+\overline{Q-\alpha\beta\gamma\,I_{22}})=0. \quad (29)$$

Hence, between $\lambda$ and $\mu$ exists a (2, 3) correspondence defined by the above relationship.

Putting $I_{22}\,\lambda^2+2I_{12}\,\lambda+I_{11}$ equal to zero, we see that to either of the self-conjugate conics correspond themselves (taken twice over) and another conic whose parameter is given by

$$(P-I_{11})\,\mu+(Q-\alpha\beta\gamma\,I_{22})=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

Conversely, to the conic (30) correspond the self-conjugate conics.

The conic (30) is thus uniquely defined, and we shall call it the "*Polo-Reciprocal Conic*" of the $S_1$, $S_2$ pencil.

4. *The Apolar Locus.*

If $\phi_\iota^4=0$ and $\psi_\iota^4=0$ be the equations of two class-quartics, the equation to the locus of a point $X$, moving so that the four tangents from $X$ to $\phi_\iota^4$ apolarly separate those from $X$ to $\psi_\iota^4$, is

$$(\phi\,\psi\,x)^4=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

We shall call the quartic (31) the "*Apolar Locus*" of $\phi_\iota^4$ and $\psi_\iota^4$.

If, again, $\phi_\iota^4$ consist of four points, we may regard these points as the intersection of a pencil of conics, viz.,

$$\begin{aligned} S_1 &\equiv a_x^{\,2}\equiv b_x^{\,2}=0 \\ S_2 &\equiv a_x^{\,2}\equiv b_x^{\,2}=0. \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$

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In this case

$$\phi_i^4 \equiv (aa'l)^2(bb'l)^2 - (abl)^2(a'b'l)^2 = 0. \quad \ldots\ldots\ldots\ldots(33)$$

Hence, by (31) and (33) the apolar locus is

$$(aa'\widehat{\psi x})^2(bb'\widehat{\psi x})^2 - (ab\widehat{\psi x})^2(a'b'\widehat{\psi x})^2 = 0, \quad \ldots\ldots\ldots\ldots(34)$$

where $\widehat{\psi x}$ typifies $(\psi_2 x_3 - \psi_3 x_2)$ in the usual way.

On multiplying out and reducing (34), we obtain

$$\begin{aligned}&\phi_a^2\phi_b^2 S_2^2 - 2\phi_a^2\phi_{a'}^2 S_1 S_2 + \phi_{a'}^2\phi_{b'}^2 S_1^2 \\ &= 4S_1(\phi_{b'}^2\phi_a\phi_{a'}a_x a_x' - \phi_b^2\phi_{a'}\phi_{b'}a_x' b_x') \quad \ldots\ldots\ldots\ldots(35)\\ &+ 4S_2(\phi_{a'}^2\phi_b\phi_{b'}b_x b_x' - \phi_{b'}^2\phi_a\phi_b a_x b_x).\end{aligned}$$

In accordance with the notation of Lemma IV., the equation (35) may be written

$$\begin{aligned}&(I_{11}S_2^2 - 2I_{12}S_1S_2 + I_{22}S_1^2)\\ &= S_1\left\{\begin{array}{l}(A_2\alpha x^2 + B_2\beta y^2 + C_2\gamma z^2 + F_2\overline{\beta+\gamma}yz + G_2\overline{\gamma+\alpha}zx + H_2\overline{\alpha+\beta}xy)\\ -(A_1x^2 + B_1y^2 + C_1z^2 + 2F_1yz + 2G_1zx + 2H_1xy)\end{array}\right\}\\ &+ S_2\left\{\begin{array}{l}(A_1\alpha x^2 + B_1\beta y^2 + C_1\gamma z^2 + F_1\overline{\beta+\gamma}yz + G_1\overline{\gamma+\alpha}zx + H_1\overline{\alpha+\beta}xy)\\ -(A_2\alpha^2x^2 + B_2\beta^2y^2 + C_2\gamma^2z^2 + 2F_2\beta\gamma yz + 2G_2\gamma\alpha zx + 2H_2\alpha\beta xy\end{array}\right\}\end{aligned} \quad (36)$$

or in the still shorter form

$$(I_{11}S_2^2 - 2I_{12}S_1S_2 + I_{22}S_1^2) = S_1V_2 + S_2V_1. \quad \ldots\ldots\ldots(37)$$

It is plain that the right-hand side of (36) represents a quartic passing through the four points of the $S_1$, $S_2$ pencil, and also through the vertices of their self-polar triangle, inasmuch as the terms $x^4$, $y^4$, $z^4$ are absent. Furthermore, the left-hand side represents the combined equation to two conics of the $S_1$, $S_2$ pencil. We therefore have the following results :—

