SEMIPRIME SEMIGROUP RINGS AND A PROBLEM OF J. WEISSGLASS

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If R is a ring and S is a semigroup, the corresponding semigroup ring is denoted by $R[S]$. A ring is semiprime if it has no nonzero nilpotent ideals. A semigroup S is a semilattice P of semigroups S_α if there exists a homomorphism φ of S onto the semilattice P such that $S_\alpha = \alpha\varphi^{-1}$ for each $\alpha \in P$.

In [**4**] J. Weissglass proves the following result.

THEOREM. *Suppose that R is a commutative ring with identity element and that S is a commutative semigroup such that a power of each element lies in a subgroup. Then $R[S]$ is semiprime if and only if S is a semilattice P of groups S_α, and $R[S_\alpha]$ is semiprime for each $\alpha \in P$.*

Then Weissglass asks [**4**, Question 9, page 477] if the commutativity of R can be removed from the hypothesis of his theorem. The purpose of this note is to answer his question affirmatively.

Given a ring R and a semigroup S, the support of $x = \sum_{s\in S} r_s s \in R[S]$, denoted by supp x, is defined to be the set $\{s \in S \mid r_s \neq 0\}$. For a set X, $|X|$ denotes the cardinality of X.

LEMMA 1. *Let R be a ring with identity element, and let S be a commutative semigroup. Assume that the group G is an ideal of S and that every element of S has a power in G. Let A be a nonzero ideal of $R[S]$ such that $A \cap R[G] = 0$. Then there exists a nonzero element $y = \sum_{i=1}^{n} r_i s_i \in A$ $(r_i \in R, s_i \in S)$ such that $yrs_j = 0$ for each $r \in R$ and each $j \leq n$.*

Proof. Let $m = \min\{j \mid 0 \neq x \in A$ and $|(\text{supp } x) \cap (S - G)| = j\}$. Since $A \cap R[G] = 0$, then $m \geq 1$. Let $y = \sum_{i=1}^{n} r_i s_i \in A - \{0\}$ be chosen such that

$$\{s_1, s_2, \ldots, s_m\} = (\text{supp } y) \cap (S - G)$$

and

$$\{s_{m+1}, \ldots, s_n\} \subseteq G \quad \text{if} \quad m < n.$$

Let k be a positive integer such that $k \leq m$, and consider the condition $ys_j = 0$ for $j < k$. This condition is vacuously satisfied when $k = 1$; so assume that the condition holds for some $k \geq 1$. Since a power, say s_k^t, of s_k is in the ideal G of S, then $ys_k^t \in R[G]$. But $ys_k^t \in A$, since $y \in A$. Hence $ys_k^t = 0$. Thus there is a least nonnegative integer h such that $ys_k^{h+1} = 0$. (If $h = 0$, let $s_k^0 = 1 \in R$ for notational convenience.) Then by the choice of m and h, we have that $s_1 s_k^h, s_2 s_k^h, \ldots, s_m s_k^h$ are distinct elements of $S - G$, and $s_i s_k^h \in G$ for

$i > m$. Since S is commutative,

$$\left(\sum_{i=1}^{n} r_i s_i s_k^h\right) s_j s_k^h = \left(\sum_{i=1}^{n} r_i s_i\right) s_j s_k^{2h} = y s_j s_k^{2h} = 0$$

for $j \leq k$. Thus, if we replace s_i by $s_i s_k^h$ in our original expression for y, we may assume that $y = \sum_{i=1}^{n} r_i s_i \in A - \{0\}$, $s_i \in S - G$ for $i \leq m$, $s_i \in G$ for $i > m$, and $ys_j = 0$ for $j \leq k$. By induction, we may assume that $ys_j = 0$ for $j \leq m$. Since G is an ideal of S, we also have $ys_j \in A \cap R[G] = 0$ for each $j > m$.

Let $j \in \{1, \ldots, n\}$. Write $T = \{s_i s_j \mid i = 1, \ldots, n\}$ and, for each $t \in T$, let $I_t = \{i \mid s_i s_j = t\}$. Since $ys_j = 0$, we have that, for all $t \in T$, $\sum_{i \in I_t} r_i = 0$. Hence, for all $r \in R$,

$$0 = \sum_{t \in T}\left(\sum_{i \in I_t} r_i\right) rt = \sum_{i=1}^{n} r_i r(s_i s_j) = \left(\sum_{i=1}^{n} r_i s_i\right) rs_j = yrs_j.$$

Let P be a semilattice whose natural order is indicated by $\leq$, and let S be a semilattice P of semigroups S_α. Then there exist ideal extensions D_α of S_α $(\alpha \in P)$ and homomorphisms $\varphi_{\alpha,\beta} : S_\alpha \to D_\beta$ $(\beta \leq \alpha)$ satisfying the following conditions:

(a) $\varphi_{\alpha,\alpha}$ is the identity map on S_α;
(b) $(S_\alpha \varphi_{\alpha,\alpha\beta})(S_\beta \varphi_{\beta,\alpha\beta}) \subseteq S_{\alpha\beta}$;
(c) if $\alpha\beta > \gamma$, then for all $a \in S_\alpha$ and $b \in S_\beta$, $[(a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta})]\varphi_{\alpha\beta,\gamma} = (a\varphi_{\alpha,\gamma})(b\varphi_{\beta,\gamma})$;
(d) S is the disjoint union of the S_α $(\alpha \in P)$;
(e) if $a \in S_\alpha$ and $b \in S_\beta$, then multiplication in S is determined by

$$ab = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) \in S_{\alpha\beta}.$$

For more details, see Section III.7 of [**2**]. We note that each $\varphi_{\alpha,\beta}$ has a natural extension to a ring homomorphism from $R[S_\alpha]$ to $R[D_\beta]$:

$$\sum_{s \in S_\alpha} r_s s \to \sum_{s \in S_\alpha} r_s (s\varphi_{\alpha,\beta}).$$

We also denote this extension by $\varphi_{\alpha,\beta}$ for convenience.

