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# FUNCTIONAL EQUATIONS, DISTRIBUTIONS AND APPROXIMATE IDENTITIES

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**1. Introduction.** The subject of this paper is the use of the theory of Schwartz distributions and approximate identities in studying the functional equation

$$\sum_{j=1}^{N} a_j(s)f_j(x + h_j(s)) = b(s)g(x). \tag{1}$$

The $a_j$'s and $b$ are complex-valued functions defined on a neighbourhood, $U$, of 0 in $\mathbf{R}^m$, $h_j$: $U \longrightarrow \mathbf{R}^n$ with $h_j(0) = 0$ and $f_j$, $g$: $\mathbf{R}^n \longrightarrow \mathbf{C}$ for $1 \leqq j \leqq N$. In most of what follows the $a_j$'s and $h_j$'s are assumed smooth and may be thought of as given. The $f_j$'s, $b$ and $g$ may be thought of as the unknowns. Typically we are concerned with locally integrable functions $f_1, \ldots, f_N$ such that, for each $s$ in $U$, (1) holds for a.e. (almost every) $x \in \mathbf{R}^n$, in the sense of Lebesgue measure. Such general equations have been studied extensively, see e.g. [**2**], [**3**], [**4**], [**8**], [**9**] and [**12**].

To motivate the method we aim to expose, suppose $m = 1$ and all our functions are of class $C^p$, $f = f_j = g$ for $1 \leqq j \leqq N$ and (1) holds for all $(x, s) \in \mathbf{R}^n \times U$. Differentiating (1) with respect to $s$ and setting $s = 0$ in the resulting equation we find that

$$\sum_{j=1}^{N}\{a_j'(0)f(x) + a_j(0)[h_j'(0) \cdot \nabla f(x)]\} = b'(0)f(x) \quad \text{for all } x \in \mathbf{R}^n.$$

This is a first order, linear, homogeneous differential equation with constant coefficients which we may write simply as $T_1 f = 0$. More generally, if for $1 \leqq l \leqq p$ we differentiate (1) $l$ times with respect to $s$ and set $s = 0$ in the resulting equation we obtain a linear, homogeneous differential equation with constant coefficients of order at most $l$:

$$T_l f = 0, \quad 0 \leqq l \leqq p. \tag{2}$$

It sometimes happens, as illustrated in [**3**], that the system (2) can be "solved" and by substituting back into (1) we are thus able to find the smooth solutions of (1). If one of the equations in (2) is elliptic then one can conclude that $f$ is in fact of class $c^{\infty}$. Examples will be given below.

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An evident deficiency of the above method is that the differentiability assumptions are unnatural (although in many, if not most, of the cases in which (1) has been studied, the $a_j$'s are constant and the $h_j$'s are linear). In many interesting cases this deficiency can be overcome, to a large extent, by appealing to the theory of distributions as was done, for example, in [**2**], [**4**] and **12**].

The main technique of this paper is that of smoothing and approximation by convolution. Our aim is to refine a method which was used in [**3**] to find the continuous solutions of certain special cases of (1). It will be used here to consider distributional analogues of (1) and thereby obtain regularity results closely related to those of [**12**].

Unless otherwise indicated, our notation and terminology is that of Rudin [**10**].

**2. Background.** If $h \in \mathbf{R}^n$ and $f\colon \mathbf{R}^n \longrightarrow \mathbf{C}$ define $\tau_h f\colon \mathbf{R}^n \longrightarrow \mathbf{C}$ by $(\tau_h f)(x) = f(x+h)$ for $x \in \mathbf{R}^n$. Note that (1) can then be written

$$(3) \qquad \sum_{j=1}^{N} a_j(s)\tau_{h_j(s)}f_j = b(s)g \quad \text{for } s \in U.$$

A function $f\colon \mathbf{R}^n \longrightarrow \mathbf{C}$ is called *locally integrable* provided it is Lebesgue measurable and $\int_K |f(x)|dx < \infty$ for every compact subset, $K$, of $\mathbf{R}^n$. Denote the set of all such functions by $L^1_{loc}(\mathbf{R}^n)$. If $f \in L^1_{loc}(\mathbf{R}^n)$ and $h \in \mathbf{R}^n$ then $\tau_h f \in L^1_{loc}(\mathbf{R}^n)$.

Let $\mathcal{D}_n$ denote the space of all test functions on $\mathbf{R}^n$ and let $\mathcal{D}_n'$ denote the space of all Schwartz distributions on $\mathbf{R}^n$. If $\phi \in \mathcal{D}_n$ and $h \in \mathbf{R}^n$ then $\tau_h\phi \in \mathcal{D}_n$.

If $f \in L^1_{loc}(\mathbf{R}^n)$ then the mapping $\phi \longrightarrow \int_{\mathbf{R}^n} f(x)\phi(x)dx$ (for $\phi \in \mathcal{D}_n$) belongs to $\mathcal{D}_n'$; it will be called the *regular* distribution determined by $f$ and denoted by $\Lambda_f$. For $f, g \in L^1_{loc}(\mathbf{R}^n)$ we have $\Lambda_f = \Lambda_g$ if and only if $f(x) = g(x)$ for a.e. $x \in \mathbf{R}^n$ (see [**10**], page 136).

