LAPLACE TRANSFORMATIONS OF DISTRIBUTIONS

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1. Introduction. In a previous paper **(2)**, I discussed conditions in order that a holomorphic function $f(w)$ be a Laplace transform of a function $F(t)$ such that, for some a, $F(t)e^{-at} \in L^p(0, \infty)$. These conditions involved (a) the behaviour of integral transforms of the values of the function on a vertical line $w =$ const. and (b) conditions involving the order of magnitude of the function at infinity. The present article discusses the corresponding question for Laplace transforms of distributions.

Such Laplace transforms are discussed by L. Schwartz **(5)**. The notation and terminology in this article follow that of Schwartz's treatise **(4)** and of **(5)** except that whereas in **(5)** the expression "Laplace transform" is used for two-sided transforms, in the present paper it is used in what is equivalent to its classical sense, as the transform of a distribution with support in $(0, \infty)$. A discussion of conditions for Laplace transforms of functions based on a discussion of special integral transforms (the first Cesáro means) occurs in **(1)**; this imposes stronger conditions than here, for example requiring both stronger inequalities at infinity and boundedness of norms over $L^p(-\infty, \infty)$.

It turns out that the function-theoretic conditions of type (a) are sufficient by themselves to ensure that $f(w)$ be a Laplace transform of a distribution of the type $\mathfrak{S}'$ to which the Schwartz theory of Fourier transforms applies. Conditions of type (a) which involve boundedness of norms of integral transforms over $L^p(0, \infty)$ have the role of governing the order of the distribution and consequently, in the situation of **(2)**, are essential in order to ensure that these distributions are ordinary functions. It appears also that it is possible to determine the exact half-plane in which the function is a Laplace transform in the sense of the theory of distributions—the analogue of the half-plane of absolute convergence in the classical theory.

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2. Function-theoretic conditions. If T_t is a distribution with support in $(0, \infty)$, we shall say that a holomorphic function $f(w)$ is the Laplace transform of T for $\operatorname{Re} w \geqslant c$ if, for every $u \geqslant c$, $\sqrt{(2\pi)}\, T_t e^{-ut}$ is in $\mathfrak{S}'$, and has $f(u + iv)$, considered as a function of v, as its Fourier transform. (Notation regarding spaces of distributions and of test functions is as in Schwartz's

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treatise **(4)**.) It is easy to see (and is proved, with more precise results, in **(5)**) that if Te^{-ut} is in $\mathfrak{S}'$ for a given value of u, then it lies in $\mathfrak{S}'$ for any larger u.

THEOREM 1. *Let $f(w)$ be holomorphic for* $\operatorname{Re} w > c$ *and continuous for* $\operatorname{Re} w \geqslant c$. *In order that $f(w)$ be the Laplace transform of a distribution it is necessary and sufficient that the following conditions hold:*

(a) *for every $\delta > 0$, there is an A_δ such that $|f(w)| < A_\delta e^{\delta|w|^2}$ throughout the half-plane* $\operatorname{Re} w > c$;

(b) *for some m, $|f(c + iv)| = O(|v|^m)$ as $|v| \to \infty$ and $w^{-m}f(w)$ is holomorphic in* $\operatorname{Re} w > c$;

(c) *for every $\delta > 0$, $f(u) = O(e^{\delta u})$ as $u \to \infty$.*

The necessity of these conditions is easily proved; in fact it is easy to show that if $f(w)$ is a Laplace transform, it must satisfy a condition $|f(w)| < A|w|^m$.

In **(4)** it is shown that (b) must hold uniformly for any compact set of c in the domain in which $f(w)$ is a Laplace transform. The conditions (a) and (c) improve on the conditions laid down in **(2)** for the case of Laplace transforms of functions, and this answers a problem posed at the end of **(2)** for such transforms.