*The Principal Conics of the Apolar Locus with respect to the generating tetrad of points $\phi_i^4$ are the Self-Conjugate Conics of the $S_1$, $S_2$ pencil with respect to $\psi_i^4$; and vice-versa with respect to $\psi_i^4$ and $\phi_i^4$.* (38)

*The right-hand side of* (36) *is the Auxiliary Quartic of the Apolar Locus with respect to the $\phi$-tetrad.*

To find the Generating Conic of the above Auxiliary Quartic, we note that its equation in (36) or (37) is in normal form, inasmuch as $S_1$ (envelope) is apolar to the conic $V_1$ (locus) and

$S_2$ to $V_2$, and we have therefore only to express the condition that $S_1 + \mu S_2$ is apolar to every conic of the $V_1 + \lambda V_2$ pencil.

The condition required is

$$\Sigma (\beta + \mu)(\gamma + \mu)\{(A_1 \alpha - A_2 \alpha^2) + \lambda (A_2 \alpha - A_1)\} = 0,$$

*i.e.*
$$\Sigma (\beta + \mu)(\gamma + \mu)(\alpha - \lambda)(A_1 - A_2 \alpha) = 0,$$

*i.e.*
$$(\lambda + \mu)\{\mu (P - I_{11}) + (Q - \alpha\beta\gamma I_{22})\} = 0. \quad \text{.................(39)}$$

Neglecting the irrelevant factor $(\lambda + \mu)$ and comparing (39) with (30), we obtain that the Generating Conic of the Apolar Locus relative to the $\phi$-tetrad is the Polo-Reciprocal Conic of $S_1$, $S_2$ pencil relative to $\psi_i^4$.

*Hence the generating conic of the Apolar Locus with respect to either generating tetrad is the Polo-Reciprocal Conic of that tetrad with respect to the other generating tetrad.* (40)

5. We are now in a position to attack the fundamental problem of this paper, viz., the necessary and sufficient conditions that a quartic must satisfy, in order that it may be the Apolar Locus of two assigned tetrads.

We have to show that *if a quartic curve pass through two tetrads of points (which we shall call the $\phi$- and the $\psi$-tetrads respectively), and is such that :—*

(1) *the $\phi$-Principal Conics are the Self-Conjugate Conics of $\phi$ with respect to $\psi_i^4$;*

(2) *the $\psi$-Principal-Conics are the Self-Conjugate Conics of $\psi$ with respect to $\phi_i^4$;*

(3) *the $\phi$-Generating-Conic is the Polo-Reciprocal Conic of $\phi$ with respect to $\psi_i^4$;*

(4) *the $\psi$-Generating Conic is the Polo-Reciprocal Conic of $\psi$ with respect to $\phi_i^4$;*

*then, subject to these conditions, the quartic curve is the Apolar Locus of the $\phi$- and $\psi$-tetrads.*

We have already seen under Lemmas I. and II. that if the most general form of quartic passing through the four intersections of

$$S_1 \equiv \alpha x^2 + \beta y^2 + \gamma z^2 = 0$$
$$S_2 \equiv x^2 + y^2 + z^2 = 0$$

(*i.e.* the $\phi_t^4$-tetrad) and possessing given Principal Conics, viz. $I_{11} S_2^2 - 2I_{12} S_1 S_2 + I_{22} S_1^2 = 0$ (the notation being that of (22), (23), (24), ) is

$$\rho\,(I_{11} S_2^2 - 2I_{12} S_1 S_2 + I_{22} S_1^2)$$
$$= S_1\,(a_2 x^2 + b_2 y^2 + c_2 z^2 + 2f_2 yz + 2g_2 zx + 2h_2 xy) \qquad \ldots\ldots(41)$$
$$- S_2\,(\alpha\, a_2 x^2 + \beta\, b_2 y^2 + \gamma\, c_2 z^2 + 2f_1 yz + 2g_1 zx + 2h_1 xy),$$

where the right-hand side is the Auxiliary Quartic expressed in normal form according to (10), if

$$a_2 + b_2 + c_2 = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(42)$$

The generating conic of the right-hand side is obtained by expressing the condition that $S_1 + \mu S_2$ is apolar to every conic of the system

$$(\alpha a_2 x^2 + \beta\, b_2 y^2 + \gamma\, c_2 z^2 + 2f_1 yz + 2g_1 zx + 2h_1 xy)$$
$$+ \lambda\,(a_2 x^2 + b_2 y^2 + c_2 z^2 + 2f_2 yz + 2g_2 zx + 2h_2 xy) = 0.$$

The condition required is

$$\Sigma\,(\beta + \mu)\,(\gamma + \mu)\,(\alpha + \lambda)\, a_2 = 0,$$

*i.e.* $\quad (\lambda - \mu)\,\{\mu\,(\alpha\, a_2 + \beta b_2 + \gamma c_2) - (\beta\gamma\, a_2 + \gamma\alpha\, b_2 + \alpha\beta\, c_2)\} = 0.$

