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AN EQUALITY OF DISTRIBUTIONS ASSOCIATED TO FAMILIES OF THETA SERIES[1]

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Introduction

Let G be a connected algebraic group, ρ a finite dimensional representation of G in a vector space V, all defined over a number field k. To the pair (G, ρ) we can associate the family of theta series

$$\Theta(G, \rho, \Phi, g) = \sum_{\xi \in V_k} \Phi(\rho(g)\xi)$$

where $g \in G_A$, the adelisation of G, gives the parametrization of the family. Here $\Phi \in \mathscr{S}(V_A)$, is a Schwartz-Bruhat function on the adelisation of the vector space V. If (G, ρ) is admissible, that is, if the function $g \to \Theta(G, \rho, \Phi, g)$ is an L^1-function on G_A/G_k, it gives rise to a tempered distribution on V_A,

$$\Phi \longrightarrow I_\rho(\Phi) = \int_{G_A/G_k} \sum_{\xi \in V_k} \Phi(\rho(g)\xi) dg \ .$$

Such distributions occur in the Siegel formula. In fact, the distribution

$$\Phi \longrightarrow I'_\rho(\Phi) = \int_{G_A/G_k} \sum_{\xi \in V'_k} \Phi(\rho(g)\xi) dg$$

where the sum is over a principal subset $V' \subset V$, is the main part of I_ρ in the complete Siegel Formula, as given by Weil [9].

Suppose now that $\tilde{G} \xrightarrow{\pi} G$ is an isogeny of groups, where $\tilde{G}$ is simply connected, semi-simple. Then $\tilde{\rho} = \pi \circ g$ is absolutely admissible$/k$ if and only if (G, ρ) is absolutely admissible$/k$ ([4]). Hence, in this case, there are two distributions that we can associate to V_A, namely I'_ρ, $I'_{\tilde{\rho}}$. Our main result is that these are the same. Actually, it is expected that $I_\rho = I_\rho$ on V_A, but at the moment we need information on the orbits that are outside the principal subset V' (see § 5 for the definition of V').

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In any case, the family of theta series associated to $(\tilde{G}, \tilde{\rho})$ is a subfamily of the series associated to (G, ρ).

Such an equality of distributions is implied by the conjectural Siegel Formula of Igusa, since the distributions that arise from the associated Eisenstein series are easily seen to be the same. However, here we give a direct proof, without passing through the Siegel Formula. In a sense our result can be viewed as further evidence for the validity of these conjectural formulae.

The central tool is the crossed diagram, made explicit by Ono [7]. In § 2 we list a set of axioms that we suppose $(\tilde{G}, \tilde{\rho})$ to satisfy. Under these hypotheses, we obtain an estimate of the ratios of the Tamagawa numbers of the stabilisers of points, in terms that do not depend on the point. Then, the special features of admissible representations, studied in detail by Igusa [3], enable us to verify the axioms and so obtain the equality of distributions. The author wishes to thank Professor Igusa for suggesting this problem.

Index of notations

$\boldsymbol{Z}, \boldsymbol{R}$: the integers, real numbers (respectively)

$\boldsymbol{F}_q$: the finite field with q elements

G_m: the multiplicative group of the universal domain

k: algebraic number field

k_v: completion of k with respect to a valuation v

$\bar{k}$: the algebraic closure of k

$\mathrm{Gal}(\bar{k}/k)$: the Galois group of the extension $\bar{k}/k$

$[G:H]$: the index of the group G over the subgroup H

$[S]$: the cardinality of a set S

For an algebraic group G defined over a field k

G_0: the identity component of G

G_k: the subgroup of G consisting of the points rational over k

G_A: the adelisation of G

$X(G) = \hat{G} = \mathrm{Hom}(G, G_m)$: the module of rational characters of G

$X_k(G)$: the module of k-rational characters of G

$G \xrightarrow{\lambda} G', \hat{G}' \xrightarrow{\hat{\lambda}} \hat{G}$: the corresponding map of the character modules

$R_{K/k}(G)$: the algebraic group defined over k obtained by restricting the field of definition from K to k

$H^q(k, G) = H^q(\mathrm{Gal}(\bar{k}/k), G_{\bar{k}})$ $(q \geq 0)$: the qth Galois cohomology set

$h_v^q(G) = [H^q(k_v, G)]$
$i^q(G) = [\mathrm{Ker}\,(H^q(k, G) \to \prod_v H^q(k_v, G))]$.

1. The Tamagawa number

Ono [7], has defined the notation of a Tamagawa number for connected, reductive groups. For our purposes, we need to extend this notion to non-connected groups. In all that follows we suppose that G is an algebraic group, defined over a number field k, subject to the following two restrictions:

(R1) all the connected components of G have k-rational coset representatives, for the cosets with respect to the identity component G_0.

(R2) G has no k-defect. That is, the restriction homomorphism $X_k(G) \xrightarrow{\mathrm{Res}} X_k(G_0)$, has finite cokernel.

For such G, we will define $\tau_\lambda(G)$, its Tamagawa number.