In order to prove sufficiency, we first observe that if

$$Q_{m+2}(u, t) = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} \frac{f(w)}{w^{m+2}} e^{wt}\, dw, \tag{1}$$

then $Q_{m+2}(c, t)$ is a continuous function of t which lies in $L(-\infty, \infty)$ and has $(2\pi)^{-\frac{1}{2}}f(c + iv)/(c + iv)^{m+2}$ as its Fourier transform, and that correspondingly $D_t^{m+2}\, Q_{m+2}(c, t)$ has $f(w)$ as Laplace transform for $\operatorname{Re} w = c$. It remains to prove the existence of $Q_{m+2}(u, t)$ and its independence of u for $u \geqslant c$.

For any positive ϵ let

$$g(w) = e^{-\epsilon w}f(w)(w - c + 1)^{-m} = e^{-\epsilon w}h(w).$$

Then $g(w)$ is bounded on the line $u = c$ and on the part of the real axis with $u \geqslant c$, and for any $\delta > 0$ it satisfies the inequality $g(w) < A_\delta \exp(\delta|w|^2)$ for all sufficiently large $|w|$ throughout the half-plane. It follows from a version of the Phragmén-Lindelöf theorem **(3**, III, 6, § 3, No. 325**)** that $g(w)$ is bounded throughout the half-plane. Now $|h(w)| < Ae^{\epsilon|w|}$ throughout the half-plane, and $h(w)$ is bounded on the line $\operatorname{Re} w = c$, so that by another form of the Phragmén–Lindelöf principle **(3**, III, 6, § 3, No. 326**)** it is bounded throughout the half-plane, i.e. $|f(w)| < |A(w - c + 1)^m|$ throughout the half-plane.

The integral in (1) is therefore absolutely convergent for any non-zero $u \geqslant c$, and by Cauchy's theorem it is independent of u for $u > 0$: we can therefore write $Q_{m+2}(t)$ for $Q_{m+2}(u, t)$. $Q_{m+2}(t)$ is bounded. Now $Q_{m+2}(u, t) = O(e^{ut})$ as $u \to \infty$ and so, since it is independent of u, it follows that the support

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of $Q_{m+2}(t)$ is in $(0, \infty)$. Now if ϕ is any function of $\mathfrak{D}$ we have for $u > 0$

$$\frac{1}{\sqrt{(2\pi)}} \int f(u + iv)\bar{\tilde{\phi}}(v)\, dv = \frac{1}{\sqrt{(2\pi)}} \int (u + iv)^{m+2} \overline{\tilde{\phi}(v)}\, dv \int Q_{m+2}(t) e^{-(u+iv)t}\, dt$$

$$= \frac{(-1)^{m+2}}{\sqrt{(2\pi)}} \int Q_{m+2}(t)\, dt\, D_t^{m+2} \int \overline{\tilde{\phi}(v)} e^{-ut-ivt}\, dt$$

$$= \frac{1}{\sqrt{(2\pi)}} \int D^{m+2} Q_{m+2}(t)\, e^{-ut} \bar{\phi}(t)\, dt.$$

$\tilde{\phi}$ denotes the Fourier transform of ϕ. We conclude that $f(u + iv)$ is $\sqrt{(2\pi)}$ times the Fourier transform of $e^{-ut}Q(t)$ where $Q(t) = D^{m+2}Q_{m+2}(t)$, that is, $f(w)$ is the Laplace transform of $Q(t)$.

3. The domain of absolute convergence. Let $T = T_t$ be a distribution such that for some real a the distribution $T_t e^{-at}$ is either in $\mathfrak{S}'$ or in $\mathfrak{D}'_{L^p}$. Necessary and sufficient conditions that this be so are, for $\mathfrak{S}'$, that there be a function $q(t)$ which is continuous and $O(t^k)$ at infinity and such that $T_t e^{-at}$ is the nth derivative of $q(t)$, or, in the case of $\mathfrak{D}'_{L^p}$, that $T_t e^{-at}$ be the sum of derivatives of functions of L^p (**4**, II, p. 57, Chap. VI, théorème XXV). If $T_t e^{-at} = D^n q(t)$, then for any $\phi \in \mathfrak{S}$ and any $c \geqslant a$.

