Let $X=(BC,\ B'C')$, $Y=(AC,\ A'C')$, $Z=(AB,\ A'B')$, $B''=(AC,\ OB')$, $X''=(BC,\ B''C')$, $Z''=(AB,\ A'B'')$; then, by Lemma 4 applied to the triangles ABC and $A'B''C'$, the points $X''YZ''$ are collinear. Also $ZABZ''$ projects from A' into $B'OBB''$, which projects from C' into $XCBX''$, and hence ZX, AC, $Z''X''$ are concurrent in Y, that is, XYZ are collinear.

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On a Chain of Circle Theorems.

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If P_1, P_2, P_3, P_4 are four points on a circle C, and P_{234} is the orthocentre of triangle $P_2P_3P_4$, P_{134} the orthocentre of triangle $P_1P_3P_4$ and so on, then the quadrilateral $P_{234}P_{134}P_{124}P_{123}$ is congruent to the quadrilateral $P_1P_2P_3P_4$. This theorem seems to be due to Steiner (*Ges. Werke*, 1, p. 128; see H. F. Baker, *Introduction to Plane Geometry*, 1943, p. 332) and has appeared frequently since in collections of riders on the elementary circle theorems.

It is clear that $P_{234}P_{134}P_{124}P_{123}$ lie on a circle C_{1234} equal to the original circle C. But also angle $P_3P_{134}P_4 = P_4P_1P_3 = P_4P_2P_3 = P_3P_{234}P_4$ (with angles directed and equations modulo π), and hence $P_3P_4P_{134}P_{234}$ lie on a circle C_{34} equal to C, and which is in fact the mirror image of C in P_3P_4. Similarly we obtain circles C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, so that we have in all eight circles with four points on each. If any one of these be taken as the original circle, the same system of eight circles is obtained; if, *e.g.*, we begin with $P_3P_4P_{134}P_{234}$ on the circle C_{34}, the four orthocentres are P_1, P_2, P_{123}, P_{124} lying on C_{12} and the remaining circles are the images of C_{34} in the six sides of the quadrangle $P_3P_4P_{134}P_{234}$. Call this configuration K_4.

Let us now take a fifth point P_5 on C. Then any four of $P_1P_2P_3P_4P_5$ give a K_4. We have in fact five points $P_1 \ldots . P_5$, ten points $P_{123} \ldots . P_{345}$, a circle C, ten circles $C_{12} \ldots . C_{45}$ and five circles $C_{1234} \ldots . C_{2345}$. *Then the circles $C_{1234}C_{1235}C_{1245}C_{1345}C_{2345}$ all pass through a point P_{12345}, completing a system of 16 points and 16 circles, five points on each circle and five circles through each point.* We may show this by taking the circle C_{12}, *e.g.*, on which lie the five points $P_1P_2P_{123}P_{124}P_{125}$ and build up the K_4's obtained by taking these four at a time. Use a parallel notation and write $Q_1 = P_1$, $Q_2 = P_2$,

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$Q_3 = P_{123}$, $Q_4 = P_{124}$, $Q_5 = P_{125}$. Then by picking out orthocentres we obtain $Q_{123} = P_3$ (*i.e.*, the orthocentre of $Q_1 Q_2 Q_3$ is P_3), $Q_{124} = P_4$, $Q_{125} = P_5$; $Q_{134} = P_{134}$, $Q_{135} = P_{135}$ down to $Q_{245} = P_{245}$; but the final Q point Q_{345} is not the final P point P_{345}. Then picking out the circles through these points, $D = C_{12}$, $D_{12} = C$, $D_{13} = C_{13}$, $D_{14} = C_{14}$, $D_{15} = C_{15}$, $D_{23} = C_{23}$, $D_{24} = C_{24}$, $D_{25} = C_{25}$; $D_{34} = C_{1234}$, $D_{35} = C_{1235}$, $D_{45} = C_{1245}$; $D_{1234} = C_{34}$, $D_{1235} = C_{35}$, $D_{1245} = C_{45}$, $D_{1345} = C_{1345}$, $D_{2345} = C_{2345}$. Now through Q_{345} go $D_{34} D_{35} D_{45} D_{1345} D_{2345}$; *i.e.* the circles $C_{2345} C_{1345} C_{1245} C_{1235} C_{1234}$ all pass through one point $Q_{345} = P_{12345}$ as required. It is clear that the figure obtained in this way is symmetrical; any one of the 16 circles may be taken as the original circle with the five points on it as the original points; call it K_5.

If we next take a sixth point P_6 on C and form the six K_5's by taking five points from $P_1 \ldots . P_6$, we have six points $P_1 \ldots . P_6$, 20 points $P_{123} \ldots . P_{456}$, and six points $P_{12345} \ldots . P_{23456}$. These lie on a circle C, 15 circles $C_{12} \ldots . C_{56}$, and 15 circles $C_{1234} \ldots . C_{3456}$. Then the six points $P_{12345} \ldots . P_{23456}$ lie on a circle C_{123456}. *This completes a symmetrical K_6 of 32 points and 32 circles, six points on a circle and six circles through a point.* A proof of the existence of C_{123456} on very similiar lines to that given above for the existence of P_{12345} may be readily supplied by the reader.

It is now obvious that a chain of theorems may be constructed, the addition of every point P_i on C involving the existence alternately of a point and a circle, so that where there are n points P_i, there are in all 2^{n-1} points and 2^{n-1} circles.

We should link this chain of theorems with the long catenation begun by Wallace, de Longchamps and Clifford. (See Baker, *l.c.*, pp. 337-344, and Richmond, *Proc. Edin. Math. Soc.* 2, 6 (1939), 78 where a further bibliography is given). The chain given here is not in the normal de Longchamps' chain, but is *a special case of Richmond's extension.* Consider for simplicity a K_4, and let a general line l_1 through P_1 cut C_{12} in H_{12}, C_{13} in H_{13}, C_{14} in H_{14}; let $l_2 = H_{12} P_2$, $l_3 = H_{13} P_3$, $l_4 = H_{14} P_4$; let $l_2 l_3$ cut in H_{23}, $l_2 l_4$ in H_{24}, $l_3 l_4$ in H_{34}. Then angle $P_4 H_{34} P_3 = P_4 H_{14} H_{12} + H_{12} H_{13} P_3 = P_4 P_{134} P_1 + P_1 P_{134} P_3 = P_4 P_{134} P_3 \pmod{\pi}$. Hence H_{34} lies on C_{34} and similarly H_{23} lies on C_{23} and H_{24} on C_{24}. Then if $P_1 P_2 P_3 P_4$ and $l_1 l_2 l_3 l_4$ be taken as the base points and lines of Richmond's extension the configuration is obtained as the 4-point case of this.

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