Let G_A be the adelisation of G,

$$G'_A = \{g \in G_A \mid \|\chi(g)\| = 1, \text{ for all } \chi \in X_k(G)\} ,$$

where $\|t\|$ is the idèle norm. Then by Borel [1, 5.6], G'_A/G_k has finite invariant measure. Let $\chi_1, \cdots, \chi_r$ form a $\boldsymbol{Z}$-basis for the torsion free part of $X_k(G)$. Then the mapping

$$\begin{array}{rcl} \psi : G_A & \longrightarrow & \boldsymbol{R}^r \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \longrightarrow & (\log \|\chi(g)\|, \cdots, \log \|\chi_r(g)\|) \end{array}$$

induces the isomorphism $\psi : G_A/G'_A \xrightarrow{\approx} \boldsymbol{R}^r$.

Let ω be an invariant gauge form on G, defined over k. It induces a Haar measure $|\omega_V|$ on G_v, the k_v-rational points of G. Let

$$\mu_p = \int_{G_{O_p}} |\omega_p|$$

be the volume of the O_p-rational points of G. Then, the standard arguments and the assumption (R1) imply that

$$\mu_p = q^{-\dim G}[G^{(p)}_{F_p}] ,$$

for almost all p ($q = Np$, $G^{(p)}$ is the algebraic group defined over the finite field F_q obtained from G by reduction modulo p [8]). Let $\lambda = (\lambda_v)$ be a family of positive real numbers, one for each valuation on k, such that $\prod \lambda_p \mu_p$ is convergent. We call λ a system of convergence factors for G. For Δ_k the discriminant of k, set $d_\lambda G_A = |\Delta_k|^{-\dim G/2} \prod_v \lambda_v \omega_v$. This gives a Haar measure on G_A, so by a suitable normalization, $d_\lambda G_A = d(G_A/G'_A)d(G'_A/G_k)dG_k$ where $d(G_A/G'_A) = dt$ is the usual measure on $\boldsymbol{R}^r$ we obtain via the mapping ψ, and dG_k is the discrete measure. $\tau_\lambda(G) =$ the measure of G'_A/G_k, for the measures so normalised. One has the following useful characterization:

$$\tau_\lambda(G) = \frac{\int_{G_A/G_k} F(\psi(g))dG_A}{\int_{\boldsymbol{R}^r} F(t)dt}$$

for F an integrable function on $\boldsymbol{R}^r$ ([7]). Ono's idea was to use this latter characterization even when G has no characters, by interwining it with a torus in the "crossed diagram." Clearly $\tau_\lambda(G)$ is the usual Tamagawa measure for G when G is connected, reductive. Note that for an isogeny of groups $\tilde{G} \xrightarrow{\pi} G$, both, subject to R1, R2, we can use the same system of convergence factors ([8]). Further, if $X_k(G_0) = \{1\}$, then G has no k-defect and we may take $\lambda = 1$ as a system of convergence for G, since in this case $G_A = G'_A$ and G_A/G_k has finite invariant measure.

LEMMA 1. *Let* $0 \longrightarrow G' \longrightarrow G \xrightarrow{\kappa} G'' \longrightarrow 0$ *be an exact sequence of groups, all the maps and groups defined over* k. *Then, the index* $[\kappa(G_A) \cap G''_k : \kappa(G_k)]$ *is finite.*

Remark. This is a strengthened version of Ono's result [7], where he assumed all the groups to be connected, reductive. We are able to strengthen his result, because of the Borel-Serre theorem that has since appeared.

Proof (Ono). For $x \in \kappa(G_A) \cap G''_k$, $\kappa^{-1}(x)$ is a principal homogeneous space for G', defined over k, which has a k_v-rational point for every place v of k. Now by Borel-Serre [2], there are only finitely many isomorphism classes of these. Thus it suffices to show that for $x, y \in G''_k$, $x^{-1}y \in \kappa(G_k)$ if and only if $\kappa^{-1}(x)$ is isomorphic to $\kappa^{-1}(y)$ as principal homogeneous spaces for G'. If $y = \kappa(b)x$, for $b \in G_k$, then $\xi \to \xi b$ gives $\kappa^{-1}(x) \xrightarrow{\approx} \kappa^{-1}(y)$. Conversely, if $f: \kappa^{-1}(x)$ $f: \kappa^{-1}(x) \to \kappa^{-1}(y)$ is an isomorphism over k, pick $\xi \in \kappa^{-1}(x)$ which is algebraic over k. Then, since ${}^\sigma\xi \in \kappa^{-1}(x)$

$${}^\sigma\xi = \xi a_\sigma\,, \qquad \text{for } a_\sigma \in G',\ \sigma \in \operatorname{Gal}(\bar{k}/k)\,.$$

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Hence, ${}^{\sigma}(f(\xi)) = f({}^{\sigma}\xi) = f(\xi)a_{\sigma}$, whence

$$b = {}^{\sigma}(f(\xi)\xi^{-1}) = f(\xi)\xi^{-1} \in G_k \quad \text{and} \quad f(\xi) = \xi b \ ,$$

so $y = \kappa(b)x$.