$$\langle Te^{-ct}, \phi \rangle = \langle Te^{-at}, e^{-(c-a)t}\phi(t) \rangle$$

$$= (-1)^n \int q(t)\, D^n[e^{-(c-a)t}\phi(t)]\, dt$$

$$= (-1)^n \sum \binom{n}{s} \int q(t)[-(c-a)^s] e^{-(c-a)t} D^{n-s}\phi(t)\, dt$$

$$= \sum_s \binom{n}{s} (c-a)^s \int \phi(t) D^{n-s}[e^{-(c-a)t} q(t)]\, dt$$

$$= \sum_s \int \phi(t) D^n h_s(t)\, dt$$

where, for a constant C_s,

$$h_s(t) = C_s \int_0^t (t-u)^s e^{-(c-a)u} q(u)\, du. \tag{2}$$

If $Te^{-at} \in \mathfrak{S}'$ and has its support in $(0, \infty)$, then $q(t)$ is a polynomial, of degree $n - 1$ at most, in $(-\infty, 0)$ and by subtracting this polynomial from $q(t)$ we see that we can suppose $q(t)$ zero for negative t. In this case the functions $h_s(t)$ are all continuous and of polynomial order at infinity whenever $c \geqslant a$, and so we have

$$Te^{-ct} = D^n Q(t) \tag{3}$$

with $Q(t)$ continuous and of polynomial order at infinity if $c \geqslant a$. If $c > a$, $e^{-(c-a)t}q(t) \in L^p$ for any $p \geqslant 1$, and consequently $T_t e^{-ct}$ is a sum of a finite number of derivatives of functions of L^p and so is in $\mathfrak{D}'_{L^p}$.

On the other hand, if we take $c = 0$ in the formulae above, then $h_s(t)$ is a constant multiple of $t^r * e^{at}q(t)$, which is bounded by an expression

$$A \int_0^t (t-u)^r e^{au} u^k \, du < A e^{at} \int_0^t (t-u)^r u^k \, du$$
$$< A' t^{k+r+1} e^{at}$$

so that

$$T = D^n H(t) \tag{4}$$

where $H(t) = e^{at} r(t)$ with $r(t)$ of polynomial order at infinity and support in $(0, \infty)$.

Now suppose that, for some a, $T_t e^{-at}$ is in $\mathfrak{D}'_L p$ for a $p \geqslant 1$, and so is a sum of terms $D^r q_r(t)$, $q_r \in L^p$, and that the support of T is in $(0, \infty)$. $T = T_1 + T_2$, where T_1 is the distribution derived by replacing each q_r by the function equal to q_r over $(0, \infty)$ and zero elsewhere. If $Q \in \mathfrak{D}$ and the support of ϕ is in $(0, \infty)$, $\langle T, \phi \rangle = \langle T_1, \phi \rangle$ while if the support of ϕ is disjoint to $(0, \infty)$, $\langle T, \phi \rangle = \langle T_1, \phi \rangle = 0$. It follows that the support of T_2 is concentrated at zero; that is, T can be written

$$Te^{-at} = D^r q_r(t) + T_2 \tag{5}$$

with q_r functions of L^p with supports in $(0, \infty)$ and with T_2 a distribution concentrated at 0.

If $c > a$, and m is any number not less than the largest r in the sum in (4), we can show that $D^r q_r$ is a sum of terms of the form

$$D^m h_{m,r,s} (m \geqslant r \geqslant s), \, h_{m,r,s}(t) = C_{m,r,s} \int_0^t (t-u)^{m-r+s} q_r(u) e^{-(c-a)u} \, du.$$

$h_{m,r,s}(t)$ is of polynomial order at infinity, and it is therefore clear that if $Te^{-at} \in \mathfrak{D}'_L p$, then $Te^{-ct} \in \mathfrak{S}'$ for all $c > a$ and so is in $\mathfrak{D}'_L q$ for any $c > a$ and any $q \geqslant 1$.