Casting aside the irrelevant factor $\lambda - \mu$, we obtain that the generating-conic is

$$(\alpha\, a_2 + \beta\, b_2 + \gamma c_2)\, S_1 + (\beta\gamma\, a_2 + \gamma\alpha\, b_2 + \alpha\beta\, c_2)\, S_2 = 0. \quad \ldots\ldots(43)$$

Now the generating-conic is to be (39), namely,

$$(P - I_{11})\, S_1 + (\alpha\beta\gamma\, I_{22} - Q)\, S_2 = 0,$$

*i.e.* $\quad \{\alpha\,(\alpha\, A_2 - A_1) + \beta\,(\beta\, B_2 - B_1) + \gamma\,(\gamma\, C_2 - C_1)\}\, S_1$
$$+ \{\beta\gamma\,(\alpha\, A_2 - A_1) + \gamma\alpha\,(\beta B_2 - B_1) + \alpha\beta\,(\gamma C_2 - C_1)\}\, S_2 = 0. \quad (44)$$

Identifying (43) with (44) and using (42), we find that

$$a_2 = -\sigma\,(\alpha A_2 - A_1)$$
$$b_2 = -\sigma\,(\beta B_2 - B_1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(45)$$
$$c_2 = -\sigma\,(\gamma\, C_2 - C_1),$$

The most general quartic, therefore, (1) which passes through the $\phi$-tetrad, (2) which has as its $\phi$-Principal-Conics the Self-Conjugate

Conics of $\phi$ with respect to $\psi_t^4$, (3) which has as its $\phi$-Generating-Conic the Polo-Reciprocal of $\phi$ with respect to $\psi_t^4$, is

$$\rho\,(I_{11}\,S_2^2 - 2I_{12}\,S_1\,S_2 + I_{22}\,S_1^2)$$
$$+\,\sigma\,[\{(\alpha A_2 - A_1)\,x^2 + (\beta B_2 - B_1)\,y^2 + (\gamma C_2 - C_1)\,z^2\}S_1$$
$$+\,\{\alpha\,(A_1 - \alpha A_2)\,x^2 + \beta\,(B_1 - \beta B_2)\,y^2 + \gamma\,(C_1 - \gamma C_2)\,z^2\}]$$
$$+\,S_1\,(p_2\,yz + q_2\,zx + r_2\,xy) + S_2\,(p_1\,yz + q_1\,zx + r_1\,xy) = 0,$$

where $\rho$, $\sigma$, $p_1$, $q_1$, $r_1$, $p_2$, $q_2$, $r_2$ are arbitrary constants.

We thus see that for a quartic to pass through four specified points, and have thereat the given conics as Principal and Generating Conics respectively, is tantamount to *seven* linear conditions. Now, if we make this quartic satisfy the corresponding conditions with regard to the $\psi$-tetrad of points, we shall obtain other seven linear conditions, *i.e.* seven linear equations connecting $\rho$, $\sigma$, $p_1$, $q_1$, $r_1$, $p_2$, $q_2$, $r_2$. Let these linear conditions be typified by

$$R_\kappa''\rho + S_\kappa\,\sigma + P_\kappa\,p_1 + Q_\kappa\,q_1 + R_\kappa\,r_1 + P_\kappa' p_2 + Q_\kappa' q_2 + R_\kappa' r_2 = 0$$
$$(\kappa = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7).$$

These equations will yield a unique solution for

$$\rho : \sigma : p_1 : q_1 : r_1 : p_2 : q_2 : r_2$$

unless the determinants of the matrix

$$\|\ R_\kappa'',\ S_\kappa,\ P_\kappa,\ Q_\kappa,\ R_\kappa,\ P_\kappa',\ Q_\kappa',\ R_\kappa'\ \|$$

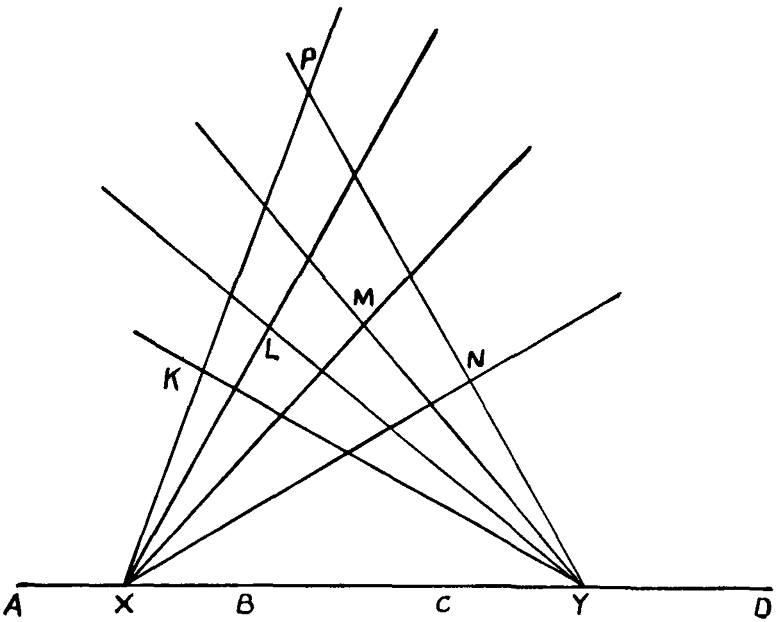
severally vanish. To show that they do not vanish *in general*, it will be sufficient to show that they do not vanish for a particular case. I have worked out in detail the test case of $\psi_t^4$ the intersections of the two conics