§ 2. The crossed diagram

Let

$$\begin{array}{l} \tilde{G} \\ \big\downarrow{\scriptstyle\pi} \\ G \xrightarrow{\ \rho\ } \operatorname{Aut}(V) \end{array}$$

be a representation of $\tilde{G}$ (a connected group), where π is as isogeny, all the groups, mappings and the finite dimensional vector space V being defined over k. Fix $v \in V_k$ and set

$$\begin{aligned} \tilde{H} &= \tilde{H}(v) = \{\tilde{g} \in \tilde{G} \mid \rho\pi(\tilde{g})v = v\} \\ H &= H(v) = \{g \in G \mid \rho(g)v = v\} \ . \end{aligned}$$

Both of these stabilisers are defined over k, and π induces the isogeny $\pi|_{\tilde{H}} : \tilde{H} \to H$. If $M = \operatorname{Ker}(\pi : \tilde{G} \to G)$, then clearly $M \subset \tilde{H}$ for every stabiliser. Let $\mathfrak{G} = \operatorname{Gal}(\bar{k}/k)$, then M is a finite continuous $\mathfrak{G}$-module (Krull topology on $\mathfrak{G}$, discrete topology on M). The character module $\hat{M} = \operatorname{Hom}(M, \bar{k}^x)$ can also be considered as a continuous $\mathfrak{G}$-module, by $({}^{\sigma}\xi)(x) = \sigma(\xi(\sigma^{-1}x))$. Let $\mathfrak{H}_{\hat{M}} = \{\sigma \in \mathfrak{G} \mid {}^{\sigma}\xi = \xi \text{ for all } \xi \in \hat{M}\}$. This is an open, nomal subgroup of $\mathfrak{G}$, hence its fixed field $K_{\hat{M}}$ is a finite Galois extension of k and $\hat{M}$ is a $\operatorname{Gal}(K_{\hat{M}/k})$-module. Let $\Gamma =$ the integral group ring of $\operatorname{Gal}(K_{\hat{M}/k})$. We have the following exact sequence

$$0 \longrightarrow W \longrightarrow \underbrace{\Gamma + \cdots + \Gamma}_{s\text{-times}} \longrightarrow \hat{M} \longrightarrow 0$$

of Γ-modules. By canonical duality, we have therefore an exact sequence of groups, defined over k:

$$0 \longrightarrow M \longrightarrow T' \xrightarrow{\ \lambda\ } T \longrightarrow 0 \tag{1}$$

where M is viewed as a zero dimensional group. T, T' are tori, defined$/k$, and

$$T' = \underbrace{\boldsymbol{R}_{K_{\hat{M}/k}}(G_m) \times \cdots \times \boldsymbol{R}_{K_{\hat{M}/k}}(G_m)}_{s\text{-times}}$$

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so that $\hat{T}' \rightrightarrows \Gamma^s$ is a projective Gal $(K_{\hat{M}/k})$-module. We fix such a sequence (1) and it is important in what follows that we can use the same sequence (1) for every stabiliser $\tilde{H}$. Consider now the following crossed diagram:

$(\underline{V})$:

$$(\underline{H}): \qquad \begin{tikzcd} & & 0 \arrow[d] & & \\ & & T' \arrow[d, "\iota"'] \arrow[dr, "\lambda"] & & \\ 0 \arrow[r] & \tilde{H} \arrow[r] \arrow[dr, "\pi"] & H^* \arrow[r, "\kappa"] \arrow[d] & T \arrow[r] & 0 \\ & & H \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

where M is diagonally embedded in $\tilde{H} \times T'$ and $H^* = \tilde{H} \times T'/M$. All the maps are the natural ones, each is defined over k and the diagram is commutative, with exact row $(\underline{H})$ and exact column $(\underline{V})$.

LEMMA 2. *Let* $0 \longrightarrow G' \xrightarrow{\iota} G \xrightarrow{\kappa} G'' \longrightarrow 0$ *be an exact sequence of groups, all defined over* k*, such that* G'' *is connected. Let* G'_0, G_0 *denote the connected components of the identity of* G', G *and* $\tilde{\kappa} = \kappa | G_0$*. Then* $[G' : G'_0] = [G : G_0][\mathrm{Ker}\, \tilde{\kappa} : (\mathrm{Ker}\, \tilde{\kappa})_0]$ *and*

$$(\mathrm{Ker}\, \tilde{\kappa})_0 = \iota(G'_0) .$$

Proof. The last statement follows by a simple dimension argument. Also, since G'' is connected, we can choose coset represetatives for G/G_0 in $\iota(G')$, say $\iota(g_1), \cdots, \iota(g_s)$. Let $\iota(\tilde{g}_1), \cdots, \iota(\tilde{g}_r)$ be representatives for $\mathrm{Ker}\, \tilde{\kappa}/(\mathrm{Ker}\, \tilde{\kappa})_0$. Then $\{g_i \tilde{g}_j\}$ form a system of coset representatives for G'/G'_0.

LEMMA 3. *Let* $0 \longrightarrow G' \longrightarrow G \xrightarrow{\kappa} G'' \longrightarrow 0$ *be an isogeny of groups, all defined over* k *where* G, G'' *are not necessarily connected. Then*

$$[G : G_0] = [G'' : G''_0][G' : G' \cap G_0] .$$

Proof. Let $g_1'', \cdots, g_s''$ be coset representatives for G_0'' in G'' and choose $g_1, \cdots, g_s$ in G, so that $\kappa(g_i) = g_i''$. If $g_1', \cdots, g_r'$ are coset representatives for $G' \cap G_0$ in G', then it is easily seen that $\{g_i g_j'\}$ form a system of coset representatives for G_0 in G.