We now consider the distribution $T_r = e^{at} D^r q_r(t)$. For any $\phi \in \mathfrak{D}$, $m \geqslant r$,

$$\begin{aligned} \langle T_r, \phi \rangle &= (-1)^r \int q_r(t) D^r (e^{at} \phi(t)) \, dt \\ &= \sum_1^r C_{s,r} \int q_r(t) e^{at} D^s \phi(t) \, dt \\ &= (-1)^m \int e^{at} p_r(t) D^m \phi(t) \, dt, \end{aligned}$$

provided that

$$e^{at} p_r(t) = \sum_s K_{r,s,m} \int_0^t (t-u)^{m-s} q_r(u) e^{au} \, du$$

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with suitable constants K. Since $t^{m-s}e^{-at} \in L(0, \infty)$, the integrands in the above sum are in $L^p(0, \infty)$; and we conclude that we can write

$$e^{at}D^r q_r(\mathrm{t}) = D^m e^{at} h_r(t) \tag{6}$$

for any $m \geqslant r$, with $h_r \in L^p(-\infty, \infty)$ and with support in $(0, \infty)$.

In order to investigate conditions under which a holomorphic function $f(w)$ is a transform of a distribution of these types, we consider summation formulae for the inversion formula for Fourier transforms involving f and kernels $k(v, \lambda)$ where

$$k(v, \lambda) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} K(t, \lambda) e^{ivt}\, dt.$$

We suppose throughout all that follows that $k(v, \lambda)$ and $K(t, \lambda)$ are in $L^p(-\infty, \infty)$ for all $p \geqslant 1$ and all λ, and that at least some of the following conditions hold:

A(0). $\displaystyle\int_{-\infty}^{\infty} |K(t, \lambda)|\, dt$ is bounded for all λ,

A(b). $\displaystyle\int_{-\infty}^{\infty} e^{-bt}|K(t, \lambda)|\, dt$ is bounded for all λ (b real).

B. $k(v, \lambda) \to 1$ as $\lambda \to \infty$, boundedly over every compact (or weakly in L^p over any finite interval).

C. For every $\delta > 0$,

$$\int_{|t|>\delta} |K(t, \lambda)|\, dt \to 0 \qquad \text{as } \lambda \to \infty.$$

Given a function $f(c + iv)$, we write

$$F(c, \lambda, n; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(c + iv) \frac{k(v, \lambda)}{(c + iv)^n} e^{(c+iv)x}\, dv$$

for what is effectively a summation formula for the inverse Laplace transform of $f(c + iv)/(c + iv)^p$.

If $\phi(t)$ is any function of $\mathfrak{S}$ and $f(c + iv)$ is the Fourier transform of $T_t e^{-ct}$,

$$\begin{aligned} \int f(c + iv)\overline{\tilde{\phi}(v)}dv &= \sqrt{(2\pi)}\, \langle Te^{-ct}, \overline{\phi(t)}\rangle \\ &= \sqrt{(2\pi)}\, \langle T, e^{-ct}\overline{\phi(t)}\rangle \\ &= (-1)^n\sqrt{(2\pi)} \int D^n[H(t)]e^{-ct}\overline{\phi(t)}\, dt \\ &= \sqrt{(2\pi)} \int H(t)D^n[e^{-ct}\overline{\phi(t)}]\, dt. \end{aligned}$$

In particular, if $k(v, \lambda)$ is in $\mathfrak{S}$, then this gives

$$F(c, \lambda, n; x) = \frac{e^x}{2\pi} \int H(t) e^{-ct} K(t - x, \lambda)\, dt. \tag{7}$$

The functions of $\mathfrak{S}$ are weakly dense in $L^p(-\infty, \infty)$, for any $p \geqslant 1$, in the weak topology imposed by the set of functionals corresponding to functions which are $O(e^{-ct})$ as $t \to \infty$, and so this formula extends to all functions k of the class considered provided that $c > a$.