If $h \in \mathbf{R}^n$, $f \in L^1_{loc}(\mathbf{R}^n)$ and $\phi \in \mathcal{D}_n$ then

$$(\Lambda_{\tau_h} f)(\phi) = \int_{\mathbf{R}^n} f(x+h)\phi(x)dx = \int_{\mathbf{R}^n} f(y)\phi(y-h)dy = \Lambda_f(\tau_{-h}\phi).$$

Given $h \in \mathbf{R}^n$ and $u \in \mathcal{D}_n'$ it is therefore natural to define $\tau_h u\colon \mathcal{D}_n \longrightarrow \mathbf{C}$ by $(\tau_h u)(\phi) = u(\tau_{-h}\phi)$ for all $\phi \in \mathcal{D}_n$. It follows that $\tau_h u \in \mathcal{D}_n'$ whenever $h \in \mathbf{R}^n$ and $u \in \mathcal{D}_n'$. Moreover $\tau_h\Lambda_f = \Lambda_{\tau_h f}$ whenever $h \in \mathbf{R}^n$ and $f \in L^1_{loc}(\mathbf{R}^n)$.

Now observe that if (3) holds with $f_1, \ldots, f_N$, $g \in L^1_{loc}(\mathbf{R}^n)$ and if we let $u_j = \Lambda_{f_j}$, $l \leqq j \leqq N$, and $v = \Lambda_g$ then we have

$$(4) \qquad \sum_{j=1}^{N} a_j(s)\tau_{h_j(s)}u_j = b(s)v \quad \text{for all } s \in U$$

– a distributional analogue of (1).

If $f \in L^1_{loc}(\mathbf{R}^n)$ and $\phi \in \mathcal{D}_n$ then $f * \phi$, the *convolution* of $f$ with $\phi$, is the function defined by

$$(f * \phi)(x) = \int_{\mathbf{R}^n} f(x-t)\phi(t)dt$$
$$= \int_{\mathbf{R}^n} f(s)\phi(x-s)ds \quad \text{for } x \in \mathbf{R}^n.$$

The following three Propositions are crucial for what follows. Their proofs may be found in [**10**], pages 155–161.

PROPOSITION 1. *If $f \in L^1_{loc}(\mathbf{R}^n)$ and $\phi \in \mathcal{D}_n$ then*
*(i) $f * \phi \in C^\infty(\mathbf{R}^n)$*
*(ii) $\tau_h(f * \phi) = (\tau_h f) * \phi = f * (\tau_h \phi)$ for all $h \in \mathbf{R}^n$*
*(iii) $D^\alpha(f * \phi) = f * (D^\alpha \phi)$ for every multi-index $\alpha$.*

It follows from $(ii)$ that if (3) holds with $f_1, \ldots, f_N, g \in L^1_{loc}(\mathbf{R}^n)$, and if $\phi \in \mathcal{D}_n$ then $\sum_{j=1}^N a_j(s)\tau_{h_j(s)}(f_j * \phi) = b(s)(g * \phi)$ for all $s \in U$. Thus (1) holds with $f_j$ replaced by $f_j * \phi$ $(1 \leqq j \leqq N)$ and $g$ replaced by $g * \phi$; moreover, all of $f_1 * \phi, \ldots, f_N * \phi$ and $g * \phi$ are $C^\infty$ functions.

If for $\phi \in \mathbf{D}_n$ we let $\phi^-(x) = \phi(-x)$ for $x \in \mathbf{R}^n$ then the mapping $\phi \longrightarrow \phi^-$ is an extremely healthy bijection of $\mathcal{D}_n$. For $f \in L^1_{loc}(\mathbf{R}^n)$ and $\phi \in \mathcal{D}_n$ we have

$$(f * \phi)(x) = \int_{\mathbf{R}^n} f(s)(\tau_{-x}\phi^-)(s)ds \quad \text{for } x \in \mathbf{R}^n.$$

This motivates the following definition ([**10**], page 155). If $\phi \in \mathcal{D}_n'$ and $\phi \in \mathcal{D}_n$ define

$$(u * \phi)(x) = u(\tau_{-x}\phi^-) \quad \text{for } x \in \mathbf{R}^n.$$

PROPOSITION 2. *If $u \in \mathcal{D}_n'$, $\phi \in \mathcal{D}_n$, $h \in \mathbf{R}^n$ and $\alpha$ is a multi-index then*
*(i) $u * \phi \in C^\infty(\mathbf{R}^n)$*
*(ii) $\tau_h(u * \phi) = u * (\tau_h \phi)$*
*(iii) $\tau_h \Lambda_{u*\phi} = (\Lambda_{\tau_h u}) * \phi$*
*(iv) $D^\alpha(u * \phi) = u * (D^\alpha \phi)$*
*(v) $D^\alpha \Lambda_{u*\phi} = (\Lambda_{D^\alpha u}) * \phi$.*

*Remark.* If we identify a locally integrable function with its corresponding regular distribution (as we will do when convenient below) then (iii) and (v) can be written as $\tau_h(u * \phi) = (\tau_h u) * \phi$ and $D^\alpha(u * \phi) = (D^\alpha u) * \phi$ respectively.