$$3x^2 - \ 2y^2 + 5z^2 + 6yz + 14zx - \ 8xy = 0$$
$$7x^2 - 11y^2 + \ z^2 - 4yz - 10zx + 12xy = 0$$

and $\phi_t^4$ the four points $(1,\ \pm 1,\ \pm 1)$. The details are too long for reproduction here. Hence *in general* we get a unique solution for $\rho : \sigma : p_1 : q_1 : r_1 : p_2 : q_2 : r_2$, and as these are known to be satisfied by the coefficients of the Apolar Locus of $\phi_t^4$ and $\psi_t^4$, the Apolar Locus must be the unique solution required.

6. *The Clebschian Quartic with a degenerate Apolar Conic.*

The Clebschian Quartic with a degenerate Apolar Conic is particularly interesting from the point of view of one particular method of its apolar generation.




Let $X$, $Y$ be the degenerate Apolar Conic of the given Clebschian Quartic. Let the curve cut the line $XY$ in the points $ABCD$. We know that the range $[ABCD]$ is equianharmonic, and has $X^2$, $Y^2$ as its Hessian Points. Now consider the four points $K$, $L$, $M$, $N$, in which the first, second, third, and fourth lines of the $X$-pencil cut the first, second, third, and fourth lines of the $Y$-pencil respectively. Consider the Apolar Locus of $A$, $B$, $C$, $D$ and $K$, $L$, $M$, $N$, which, of course, passes through these eight points. Now $[A, B, C, D]$, being equianharmonic, is apolar to $X$, $Y$ and any other two collinear points. Hence $P\,[ABCD]$ is apolar to $P\,[K, L, M, N]$. Consequently, $P$ lies on the Apolar Locus. We thus see that the Apolar Locus passes through the

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sixteen intersections of the $X$-pencil with the $Y$ pencil, and is, in fact, the Clebschian Quartic, having $X$, $Y$ as its degenerate Apolar Conic.

Conversely, if we wish to apolarly generate a given Clebschian Quartic having $X$, $Y$ as its Apolar Conic, we draw any line $x_1$ through $X$, join the four intersections of $x_1$ with the curve to $Y$, obtaining $y_1, y_2, y_3, y_4$. By joining up the remaining intersections of $y_1, y_2, y_3, y_4$ with the curve to $X$, we obtain $x_2, x_3, x_4$. Then the tetrads of points given by the following scheme will, along with $ABCD$, apolarly generate the given Clebschian Quartic.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|
| $y_1$ | $y_2$ | $y_3$ | $y_4$ |
| $y_2$ | $y_1$ | $y_4$ | $y_3$ |
| $y_3$ | $y_4$ | $y_1$ | $y_2$ |
| $y_4$ | $y_3$ | $y_2$ | $y_1$ |

The first two rows taken together give the points $x_1 y_1$, $x_2 y_2$, $x_3 y_3$, $x_4 y_4$, which is a tetrad co-apolar with $ABCD$.

Also, the first and third row taken together give the points $x_1 y_2$, $x_2 y_1$, $x_3 y_4$, $x_4 y_3$, which is another tetrad co-apolar with $ABCD$.

Again, reverting to the figure of this article, the tangents at $ABCD$ are concurrent; for, if $AT$ be the tangent at $A$, $A\,[D^3T]$ is apolar to $A\,[KLMN]$, *i.e.* $AT$ passes through the pole of $AD$ with respect to the tetrad $ABCD$.

Many properties of the Clebschian Quartic regarding the pencil of conics through $KLMN$ and all such tetrads now follow at once from our discussion of the general method of apolarly generating the general Quartic Curve.

7. *Two Tetrads on the same Conic.*

The case when the $\phi$- and the $\psi$-tetrads both lie on the same conic yields a rather unexpected result. We find on investigation that, not a unique quartic, but a pencil of quartics satisfy the criteria of § 5. I have, however, not completed the solution at the present date.