It is easy to show that, if A(0) holds, then the map

$$f \to g \qquad \text{with } g(x) = \int_{-\infty}^{\infty} K(t - x, \lambda) f(t)\, dt$$

is a bounded set of maps of $L^p(-\infty, \infty)$ to $L^p(-\infty, \infty)$ and consequently also of $L^p(0, \infty)$ to $L^p(0, \infty)$ for any $p \geqslant 1$.

THEOREM 2. *Let* A(0) *hold. If* $f(w)$ *is the Laplace transform of* T_t, *where* $T_t e^{-at} \in \mathfrak{S}'$, *and is of order* n *in* $\mathfrak{S}'$ *for some* a, *then, for all* $c > a$, *the norms of the functions* $e^{-cx} F(c, \lambda, n, x)$ *in the spaces* $L^p(0, \infty)$ (*and indeed in* $L^p(-\infty, \infty)$) *are bounded for any* $p \geqslant 1$. *If* A(b) *holds for all real* b, *then the norms of* $e^{-bx} F(c, \lambda, n, x)$ *are bounded for all* $c > a$ *and* $b > a$.

The last part follows on noticing that

$$e^{-bx} F(c, \lambda, n, x) = \frac{1}{2\pi} \int H(t)\, e^{-bt} e^{(c-b)(x-t)} K(t - x, \lambda)\, dt$$

and that if $A(c - b)$ holds, then the maps in (4) with $K(x)$ replaced by $e^{-(c-b)x} K(x)$ are a bounded set of maps from L^p to itself.

A converse theorem is as follows.

THEOREM 3. *Let* A(b) *hold for all real* b, *and let* B *hold. For* $c > 0$, *let* $f(w)$ *satisfy the conditions of Theorem* 1, *for some* m, *and let* $e^{-bx} F(c, \lambda, n; x)$ *form a bounded set of elements of* $L^p(0, \infty)$, *for some real* $b \leqslant c$. *Then* $f(w)$ *is the Laplace transform of a distribution* T_t *such that* $T_t e^{-ut}$ *is in* $\mathfrak{S}' \cap \mathfrak{D}_L p'$ *for all* $u \geqslant b$.

Note that it follows from Theorems 1 and 2 that if the conditions of Theorem 1 hold, then $e^{-cx} F(c, \lambda, m + 2, x)$ form a bounded set of functions in L^p; and on combining the present theorem with these theorems, that if the hypotheses of this theorem hold for any n and m they must hold with $m \leqslant n + 2$. The interest of the theorem, supplementing Theorem 1, is that if k is the Weierstrass kernel or any other kernel for which A(b) holds for all b, then the infimum of the values of b for which $e^{-bx} F(c, \lambda, n; x)$ is bounded in $L^p(0, \infty)$ gives the left-hand abscissa of the half-plane of absolute convergence of the Laplace transform formula for $f(w)$.

Under the hypotheses of Theorem 3 it follows, since any bounded set in $L^p(0, \infty)$ is weakly compact, that there is a $g \in L^p$ which is a weak limiting point of $e^{-bx}F(c, \lambda, n, x)$ and such that $\langle t^m e^{-wt}, e^{-bx}F(c, \lambda, n, t)\rangle$ has $\langle t^m e^{-wt}, g(t)\rangle$ as limit point for any w with $\operatorname{Re} w > 0$. By putting $H(t) = e^{bt}g(t)$ and $h(w)$ for the Laplace transform of $H(t)$, we get that if $\operatorname{Re} w > c$, then

$$h^{(m)}(w) = (-1)^m \int_0^\infty t^m e^{-wt} H(t)\, dt$$

is a limiting point of

$$(-1)^m \frac{m!}{2\pi} \int \frac{k(v, \lambda) f(c + iv)}{[w - (c + iv)]^{m+1} (c + iv)^n}\, dv$$

and so, because of the conditions on k, is equal to

$$(-1)^m \frac{m!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(z)\, dz}{(w - z)^{m+1} z^n} = \left(\frac{d}{dw}\right)^m \frac{f(w)}{w^n}.$$