Let $\phi_1 \in \mathcal{D}_n$ such that $\phi_1(x) \geqq 0$ for all $x \in \mathbf{R}^n$ and $\int_{\mathbf{R}^n} \phi_1(x)dx = 1$. Let $\phi_k(x) = k^n\phi_1(kx)$ for $x \in \mathbf{R}^n$ and $k = 1, 2, \ldots$. Then $\{\phi_k\}_{k=1}^\infty$ is sometimes called an *approximate identity* (see [**10**], page 157) because (for example) of

PROPOSITION 3. $(i)$ *If $f \in C^P(\mathbf{R}^n)$ then $\{f * \phi_k\}_{k=1}^\infty$ converges to $f$ in the sense of $C^P(\mathbf{R}^n)$ whenever $0 \leqq p \leqq +\infty$.*

*(ii) If* $f \in L^1_{loc}(\mathbf{R}^n)$ *then* $\{f * \phi_k\}_{k=1}^{\infty}$ *converges to* $f$ *in the sense of* $L^1_{loc}(\mathbf{R}^n)$, *i.e.* $\lim_{k\to\infty} \int_K |f * \phi_k - f| = 0$ *for every compact* $K \subseteq \mathbf{R}^n$.

*(iii) If* $u \in \mathcal{D}_n'$ *then* $\{u * \phi_k\}_{k=1}^{\infty}$ *converges to u in the weak topology of* $\mathcal{D}_n'$. *More precisely, the last assertion means that*

$$\lim_{k\to\infty} \Lambda_{u*\phi_k}(\phi) = u(\phi) \quad \text{for every } \phi \in \mathcal{D}_n.$$

*Remark.* The idea of approximating by convolution with an approximate identity (or something similar) goes back at least to Weierstrass [**13**] where he used it to prove his famous approximation theorem and, in the same paper, to study the heat equation. The idea is pervasive in analysis. For example, it was used by Fejér [**5**] to prove that the Fourier series of a continuous, periodic function, $f$, is uniformly Cesàro summable (to $f$) on $\mathbf{R}$. It is basically this idea that is used to show that the Poisson integral formula indeed defines a continuous function up to the boundary; see [**6**] page 20. Also see [**11**] for an amazing use by Carleman of such ideas in proving a beautiful generalization of Weierstrass' theorem.

**3. General Properties of (1) and (4).** The next three assertions involve no smoothness assumptions on the $a_j$'s or the $h_j$'s. They hold with $U$ replaced by any nonempty set and follow easily from the previous remarks.

PROPOSITION 4. *Suppose* $(u_1, \ldots, u_N, v)$ *is a solution of* (4) *(i.e.* $u_1, \ldots, u_N, v \in \mathcal{D}_n'$ *and* (4) *holds for each* $s \in U$*). Then, for every* $\phi \in \mathcal{D}_n$, $(u_1*\phi, \ldots, u_N*\phi, v*\phi)$ *is a* $C^\infty$ *solution of* (1) *(for all* $(x, s) \in \mathbf{R}^n \times U$*). If we let* $f_{jk} = u_j*\phi_k$ *and* $g_k = v * \phi_k$ *for* $1 \leqq j \leqq N$, $k = 1, 2, \ldots,$ *then* $(f_{1k}, \ldots, f_{Nk}, g_k)$ *is a* $C^\infty$ *solution of* (1) *for every* $k = 1, 2, \ldots$. *Moreover, for each* $j = 1, 2, \ldots, N$, *we have* $f_{jk} \to u_j$ *and* $g_k \to v$ *in* $\mathcal{D}_n'$ *as* $k \to \infty$.

PROPOSITION 5. *Suppose* $f_1, \ldots, f_N, g \in L^1_{loc}(\mathbf{R}^n)$ *are such that, for each* $s \in U$, (1) *holds for a.e.* $x \in \mathbf{R}^n$. *If* $\phi \in \mathcal{D}_n$ *then* $(f_1 * \phi, \ldots, f_N * \phi, g * \phi)$ *is a* $C^\infty$ *solution of* (1) *(for all* $(x, s) \in \mathbf{R}^n \times U$*). If* $f_{jk} = f_j * \phi_k$ *and* $g_k = g * \phi_k$ *for* $k = 1, 2, \ldots$ *then* $(f_{1k}, \ldots, f_{Nk}, g_k)$ *is a* $C^\infty$ *solution of* (1) *for every* $k = 1, 2, \ldots, f_{jk} \to f_j$ *in* $L^1_{loc}(\mathbf{R}^n)$ *as* $k \to \infty$ *for each* $j = 1, 2, \ldots, N$ *and* $g_k \to g$ *in* $L^1_{loc}(\mathbf{R}^n)$ *as* $k \to \infty$.

COROLLARY. *If* (4) *has a nontrivial solution in* $\mathcal{D}_n'$ *or* (1) *has a nontrivial solution in* $L^1_{loc}(\mathbf{R}^n)$ *then* (1) *has a nontrivial solution in* $C^\infty(\mathbf{R}^n)$.

*Proof.* Suppose $u_1, \ldots, u_N, v \in \mathcal{D}_n'$, (4) holds and not all of $u_1, \ldots, u_N, v$ are zero. Suppose, for example, that $v \neq 0$. Since $v * \phi_k \to v$ in $\mathcal{D}_n'$ as $k \to \infty$ we must have $v * \phi_{k_0} \neq 0$ for some natural number $k_0$. Hence, according to Proposition 4, $(u_1 * \phi_{k_0}, \ldots, u_N * \phi_{k_0}, v * \phi_{k_0})$ is a nontrivial solution of (1). The rest of the assertion can be proved in a similar manner.