LEMMA 4. *Let* $0 \longrightarrow G' \longrightarrow G \xrightarrow{\kappa} G'' \longrightarrow 0$ *be an exact sequence of connected groups, all defined over* k. *Let* G'' *be a torus, and* G' *have no characters,* $X(G') = \{1\}$. *Then the index* $[G_A'' : \kappa(G_A)G_k'']$ *is finite.*

Proof. Since the characteristic of k is zero, $G = H \cdot \boldsymbol{R}_u(G)$, a semi-direct product of the unipotent radical of G with a maximal reductive subgroup H of G; both $H, \boldsymbol{R}_u(G)$ are defined over k. Denote $\tilde{\kappa} = \kappa|_H$, so that $G' = (\mathrm{Ker}\,\tilde{\kappa}) \cdot \boldsymbol{R}_u(G)$, again a semi-direct product. This gives rise to the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\tilde{\kappa} \longrightarrow H \longrightarrow G'' \longrightarrow 0$$

of groups over k with $X(\mathrm{Ker}\,\tilde{\kappa}) = \{1\}$, from the same assumption for G'.

Let $T = \boldsymbol{Z}(H)_0$ be the identity component of the centre of H, T' a maximal torus, defined over k, of the semisimple part S of H. Then $H = TS$ and TT' is a torus, defined over k of H. Further, $S \subset \mathrm{Ker}\,\tilde{\kappa}$ and since $X(\mathrm{Ker}\,\tilde{\kappa}) = 1$, $\mathrm{Ker}\,\tilde{\kappa} = S$, (unipotent radical), semi-direct product. Therefore a maximal torus of S is a maximal torus of $\mathrm{Ker}\,\tilde{\kappa}$. $T \cap \mathrm{Ker}\,\tilde{\kappa}$ is a reductive subgroup of $\mathrm{Ker}\,\tilde{\kappa}$, hence a conjugate of it lies in S. But T, being central in H, is its own conjugate in $\mathrm{Ker}\,\tilde{\kappa}$, whence $T \cap \mathrm{Ker}\,\tilde{\kappa} \subset S$. But it clearly is in the centre of S, so that $T \cap \mathrm{Ker}\,\tilde{\kappa} \subset T'$ and we have the following exact sequence of tori:

$$0 \longrightarrow T' \longrightarrow TT' \xrightarrow{\kappa} G'' \longrightarrow 0 .$$

Clearly $[G_A'' : \kappa(G_A)G_k''] \leq [G_A'' : \kappa(TT')_A G_k'']$ and this latter is finite [6, 4.3].

Now apply these results to the exact sequence ($\underline{H}$) of the crossed diagram. Let H_0^* be the identity component of H^*, $\tilde{\kappa} = \kappa|_{H_0^*}$, whence $(\mathrm{Ker}\,\tilde{\kappa})_0 = \tilde{H}_0$. By Lemma 2,

$$[H : \tilde{H}_0] = [H^* : H_0^*][\mathrm{Ker}\,\tilde{\kappa} : (\mathrm{Ker}\,\tilde{\kappa})_0] .$$

Applying Lemma 3 to the sequence $0 \to M \to \tilde{H} \times T' \to H^* \to 0$, we have

$$\begin{aligned}[\tilde{H} : \tilde{H}_0] &= [\tilde{H} \times T' : \tilde{H}_0 \times T'] \\ &= [H^* : H_0^*][M : M \cap \tilde{H}_0 \times T'] \\ &= [H^* : H_0^*][M : M \cap \tilde{H}_0] ,\end{aligned}$$

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whence

$$[\operatorname{Ker}\tilde{\kappa}:(\operatorname{Ker}\tilde{\kappa})_0] = [M: M \cap \tilde{H}_0] .$$

We make the following assumptions on $(\tilde{G}, \tilde{\rho})$, where $\tilde{\rho} = \rho \circ \pi$. Suppose $v \in V_k$ is such that for the stabiliser $\tilde{H}(v)$,

(A1) $\tilde{H}(v)/\tilde{H}(v)_0$ has k-rational coset representatives.

(A2) $X(\tilde{H}(v)_0) = \{1\}$

(A3) $M = \operatorname{Ker}\pi \subset \tilde{H}(v)_0$

(A4) $\tilde{H}(v)_0$ is of Kneser type, i.e.

$$h_p^1(\tilde{H}(v)_0) = 1 \qquad \text{for all non-archimedian } p \text{ of } k$$
$$i^1(\tilde{H}(v)_0) = 1$$

Therefore, for the exact sequence $(\underline{H})$ for such $v \in V_k$, (A3) implies that $\operatorname{Ker}\tilde{\kappa}$ is connected and we have the exact sequence of connected groups

$$0 \longrightarrow \tilde{H}_0 \longrightarrow H_0^* \xrightarrow{\tilde{\kappa}} T \longrightarrow 0 .$$

PROPOSITION 1. *For the exact sequence* $(\underline{H})$, $[T_A : \kappa(H_A^*)T_k]$ *is finite and* $\kappa(H_A^*)$ *is open in* T_A.