It follows that

$$h(w) = \frac{f(w)}{w^m} + P_{m-1}(w),$$

where $P_{m-1}(w)$ is a polynomial in w of order $m - 1$ at most, and so

$$f(w) = w^n h(w) + w^n P_{m-1}(w),$$

if $u \geqslant c$, which is the Laplace transform of $D^n H(t)$ plus a polynomial. The polynomial is the Laplace transform of a distribution of finite order whose support is concentrated at $t = 0$. Because $f(w)$, $h(w)$, and $P_{m-1}(w)$ are holomorphic for $u \geqslant b$, $f(w)$ has the form found here for $u \geqslant b$, and this concludes the proof.

If $c < 0$, the theorem remains valid with the additional assumption that $f(w)$ has an n-fold zero at $w = 0$.

4. Conditions relating only to the values of f on a single line. We shall show finally that all complex function-theoretic conditions on f can be eliminated from the hypotheses, and that we can obtain our results by considering only integral transforms of the values of f on a single line, provided that we work with norms over $(-\infty, \infty)$.

THEOREM 4. *Let $a > 0$ and let* A(0) *hold. Then in order that $f(w)$ be, on $\operatorname{Re} w = a$, the Laplace transform of a distribution T which is such that $Te^{-at} \in \mathfrak{D}'_{L^p}$ $(p \geqslant 1)$, it is necessary that* (a) *there be an m such that the norms of $e^{-cx}F(c, \lambda, n, x)$ are bounded in λ for any $n \geqslant m$.*

If, in addition, C *holds, then it is also necessary that* (b) *the norms of $e^{-ax}F(a, \lambda, n, x)$ in $L^p(-\infty, 0)$ tend to 0 as $\lambda \to \infty$.*

If, in addition, B *holds, then those conditions are sufficient.*

Necessity. If $Te^{-at} \in \mathfrak{D}'_{L^p}$, then by (4) and (5) there is an r such that

$$T = D^r h(t) + T_2$$

where T_2 is a distribution concentrated at 0 and $e^{-at}h(t) = g(t) \in L^p(0, \infty)$ with $h(t)$ zero for $t < 0$. If f is the Laplace transform of T, we have

$$f(w) = w^m H(w) + P(w)$$

where $H(w)$ is the Laplace transform of $h(t)$ and $P(w)$ is a polynomial of degree d, say. Let $m = \max(r, d + 1)$. If $n \geqslant m$,

$$w^{-n}f(w) = w^{-n}P(w) + w^{-n}H(w).$$

The first term on the left is the sum of multiples of w^{-s}, $s \geqslant 1$; and since w^{-s} is, throughout $\operatorname{Re} w > 0$, the Laplace transform of t^{s-1}/s , its contribution to the norms of $F(a, \lambda, n, x)e^{-ax}$ satisfies (a) and (b), because of Theorems 1 and 2 of **(2)**. $w^{-n}H(w)$ is the Laplace transform of the convolution

$$n! \int_0^t (t-u)^{n-1} h(u)\, du = e^{at} \int_0^t (t-u)^{n-1} e^{-a(t-u)} g(u)\, du.$$

The integral here is the convolution of a function of L with one of L^p, so is in L^p. Because of the theorems cited, the contribution of $H(w)$ therefore also obeys conditions (a) and, under hypothesis C, (b).

On the other hand, suppose that conditions (a) and (b) hold and that k obeys B. Then, by **(1**, Theorem 5**)**, $w^{-n}f(w)$ is the Laplace transform of a function $H(t)$ which is such that $H(t)e^{-at} \in L^p(0, \infty)$, and $f(w)$ is then the Laplace transform of $D^nH(t)$, and clearly $e^{-at}D^nH(t) \in \mathfrak{D}'_{L^p}$.

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