**4. Smooth Coefficients and Differential Equations.** In this section we suppose that $a_1, \ldots, a_N \in C^P(U)$ and $h_1, \ldots, h_N$ are of class $C^P$ for some $p$ where $1 \leqq p \leqq +\infty$.

PROPOSITION 6. *(i) Suppose* $f_1, \ldots, f_N$, $g \in L^1_{loc}(\mathbf{R}^n)$ *and, for each* $s \in U$, (1) *holds for a.e.* $x \in \mathbf{R}^n$. *If* $g \neq 0$ *(i.e. it is not true that* $g(x) = 0$ *for a.e.* $x \in \mathbf{R}^n$*) then* $b \in C^p(U)$.

*(ii) If* (4) *holds for some* $u_1, \ldots, u_N$, $v \in \mathcal{D}_n'$ *and if* $v \neq 0$ *then* $b \in C^p(U)$.

*Proof.* (i) As in the proof of the last corollary, choose $\phi \in \mathcal{D}_n$ such that $g*\phi \neq 0$. Thus, for some $x_0 \in \mathbf{R}^n$, $(g*\phi)(x_0) \neq 0$. But, according to Proposition 4,

$$\sum_{j=1}^{N} a_j(s)(f_j * \phi)(x + h_j(s)) = b(s)(g * \phi)(x) \quad \text{for } (x\, s) \in \mathbf{R}^n \times U.$$

Hence $b(s) = [(g * \phi)(x_0)]^{-1} \sum_{j=1}^{N} a_j(s)(f_j * \phi)(x + h_j(s))$ for all $s \in U$ and the assertion follows since $f_j * \phi \in C^\infty(\mathbf{R}^n)$ for $1 \leqq j \leqq N$.

The proof of (ii) is similar.

If (1) were to hold with all the $a_j$'s, $f_j$'s, $h_j$'s, $b$ and $g$ of class $C^p$ and if $\alpha$ were a multi-index of order at most $p$ then by differentiating (1) $\alpha$ times with respect to $s$ and setting $s = 0$ in the resulting equation (recall $h_j(0) = 0$ for $1 \leqq j \leqq N$) we would obtain a system of linear differential equations, each of order at most $|\alpha|$:

$$(5) \qquad \sum_{j=1}^{N} L_{\alpha,j} f_j = b_\alpha g, \quad |\alpha| \leqq p.$$

For each $\alpha$ and $j$, $L_{\alpha,j}$ is a linear differential operator with constant coefficients and $b_\alpha = D^\alpha b(0)$ for $|\alpha| \leqq p$.

To illustrate how the above can be used to study functional equations we consider a generalization of the "cosine equation":

$$f(x + y) + f(x - y) = 2f(x)f(y).$$

The essence of the method may be clearer when applied to the more general equation

$$(\#) \qquad f(x + s) + f(x - s) = 2b(s)f(x).$$

Its distributional analogue is

$$(\#)' \qquad \tau_s u + \tau_{-s} u = 2b(s)u.$$

Such equations have been studied extensively (see [**1**], §3.2). Our aim here is only to illustrate a method of solution.

PROPOSITION 7. *Suppose $\delta > 0$, $b\colon (-\delta, \delta) \to \mathbf{C}$, $f \in C^\infty(\mathbf{R})$ and $b \in C^\infty(-\delta, \delta)$. Then (#) holds for all $(x, s) \in \mathbf{R} \times (-\delta, \delta)$ if and only if one of the following hold:*

*(i) $f \equiv 0$,*

*(ii) for some $\lambda$, $\alpha$, $\beta \in \mathbf{C}$ with $\lambda \neq 0$, $f(x) \equiv \alpha e^{\lambda X} + \beta e^{-\lambda X}$ for all $x \in \mathbf{R}$ and $b(s) = \cosh(\lambda s)$ for all $s \in (-\delta, \delta)$,*

*(iii) $b \equiv 1$ and, for some $\alpha$, $\beta \in \mathbf{C}$, $f(x) = \alpha + \beta x$ for all $x \in \mathbf{R}$.*

*Proof.* Putting $s = 0$ in (#) we find that $f = b(0)f$. Hence either $f \equiv 0$ or $b(0) = 1$.

Suppose $f \not\equiv 0$ so that $b(0) = 1$. If we differentiate (#) twice with respect to $s$, put $s = 0$ in the resulting equation and let $b''(0) = -\lambda^2$ then we find that

$$f''(x) = -\lambda^2 f(x) \quad \text{for all } x \in \mathbf{R}.$$

If $\lambda \neq 0$ then there exist $\alpha$, $\beta \in \mathbf{C}$ such that $f(x) = \alpha e^{\lambda x} + \beta e^{-\lambda x}$ for all $x \in \mathbf{R}$ and hence, from (#), it follows that

$$b(s) = \cosh(\lambda s) \quad \text{for all } s \in (-\delta, \delta).$$

If $\lambda = 0$ then there exist $\alpha \beta \in \mathbf{C}$ such that

$$f(x) = \alpha + \beta x \quad \text{for all } x \in \mathbf{R}.$$

In this case (#) implies that $b \equiv 1$.

COROLLARY 7. *Suppose $\delta > 0$, $b\colon (-\delta, \delta) \to \mathbf{C}$, $u \in D_1'$ and (#)$'$ holds for all $s \in (-\delta, \delta)$. Then either $u = 0$ or $b \in C^\infty$ and $u = \Lambda_f$ for some $f \in C^\infty(\mathbf{R})$ such that (#) holds.*

*Proof.* If $b(0) = 0$ then (#)$'$, with $s = 0$, implies that $u = 0$.