Proof. We have just seen that (H) gives rise to the exact sequence

$$0 \longrightarrow \tilde{H}_0 \longrightarrow H_0^* \xrightarrow{\tilde{\kappa}} T \longrightarrow 0$$

of connected groups. Hence $\tilde{\kappa}((H_0^*)_A)$ is open in T_A ([7, 1.2.1]) so that $\kappa(H_A^*) \supset \tilde{\kappa}((H_0^*)_A)$ is open in T_A. By (A2), Lemma 4 is applicable to the above sequence, whence $[T_A : \kappa(H_A^*)T_k] \leq [T_A : \tilde{\kappa}((H_0^*)_A T_k] < \infty$.

Let $\mu : \kappa(H_A^*)/\kappa(H_k^*) \to T_A/T_k$ be the natural mapping. We define

$$r(\underline{H}) = \frac{[\operatorname{coker}\mu]}{[\ker\mu]} = \frac{[T_A : \kappa(H_A^*)T_k]}{[\kappa(H_A^*) \cap T_k : \kappa(H_k^*)]}$$

By Lemma 1 and the above proposition, this is a well defined finite number.

LEMMA 5. *$H^* = \tilde{H} \times T'/M$ has no k-defect and H^*/H_0^* has k-rational coset representatives.*

Proof. By the axioms (A1) and (A2), $\tilde{H}$ is of this type. T', being a torus is connected, so clearly is of this type, whence H^*, being a

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homomorphic image has rational coset representatives. To see that H^* has no k-defect, it suffices to prove the following: G algebraic group$/k$, Γ a finite normal subgroup, defined over k. Then G has no k-defect if and only if G/Γ has no k-defect, since $G \to G/\Gamma$ induces $G_0 \to (G/\Gamma)_0$ and $X(G/\Gamma) \to X(G)$ is injective, with finite cokernel, so clearly $X_k(G/\Gamma) \to X_k(G)$ has the same properties. Hence

$$\begin{aligned}&\operatorname{rank} X_k(G/\Gamma) = \operatorname{rank} X_k(G)\\ &\operatorname{rank}(X_k(G_0)) = \operatorname{rank} X_k((G/\Gamma)_0) \text{ gives the equivalence.}\end{aligned}$$

By the axioms and Lemma 5, every group occuring in the crossed diagram had rational coset representatives for the identity component and has no k-defect, hence we can define $\tau_\lambda(G)$, for suitable convergence factors λ. For the torus T, take λ to be the canonical one given by Ono [6]. For $\tilde{H}$ we can take $\lambda = 1$ and for H^* take their product ([8]). Then, by an earlier remark we can take the same systems for T, T' and also for H, $\tilde{H}$. We shall suppress these convergence factors in our notation. Set

$$\tau(\underline{H}) = \frac{\tau(\tilde{H})\tilde{\tau}(T)}{\tau(H^*)}, \qquad \tau(\underline{V}) = \frac{\tau(T')\tau(H)}{\tau(H^*)} .$$

The crossed diagram gives rise to the following commutative diagram and exact sequences of character groups:

$$\begin{tikzcd} & X_k(T') & & \\ X_k(\tilde{H}) & X_k(H^*) \arrow[l] \arrow[u, "\hat{\iota}"] & X_k(T) \arrow[l, "\hat{\kappa}"'] \arrow[ul, "\hat{\lambda}"'] & 0 \arrow[l] \\ & X_k(H) \arrow[u] & & \\ & 0 \arrow[u] & & \end{tikzcd}$$

Further $\hat{\lambda}$ is injective with finite cokernel. Exactness of the row implies that $\operatorname{Im}(\hat{\kappa})$ has finite cokernel, so since $\operatorname{Im}(\hat{\kappa})$ is torsion free, $\operatorname{rank} X_k(T) = \operatorname{rank} X_k(H^*)$. But $\operatorname{rank} X_k(T) = \operatorname{rank} X_k(T')$, so since $\hat{\iota}$ has finite kernel, $\hat{\iota}$ must have finite cokernel. Choose generators $\xi_1^*, \cdots, \xi_r^*$ for the torsion free part of $X_k(H^*)$ and a basis $\xi_1, \cdots, \xi_r$ for $X_k(T)$, such that

$$\hat{\kappa}(\xi_i) = n_i \xi_i^* \qquad (1 \leq i \leq r) .$$

Commutativity of the diagram, $\hat{\lambda} = \hat{\iota} \circ \hat{\kappa}$ implies that $\hat{\lambda}(\xi_i) = n_i \hat{\iota}(\xi_i^*)$, whence

$$[\operatorname{coker} (\hat{\lambda})_k] = n[\operatorname{coker} (\hat{\iota})_k] , \qquad n = \prod_1^r n_i$$

and all the maps are in fact $(\hat{\lambda})_k$, etc.