Suppose $b(0) \neq 0$. Then, by (#)$'$, either $u = 0$ or $b(0) = 1$. Suppose $b(0) = 1$ and $u \neq 0$.

Let $f_j = u * \phi_j$ for $j = 1, 2, \ldots$. Then, for $j = 1, 2, \ldots, f_j \in C^\infty(\mathbf{R})$ and

$$f_j(x+s) + f_j(x-s) = 2b(s)f_j(x) \quad \text{for all } (x, s) \in \mathbf{R} \times U.$$

Since $u \neq 0$, according to Proposition 6, $b \in C^\infty(\mathbf{R})$. By Proposition 7, either $b \equiv 1$ or there exists $\lambda \in \mathbf{C}$ such that $b(s) = \cosh(\lambda s)$ for every $s \in (-\delta, \delta)$.

If $b \equiv 1$ then for each $j = 1, 2, \ldots$ there exist $\alpha_j$, $\beta_j \in \mathbf{C}$ such that

$$f_j(x) = \alpha_j + \beta_j x \quad \text{for all } x \in \mathbf{R}.$$

Let $u_0 = \Lambda_{P_0}$ where $P_0(x) = 1$ for all $x \in \mathbf{R}$ and let $u_1 = \Lambda_{P_1}$ where $P_1(x) = x$ for all $x \in \mathbf{R}$. Let $S$ be the subspace of $\mathcal{D}_1'$ spanned by $\{u_0, u_1\}$. Then $f_j \in S$ for each $j = 1, 2, \ldots$ and $S$, being finite dimensional, is closed (see [**10**] page

16). But $f_j \longrightarrow u$ in $\mathcal{D}_1'$ so that $u \in S$. Thus there exists $\alpha, \beta \in \mathbf{C}$ such that $u = \alpha u_0 + \beta u_1$. That is $u = \Lambda_f$ where $f(x) = \alpha + \beta x$ for all $x \in \mathbf{R}$.

If $b \not\equiv 1$ then, by Proposition 7, there exists $\lambda \in \mathbf{C}$ such that $b(s) = \cosh(\lambda s)$ for all $s \in (-\delta, \delta)$. By Proposition 7, for each $j = 1, 2, \ldots,$ there exist $\alpha_j, \beta_j \in \mathbf{C}$ such that

$$f_j(x) = \alpha_j e^{\lambda x} + \beta_j e^{-\lambda x} \quad \text{for all } x \in \mathbf{R}.$$

By using an argument like that employed in the last paragraph, we conclude that $u = \Lambda_f$ where, for some $\alpha, \beta \in \mathbf{C}$, $f(x) = \alpha e^{\lambda x} + \beta e^{-\lambda x}$ for all $x \in \mathbf{R}$.

Similarly, it is possible to prove

COROLLARY 2. *Suppose $\delta > 0$, $b$: $(-\delta, \delta) \longrightarrow \mathbf{C}$, $F \in L^1_{loc}(\mathbf{R})$ and, for each $s \in (-\delta, \delta)$,*

$$F(x+s) + F(x-s) = 2b(s)F(x) \quad \textit{for a.e. } x \in \mathbf{R}.$$

*Then either $F(x) = 0$ for a.e. $x \in \mathbf{R}$ or $b \in C^\infty(\mathbf{R})$ and there exists $f \in C^\infty(\mathbf{R})$ such that (#) holds and $F(x) = f(x)$ for a.e. $x \in \mathbf{R}$.*

**5. Regularity Properties.** In the remainder of the paper we will consider (1) with $f_1 = \cdots = f_N = f$, $b \equiv 0$ and (4) with $u_1 = \cdots = u_N = u$, $b \equiv 0$. that is, we consider the equation

$$(1)' \qquad \sum_{j=1}^{N} a_j(s) f(x + h_j(s)) = 0$$

and the associated distributional equation

$$(4)' \qquad \sum_{j=1}^{N} a_j(s) \tau_{h_j(s)} u = 0.$$

We assume that $a_1, \ldots, a_N \in C^p(U)$, for some natural number $p$. We also assume that $h_j$ is of class $C^p$ on $U$ and $h_j(0) = 0$ for $1 \leqq j \leqq N$. In the examples below $m = 1$, the $a_j$'s are constant and the $h_j$'s are linear. If we formally differentiate $(1)'$ $\alpha$ times with respect to $s$ ($\alpha$ being a multi-index of order at most $p$) and set $s = 0$ in the resulting equation we obtain a system of linear differential equations

$$(5)' \qquad T_\alpha f = 0, \quad |\alpha| \leqq p.$$

The following result is a particular case of Theorem 1 of [**12**] which is applicable to many particular cases of $(1)'$ that have been extensively studied, as illustrated below.

PROPOSITION 8. *Suppose one of the operators, $T_\alpha$, in* (5)$'$ *is elliptic.*

*(i) If $u \in \mathcal{D}_n'$ and* (4)$'$ *holds for all $s \in U$ then there exists $f \in C^\infty(\mathbf{R}^n)$ such that* (1)$'$ *holds for all $(x, s) \in \mathbf{R}^n \times U$ and $u = \Lambda_f$.*

*(ii) If $f \in L^1_{loc}(\mathbf{R}^n)$ and, for each $s \in U$,* (1)$'$ *holds for a.e. $x \in \mathbf{R}^n$ then $f$ is almost everywhere equal to a $C^\infty$ solution of* (1)$'$.

*Proof.* (i) Let $f_k = u * \phi_k$ for $k = 1, 2, \ldots$. Then $f_k \in C^\infty(\mathbf{R}^n)$ for each $k = 1, 2, \ldots$ and, according to (iii) of Proposition 2, $\sum_{j=1}^N a_j(s)\tau_{h_j(s)} f_k = 0$ for all $s \in U$. Hence $\sum_{j=1}^N a_j(s) f_k(x + h_j(s)) = 0$ for $x \in \mathbf{R}^n$, $s \in U$ and $k = 1, 2, \ldots$ Thus $T_\alpha(f_k) = 0$ for all $k = 1, 2, \ldots$. But $f_k \longrightarrow u$ in $\mathcal{D}'(\mathbf{R}^n)$ as $k \longrightarrow \infty$ and $T_\alpha$ is a continuous mapping of $\mathcal{D}_n'$ to itself (see [**10**] page 146). Hence $T_\alpha(u) = \lim_{k\to\infty} T_\alpha(f_k) = 0$. Since $T_\alpha$ is elliptic, (see [**10**], page 201) there exists $f \in C^\infty(\mathbf{R}^n)$ such that $u = \Lambda_f$. Hence, because of (4)$'$ and the continuity of $f$, (1)$'$ holds for all $(x, s) \in \mathbf{R}^n \times U$.

(ii) Let $u = \Lambda_f$. Since (1)$'$ holds so does (4)$'$. Hence, by (i), there exists a $C^\infty$ solution, say $\tilde{f}$, of (1) such that $u = \Lambda_{\tilde{f}}$. Thus $f(x) = \tilde{f}(x)$ for a.e. $x \in \mathbf{R}^n$. The continuity of $\tilde{f}$ implies the rest.

**6. Functional Equations Analogues of PDE's.** Let $b_1, \ldots, b_n$ denote the usual basis of $\mathbf{R}^n$. For $f$: $\mathbf{R}^n \longrightarrow \mathbf{C}$, $h \in \mathbf{R}$ and $l \leqq k \leqq n$ let

$$\underset{h}{\Delta}_k f(x) = f(x + hb_k) - f(x)$$

and

$$\underset{h}{\bar{\Delta}}_k f(x) = f(x + (h/2)b_k) - f(x - (h/2)b_k) \quad \text{for } x \in \mathbf{R}^n.$$

We may regard each $\underset{h}{\Delta}_k$ and $\underset{h}{\bar{\Delta}}_k$ as a linear "difference" operators. Notice that $\underset{h}{\Delta}_k f = \tau_{hb_k} f - f$, etc. If $f \in C^1(\mathbf{R}^n)$ then, for $x \in \mathbf{R}^n$ and $1 \leqq k \leqq n$,

$$D_k f(x) = \lim_{h\to 0}(1/h)\underset{h}{\Delta}_k f(x) = \lim_{n\to 0}(1/h)\underset{h}{\bar{\Delta}}_k f(x).$$

This suggests that if $0 \neq h \in \mathbf{R}^n$, $h$ is small and $1 \leqq k \leqq n$ then $h^{-1}\underset{h}{\Delta}_k$ and $h^{-1}\underset{h}{\bar{\Delta}}_k$ are good approximations to $D_k$ (see [**10**], page 178, for a distributional analogue of this heuristic).

Given a linear, homogeneous, constant coefficient differential equation of order $p, LF = 0$, one may replace $D_k$ by $s^{-1}\underset{s}{\Delta}_k$ (or $s^{-1}\underset{s}{\bar{\Delta}}_k$) for $1 \leqq k \leqq n$, multiply the resulting equation by $s^p$ and obtain an equation of the type (1)$'$ with polynomial $a_j$'s and linear $h_j$'s. One may then study the resulting equation. This was done for the one-dimensional wave equation in [**3**] using $\underset{h}{\bar{\Delta}}_k$ and it was found that, roughly speaking, the solutions of the resulting functional equation

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are the same as those of the original wave equation. In contrast, the solutions of the functional analogue of the Laplace equation are harmonic polynomials of degree depending on the dimension, $n$, (see [**2**] and [**4**]).

We aim to examine the Cauchy-Riemann equations and the heat equation in a similar spirit. From now on $n = 2$.

The Cauchy-Riemann equations can be written as the single equation

$$\text{(C-R)} \qquad D_1 f(x, y) + iD_2 f(x, y) = 0, \quad (x, y) \in \mathbf{R}^n.$$

If in $(C\text{-}R)$ we replace $D_1$ by $s^{-1}\underset{s}{\Delta}_1$, replace $D_2$ by $s^{-1}\underset{s}{\Delta}_2$ and multiply the resulting equation by $s$ we arrive at the equation

$$(6) \qquad [f(x+s, y) - f(x, y)] + i[f(x, y+s) - f(x, y)] = 0.$$

Notice that (6) holds trivially when $s = 0$. The distributional analogue of (6) is

$$(6)' \qquad \tau_{(s,0)}u + i\tau_{(0,s)}u - (1+i)\tau_{(0,0)}u = 0,$$

a special case of $(4)'$.

PROPOSITION 9. *Let* $\delta > 0$.