PROPOSITION 2. *For the exact sequence* $(\underline{H})$: $0 \longrightarrow \tilde{H} \longrightarrow H^* \xrightarrow{\kappa} T \longrightarrow 0$

$$\tau(\underline{H}) = r(\underline{H}) \frac{[\operatorname{coker} (\hat{\iota})_k]}{[\operatorname{coker} (\hat{\lambda})_k]} .$$

Proof. Consider

$$\begin{array}{ccc} \psi'' : T_A & \longrightarrow & \boldsymbol{R}^r \\ \cup\!\!\!| & & \cup\!\!\!| \\ x & \longrightarrow & (\log \|\xi_1(x)\|, \cdots, \log \|\xi_r(x)\|) \end{array}$$

for the basis $\xi_1, \cdots, \xi_r$ of $X_k(T)$ chosen earlier. Then

$$\kappa(H_A^*)/\kappa(H_k^*) \xrightarrow{\mu} T_A/T_k \xrightarrow{\psi''} \boldsymbol{R}^r$$

is surjective and the image under μ is open, by Proposition 1.

$$\begin{aligned} &\int_{\kappa(H_A^*)/\kappa(H_k^*)} F(\psi''\mu(x)) d(\kappa(H_A^*)/\kappa(H_k^*)) \\ &\quad = \int_{(\kappa(H_A^*)/\kappa(H_k^*))/\operatorname{Ker} \mu} \int_{\operatorname{Ker} \mu} \cdots \\ &\quad = \frac{[\ker \mu]}{[\operatorname{coker} \mu]} \int_{T_A/T_k} F(\psi''(x'')) d(T_A/T_k) . \end{aligned}$$

The groups occuring in $(\underline{H})$ are unimodular, so by Weil [8, 2.4.4], we can rewrite the following integral,

$$\begin{aligned} I &= \int_{H_A^*/H_k^*} F(\psi''(\kappa_A(x))) d(H_A^*/H_k^*) \\ &= \tau(\tilde{H}) \int_{\kappa(H_A^*)/\kappa(H_k^*)} F(\psi''\mu(x)) d(\kappa(H_A^*)/\kappa(H_k^*)) \\ &= \frac{\tau(\tilde{H})}{r(\underline{H})} \int_{T_A/T_k} F(\psi''(x'')) d(T_A/T_k) \\ &= \frac{\tau(\tilde{H})\tau(T)}{r(\underline{H})} \int_{\boldsymbol{R}^r} F(t) dt . \end{aligned}$$

Choosing the basis for the torsion free part of $X_k(H^*)$ as before and changing variables suitably in $\boldsymbol{R}^r$, we have also

$$\begin{aligned} I &= \frac{1}{n_1 \cdots n_r} \int_{H_A^*/H_k^*} F(\psi(x)) d(H_A^*/H_k^*) \\ &= \frac{\tau(H^*)}{n} \int_{\boldsymbol{R}^r} F(t) dt \end{aligned}$$

where

$$\begin{array}{cccc} \psi : & H_A^* & \longrightarrow & \boldsymbol{R}^r \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & x & \longrightarrow & (\log \|\xi_1^*(x)\|, \cdots, \log \|\xi_r^*(x)\|) . \end{array}$$

LEMMA 6. *The exact sequence* $(\underline{V}) : 0 \longrightarrow T' \xrightarrow{\iota} H^* \longrightarrow H \longrightarrow 0$ *has a generic cross section defined over k. This is precisely Ono* [7, 2.1.1]. *This is the reason for choosing $\hat{T}'$ to be a* Gal $(K_{M/k})$*-projective module. The fact that this also gives $\tau(T') = 1$ is not needed for our purpose.*

PROPOSITION 3. *For the exact sequence* $(\underline{V}) : 0 \longrightarrow T' \xrightarrow{\iota} H^* \longrightarrow H \longrightarrow 0$

$$\tau(\underline{V}) = [\operatorname{coker} (\hat{\iota})_k] .$$

Proof. This proposition is a variant of a result of Ono. We must modify his result to account for the non-connectedness of H^*, H and for H not being semi-simple. Choose bases $\bar{\xi}_1, \cdots, \bar{\xi}_r$ for the torsion free part of $X_k(H^*)$ and $\xi_1', \cdots, \xi_r'$ for $X_k(T')$ such that $\hat{\iota}(\bar{\xi}_i) = n_i \xi_i'$ $(1 \leq i \leq r)$. Let $\bar{\psi} : H^* \to \boldsymbol{R}^r$ be the corresponding mapping for this choice of basis. Then

$$\begin{aligned} &\int_{T_A'/T_k'} F(\psi(\iota x')) d(T_A'/T_k') \\ &\qquad = \tau(T') \int_{\boldsymbol{R}^r} F(n_1 t_1, \cdots, n_r t_r) dt \\ &\qquad = \frac{1}{[\operatorname{coker} (\hat{\iota})_k]} \int_{T_A'/T_k'} F(\psi'(x')) d(T_A'/T_k') . \end{aligned}$$

Now, by Lemma 6, we can apply a theorem of Fubini type ([8, Lemma 3.2.1]) to the integral

$$I = \tau(H^*) \int_{\boldsymbol{R}^r} F(t) dt$$

$$= \int_{H_A^*/H_k^*} F(\bar{\psi}(x)) d(H_A^*/H_k^*)$$
$$= \int_{H_A/H_k} d(H_A/H_k) \int_{T_A'/T_k'} F(\bar{\psi}((\iota x')x) d(T_A'/T_k') \, .$$