*(i) If* $f \in C^2(\mathbf{R}^2)$ *then* (6) *holds for all* $(x, y) \in \mathbf{R}^2$ *and all* $s \in (-\delta, \delta)$ *if and only if there exist* $a_0, b_0 \in \mathbf{C}$ *such that*

$$(7) \qquad f(x, y) = a_0 + b_0(x + iy) \quad \textit{for all } (x, y) \in \mathbf{R}^2.$$

*(ii) If* $u \in \mathcal{D}_2'$ *then* $(6)'$ *holds for all* $s \in (-\delta, \delta)$ *if and only if* $u = \Lambda_f$ *for some* $f \in C^\infty(\mathbf{R}^2)$ *satisfying* (6).

*(iii) If* $f \in L^1_{loc}(\mathbf{R}^2)$ *and, for each* $s \in (-\delta, \delta)$, (6) *holds for a.e.* $(x, y) \in \mathbf{R}^2$ *then there exist* $a_0, b_0 \in \mathbf{C}$ *such that* $f(x, y) = a_0 + b_0(x+iy)$ *for a.e.* $(x, y) \in \mathbf{R}^2$.

*(iv) If* $f \in C(\mathbf{R}^2)$ *and* (6) *holds for all* $(x, y, s)$ *belonging to a dense subset of* $\mathbf{R}^3$ *then there exist* $a_0, b_0 \in \mathbf{C}$ *such that* (7) *holds.*

*Proof.* (i) If (7) holds for some $a_0, b_0 \in \mathbf{C}$ then it is easy to check that (6) holds for all $x, y, s \in \mathbf{R}$.

Suppose $f \in C^2(\mathbf{R}^2)$ and (6) holds for $(x, y) \in \mathbf{R}^2$ and $-\delta < s < \delta$. Differentiating (6) twice with respect to $s$ and setting $s = 0$ we find that

$$(8) \qquad D_1 f + iD_2 f = 0$$

and

$$(9) \qquad D_1^2 + iD_2^2 f = 0.$$

Using (8) twice, we deduce that

$$D_1^2 f = D_1(D_1 f) = D_1(-iD_2 f) = -iD_2(D_1 f) = -iD_2(-iD_2 f) = -D_2^2 f$$

or

(10) $\quad D_1^2 f + D_2^2 f = 0.$

From (9) and (10) it follows that $D_1^2 f = D_2^2 f = 0$. Since $D_1^2 f = 0$, for each $y \in \mathbf{R}$ there exist $a(y)$, $b(y) \in \mathbf{C}$ such that

(11) $\quad f(x, y) = a(y) + b(y)x \quad$ for all $x \in \mathbf{R}$.

Thus $a(y) = f(0, y)$ and $b(y) = f(1, y) - a(y)$ for all $y \in \mathbf{R}$ so that $a, b \in C^2(\mathbf{R})$. Since $D_2^2 f = 0$, (11) implies that $a''(y) = b''(y) = 0$ for all $y \in \mathbf{R}$. Hence there exist $a_0, a_1, b_0, b_1 \in \mathbf{C}$ such that $a(y) = a_0 + a_1 y$ and $b(y) = b_0 + b_1 y$ for all $y \in \mathbf{R}$. Thus

(12) $\quad f(x, y) = a_0 + a_1 y + (b_0 + b_1 y)x \quad$ for all $x, y \in \mathbf{R}$.

Substituting (12) into (6) we find that

$$b_0 s + b_1 sy + ia_1 s + ib_1 xs = 0 \quad \text{for } x, y \in \mathbf{R} \quad \text{and} \quad -\delta < s < \delta$$

from which it follows that $a_1 = ib_0$ and $b_1 = 0$. Hence, by (12), $f(x, y) = a_0 + b_0(x + iy)$ for all $(x, y) \in \mathbf{R}^2$. Thus (i) has been proved.

(ii) Suppose $u \in \mathcal{D}_2'$ and (6)' holds for each $s \in (-\delta, \delta)$. Then $u^*\phi_k$ is a $C^\infty$ solution of (6) for each $k = 1, 2, \ldots$. From (i) it follows that for each $k = 1, 2, \ldots$ there exist $a_k, b_k \in \mathbf{C}$ such that

$$(u * \phi_k)(x, y) = a_k + b_k(x + iy) \quad \text{for all } (x, y) \in \mathbf{R}^2.$$

The sequence of distributions $\{\Lambda_{u*\phi_k}\}_{k=1}^\infty$ is therefore contained in a two dimensional subspace of $\mathcal{D}_2'$.

Again using the fact that finite dimensional subspaces of Hausdorff linear topological spaces are closed and recalling that $\{\Lambda_{u*\phi_k}\}_{k=1}^\infty$ converges to $u$ in $\mathcal{D}_2'$ it follows that $u$ belongs to the subspace in question. Hence $u = \Lambda_f$ for some $f: \mathbf{R}^2 \longrightarrow \mathbf{C}$ satisfying (7) for some $a_0, b_0 \in \mathbf{C}$. The remainder of the proof of (ii) follows from (i).

It is easy to deduce (*iii*) from (*ii*) and to deduce (*iv*) from (*iii*).

*Remark.* In the proof of (i) we could have used (8) or (10) and Proposition 8 to deduce that $f \in C^\infty(\mathbf{R}^2)$.