Now, as x' runs over T_A', $\psi'(x')$ runs over $\boldsymbol{R}^r$, so $\bar{\psi}(x) = \psi'(y')$ for some $y \in T_A'$. Hence, by the invariance of the measure $d(T_A'/T_k')$, we have

$$I = \int_{H_A/H_k} d(H_A/H_k) \int_{T_A'/T_k'} F(\bar{\psi}(x')) d(T_A'/T_k')$$
$$= \frac{\tau(T')}{[\text{coker } (\hat{\iota})_k]} \left(\int_{\boldsymbol{R}^r} F(t) dt \right) \cdot \left(\int_{H_A/H_k} d(H_A/H_k) \right)$$

and by axiom (A2), the last factor is $\tau(H)$. Hence equating the two expressions for I, we have

$$\tau(\underline{V}) = [\text{coker } (\hat{\iota})_k] \, .$$

Propositions (2), (3) have given two expressions involving $\tau(H^*)$;

$$\frac{\tau(T')\tau(H)}{\tau(H^*)} = [\text{coker } (\hat{\iota})_k]$$
$$\frac{\tau(\tilde{H})\tau(T)}{\tau(H^*)} = \frac{r(\underline{H})[\text{coker } (\hat{\iota})_k]}{[\text{coker } (\hat{\lambda})_k]} \, .$$

These imply

$$\frac{\tau(H)}{\tau(\tilde{H})} = \frac{\tau(T)}{\tau(T')} \frac{[\text{coker } (\hat{\lambda})_k]}{r(\underline{H})} \, . \tag{2}$$

Notice that since we have the same sequence (1): $0 \longrightarrow M \longrightarrow T' \xrightarrow{\lambda} T \longrightarrow 0$ for every point $v \in V_k$, the only part of the above expression that depends on v, is $r(\underline{H})$, since of course the sequence $(\underline{H})$ involves $\tilde{H}(v)$. Hence, whenever $\tilde{H}(v)$ satisfies the axioms A(1)–A(3), we have (2). Notice also that we have not yet made use of Axiom A(4).

§ 3. A consequence of axiom (A4)

The sequence $(\underline{H})$ gives rise to the exact sequence $0 \longrightarrow \tilde{H}_0 \longrightarrow H_0^* \xrightarrow{\tilde{\kappa}} T \longrightarrow 0$, of connected groups. By axiom (A4) and the theorem of Ono [7, 2.2.1], we have $T_A = \kappa((H_0^*)_A)T_k$,

$$\kappa((H_0^*)_A) \cap T_k = \kappa((H_0^*)_k) \, .$$

PROPOSITION 4. *For the exact sequence* $(\underline{H})$: $0 \to \tilde{H} \to H^* \to T \to 0$ *we have* $r(\underline{H}) = 1$.

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Proof. Since

$$r(\underline{H}) = \frac{[T_A : \kappa(H_A^*)T_k]}{[\kappa(H_A^*) \cap T_k : \kappa(H_k^*)]} \ ,$$

the remarks above show that it suffices to prove that

$$\kappa(H_A^*) \cap T_k = \kappa(H_k^*) \ .$$

Let $x = (x_v) \in H_A^*$, such that $x'' = \kappa(x) \in \kappa(H_A^*) \cap T_k$. Then $\kappa^{-1}(x'')$ is an algebraic set, defined over k, hence each of its irreducible components is defined over some finite extension k'/k, which we may take a Galois extension. The irreducible components of $\kappa^{-1}(x'')$ are cosets of $\tilde{H}_0$, and by altering (x_v) suitably, we can suppose that each x_v lies in the same coset of $\tilde{H}_0$. Hence, this particular coset is an algebraic variety, defined k', with a k_v-rational point, for every v, so is it also defined over k_v. By Chebotarev density for k, we conclude that in fact, this coset is defined over k. This coset, being a principal homogeneous space for $\tilde{H}_0$, which is defined over k, with a k_v-rational point, for every v, must therefore contain a k-rational point, by the axiom (A4) for $\tilde{H}_0$. Hence $\kappa(H_A^*) \cap T_k = \kappa(H_k^*)$ and $r(\underline{H}) = 1$. Summarising, we have

THEOREM 1. *For the stabilisers $\tilde{H}(v), H(v)$ of a point $v \in V_k$, for which $\tilde{H}(v)$ satisfies the axioms* (A1)–(A4),

$$\frac{\tau(H(v))}{\tau(\tilde{H}(v))} = \frac{\tau(T)}{\tau(T')}[\text{coker}\ (\hat{\lambda})_k] \ .$$

This is independent of the point v.

§ 4. A decomposition of the theta series

For the convenience of the reader, we recall now how to rewrite the integral for $I_\rho(\Phi)$ so as to make the Tamagawa numbers explicit [9, p. 14]. Let μ be a relatively invariant measure on the (left) for G_A, invariant on the right by G_k. The convergence of $I_\rho(\Phi)$ implies that for every stabiliser $H(v)$ of a point $v \in V_k$, the quotient $H(v)_A/H(v)_k$ has a finite volume for a suitable relatively invariant measure. In particular, it is also true for G_A/G_k. Hence $H(v)_A$ is unimodular and the quotient space $G/H(v)$ has a relatively invariant gauge form.

Further, let us normalize μ so that $\mu(G_A/G_k) = 1$. We express $I_\rho(\Phi)$ as a sum of similar integrals over the different G_k-orbits in V_k. If U

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is one such orbit, the Witt condition (§ 5) allows one to identify $U_A \equiv G_A/H(v)_A$, $U_k \equiv G_k/H(v)_k$, for $v \in U_k$. Since $H(v), G$ are unimodular, gauge forms dg, dh on them give rise to a G-invariant gauge form Θ on U. Moreover, since G is connected, Θ is unique up to a scalar multiple in k. Denoting the measures one obtains from them, with respect to suitable convergence factors, by $|\Theta|_A, |dg|_A, |dh|_A$ etc., we have

$$\frac{\tau_\lambda(H(v))}{\tau_\lambda(G)} \int_{U_A} \Phi|\Theta|_A = \int_{G_A/H(v)_k} \Phi(\rho(g)v)d\mu(g)$$
$$= \int_{G_A/G_k} \sum_{\xi \in U_k} \Phi(\rho(g)\xi)d\mu(g) .$$

Hence

$$I_\rho(\Phi) = \sum_{\omega \in \Omega} \frac{\tau_\lambda(H(v))}{\tau_\lambda(G)} \int_{U(\omega)} \Phi\,|\Theta(\omega)|_A$$

where $\Omega =$ the collection of G_k-orbits in V_k.

To obtain the equality of tempered distributions, it suffices to show (i) $G_k, \tilde{G}_k$ have the same orbits in V_k

$$\text{(ii)} \quad \frac{\tau_\lambda(\tilde{H}(\omega))}{\tau_\lambda(\tilde{G})} = \frac{\tau_\lambda(H(\omega))}{\tau_\lambda(G)} .$$

§ 5. Properties of absolutely admissible representations

Let $\tilde{G}$ be a connected, simply connected, semi-simple group, defined over k. We have mentioned that $(\tilde{G}, \tilde{\rho})$ is admissible over k if the theta series $\Theta(\tilde{G}, \tilde{\rho}, \Phi, g)$ is L^1-integrable over for every Φ. It is called absolutely admissible over k if it remains admissible over any finite algebraic extension of k. In the paper "Geometry of Absolutely Admissible Representations" [3], Igusa studies in detail the orbits in the principal subset. The principal subset V' of V is defined as follows: let Ω be a universal domain containing $k, \Omega[V]^{\tilde{G}}$, the ring of invariants. It turns out that this is generated by algebraically independent homogeneous polynomials $f_1, \cdots, f_N$ with coefficients in k. Let f denote the corresponding morphism $V \to \Omega^N$, then each fibre $f^{-1}(i)$ contains a principal orbit $U(i)$. The union of all $U(i)$ is called the principal subset of V.

The results of [3], can be summarised:

THEOREM (Igusa)

(i) *every orbit of $\tilde{G}$ in the principal subset, which is defined over k, has a k-rational point.*

(ii) *in every one of these orbits, the Witt condition holds. Namely, for each* $\xi \in V'_k$ *and every field extension* $L \supset K \supset k$, *all the points of* V'_K *which are in the orbit of* ξ *under* $\tilde{G}_L$ *are in fact in the orbit of* ξ *under* $\tilde{G}_K$.

(iii) *if* $\tilde{\rho}$ *does not contain the third fundamental representation of* Sp_6 *as an irreducible constituent, the stabiliser of every point of* V' *is simply connected, with its radical unipotent.*

Since connected unipotent groups are cohomologically trivial, it follows from (iii) above and Kneser [5] that the axioms (A1)–(A4) are satisfied for $(\tilde{G}, \tilde{\rho})$ and points $v \in V'_k$. The axioms are also satisfied for the point (0), which is never in V'_k, but has stabiliser $\tilde{G}$.

In fact, the axioms (A1), (A3) are trivially satisfied, since the stabilisers are connected. However, it is often bothersome to prove this and it is possible to check (A3) by exhibiting some torus T which contains $\operatorname{Ker}\pi$ and which leaves the point fixed, hence $\operatorname{Ker}\pi \subset T \subset \tilde{H}(v)_0$. Summarising, by the above, Theorem 1 and § 4, we have

THEOREM 2. *Let* $(\tilde{G}, \tilde{\rho})$ *be an absolutely admissible representation of* $\tilde{G}$ *which factors through an isogeny* $\tilde{G} \xrightarrow{\pi} G$. *Suppose that* $\tilde{\rho}$ *does not contain the third fundamental representation of* Sp_6. *Then the two tempered distributions* $I'_{\tilde{\rho}}, I'_{\rho}$ *are equal.*

Remark. It is expected that Igusa's theorem on the stabilisers will be true even outside V', so that in that case, we have the stronger result

$$I_{\rho}(\Phi) = I_{\tilde{\rho}}(\Phi)\ , \qquad \text{for all } \Phi \in S(V_A)\ .$$

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