We now consider the heat equation in $\mathbf{R}^2$: $D_1 f = D_2^2 f$. Replacing $D_1$ by $s^{-1}\underset{s}{\Delta}_1$ and $D_2^2$ by $s^{-2}\underset{s}{\Delta}_2^2$ we are led to

(13) $\quad s[f(x + s, y) - f(x, uy)] = f(x, y + 2s) - 2f(x, y + s) + f(x, y).$

A distributional analogue of (13) is

(13)$'$ $\quad s[\tau_{(s,0)}u - u] = \tau_{(0,2s)}u - 2\tau_{(0,s)}u + \tau_{(0,0)}u.$

PROPOSITION 10. *Let $\delta > 0$.*

*(i) If $f \in C^\infty(\mathbf{R}^2)$ then* (13) *holds for all $(x, y) \in \mathbf{R}^2$ and all $s \in (-\delta, \delta)$ if and only if there exist $\alpha, \beta \in \mathbf{C}$ such that*

$$(14) \qquad f(x, y) = \alpha + \beta y \quad \textit{for all } x, y \in \mathbf{R}.$$

*(ii) If $u \in \mathcal{D}_2'$ then* (13)$'$ *holds for every $s \in (-\delta, \delta)$ if and only if $u = \Lambda_f$ for some $f \in C^\infty(\mathbf{R}^2)$ satisfying* (14) *for some $\alpha, \beta \in \mathbf{C}$.*

*(iii) If $f \in L^1_{loc}(\mathbf{R}^2)$ and, for each $s \in (-\delta, \delta)$,* (13) *holds for a.e. $(x, y) \in \mathbf{R}^2$ then there exist $\alpha, \beta \in \mathbf{C}$ such that*

$$f(x, y) = \alpha + \beta y \quad \textit{for a.e. } (x, y) \in \mathbf{R}^2.$$

*(iv) If $f \in C(\mathbf{R}^2)$ and* (13) *holds for all $(x, y, s)$ belonging to a dense subset of $\mathbf{R}^3$ then there exist $\alpha, \beta \in \mathbf{C}$ such that* (14) *holds.*

*Proof.* (i) Suppose $f \in C^\infty(\mathbf{R}^2)$ and (13) holds whenever $x, y \in \mathbf{R}$ and $s \in (-\delta, \delta)$. By differentiating (13) three times with respect to $s$ and putting $s = 0$ in the resulting equations we conclude that

$$(15) \qquad D_1 f = D_2^2 f$$

and

$$(16) \qquad D_1^2 f = 2D_2^3 f.$$

From (15) and (16) it follows that $D_1 D_2^2 f = D_1^2 f = 2D_2^3 f$ or

$$(17) \qquad (D_1 - 2D_2) D_2^2 f = 0.$$

Hence there exists $A \in C^\infty(\mathbf{R})$ such that

$$(18) \qquad D_2^2 f(x, y) = A(2x + y) \quad \text{for all } x, y \in \mathbf{R}.$$

Comparing (15) and (18) we conclude that

$$(19) \qquad D_1 f(x, y) = A(2x + y) \quad \text{for } (x, y) \in \mathbf{R}^2.$$

Let $a(t) = (1/2) \int_0^t A(\xi) d\xi$ for $t \in \mathbf{R}$. Then (19) implies that for each $y \in \mathbf{R}$ there exists $b(y) \in \mathbf{C}$ such that, for all $x \in \mathbf{R}$,

$$(20) \qquad f(x, y) = a(2x + y) + b(y).$$

It follows that $a, b \in C^\infty(\mathbf{R})$ and, according to (20) and (15), we have, for all $x, y \in \mathbf{R}$,

$$(21) \qquad 2a'(2x + y) = a''(2x + y) + b''(y).$$

Let $2k = 2a'(0) - a''(0)$. Then, according to (21),

$$b''(y) = 2a'(2(-y/2) + y) - a''(2(-y/2 + y) = 2k \quad \text{for all } y \in \mathbf{R}.$$

Hence, by (21)

$$2a'(2x + y) - a''(2x + y) = 2k \quad \text{for all } x, y \in \mathbf{R}.$$

Thus

(22) $\quad b''(y) = 2k \quad$ for all $y \in \mathbf{R}$

and

(23) $\quad 2a'(t) - a''(t) = 2k \quad$ for all $t \in \mathbf{R}$

According to (22) and (23) there exist $b_0, b_1, c, \gamma \in \mathbf{C}$ such that

(24) $\quad b(y) = b_0 + b_1 y + ky^2 \quad$ for all $y \in \mathbf{R}$

and

(25) $\quad a(t) = c + kt + \gamma e^{2t} \quad$ for all $t \in \mathbf{R}$.

Going back to (20) we find that, for all $x, y \in \mathbf{R}$,

(26) $\quad f(x, y) = c + k(2x + y) + \gamma e^{4x+2y} + b_0 + b_1 y + ky^2.$

On substituting (26) in (13) we find, after some tedious calculation, that $\gamma = k = 0$. Thus, if $\alpha = c + b_0$ and $\beta = b_1$ we have

(27) $\quad f(x, y) = \alpha + \beta y \quad$ for all $x, y \in \mathbf{R}^2$.

Conversely, given $\alpha, \beta \in \mathbf{R}$, if we define $f$ by (27) then (13) holds. This completes the proof of (i).

The proofs of (ii), (iii) and (iv) are similar to those of the corresponding parts of Proposition 9.

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