Lemma 2. *Let R be a ring with identity element, and let S be a semilattice P of commutative semigroups S_α. Let $\sigma \in P$ and $y = \sum_{i=1}^{n} r_i s_i \in R[S_\sigma]$ be such that $yrs_j = 0$ for each $r \in R$ and each $j \leq n$. Then the principal ideal B of $R[S]$ generated by y satisfies $B^2 = 0$.*

Proof. Every element of B^2 is a sum of terms, each of which contains at least one of the following factors: y^2 or $y \,.\, rs \,.\, y$, where $r \in R$ and $s \in S_\alpha$ for some $\alpha \in P$.

But $y^2 = \sum_{i=1}^{n} yr_i s_i = 0$ by our choice of y. Moreover, if $r \in R$ and $s \in S_\alpha$ then, since $S_{\alpha\sigma}$

is commutative and $\varphi_{\sigma,\alpha\sigma}$ is a homomorphism, we have

$$\begin{aligned}
y \,.\, rs \,.\, y &= \left(\sum_{i=1}^{n} r_i s_i\right) . \, rs \,. \left(\sum_{i=1}^{n} r_i s_i\right) \\
&= \sum_{i,j} r_i r r_j (s_i \varphi_{\sigma,\alpha\sigma})(s\varphi_{\alpha,\alpha\sigma})(s_j \varphi_{\sigma,\alpha\sigma}) \\
&= \sum_{i,j} r_i r r_j ((s_i s_j)\varphi_{\sigma,\alpha\sigma})(s\varphi_{\alpha,\alpha\sigma}) \\
&= \left[\sum_{j=1}^{n} \left(\left(\sum_{i=1}^{n} r_i s_i\right) r r_j s_j\right) \varphi_{\sigma,\alpha\sigma}\right](s\varphi_{\alpha,\alpha\sigma}) \\
&= \left[\sum_{j=1}^{n} (y(rr_j)s_j)\varphi_{\sigma,\alpha\sigma})\right](s\varphi_{\alpha,\alpha\sigma}) = 0
\end{aligned}$$

by our choice of y.

It follows that $B^2 = 0$ as desired.

LEMMA 3. *Let R be a ring with identity element, and let S be a semilattice P of commutative semigroups S_α. Let $\sigma \in P$, and assume that the group G is an ideal of S_σ. Let A be a nilpotent ideal of $R[S_\sigma]$ such that $A^2 = 0$. Then the principal ideal C of $R[S]$ generated by any element of $A \cap R[G]$ satisfies $C^2 = 0$.*

Proof. Let $x = \sum_{i=1}^{n} r_i s_i \in A \cap R[G]$ with $\{s_1, s_2, \ldots, s_n\} \subseteq G$, and let x generate C. Since $x^2 = 0$, then every element of C^2 is a sum of terms, each of which contains a factor of the form $x \,.\, rs \,.\, x$, where $r \in R$ and $s \in S_\alpha$ for some $\alpha \in P$. Let e be the identity element of G, let $r \in R$, and let $s \in S_\alpha$ for some $\alpha \in P$. Since $S_{\alpha\sigma}$ is commutative and $\varphi_{\sigma,\alpha\sigma}$ is a homomorphism, we have

$$\begin{aligned}
x \,.\, rs \,.\, x &= \left(\sum_{i=1}^{n} r_i s_i\right) . \, rs \,. \left(\sum_{i=1}^{n} r_i s_i\right) \\
&= \sum_{i,j} r_i r r_j (s_i \varphi_{\sigma,\alpha\sigma})(s\varphi_{\alpha,\alpha\sigma})(s_j \varphi_{\sigma,\alpha\sigma}) \\
&= \left[\left(\sum_{i,j} r_i r r_j s_i s_j\right) \varphi_{\sigma,\alpha\sigma}\right](s\varphi_{\alpha,\alpha\sigma}) \\
&= \left[\left(\sum_{i=1}^{n} r_i s_i\right)(re)\left(\sum_{j=1}^{n} r_j s_j\right)\right] \varphi_{\sigma,\alpha\sigma} \,.\, (s\varphi_{\alpha,\alpha\sigma}) \\
&\subseteq (A^2)\varphi_{\sigma,\alpha\sigma} \,.\, (s\varphi_{\alpha,\alpha\sigma}) \\
&= 0.
\end{aligned}$$

It follows that $C^2 = 0$ as desired.

We are now ready for our main result.

THEOREM. *Let R be a ring with identity element, and let S be a commutative semigroup such that a power of each element of S lies in a subgroup. Then $R[S]$ is semiprime if and only if S is a semilattice P of groups S_α, and $R[S_\alpha]$ is semiprime for each $\alpha \in P$.*

Proof. By [**1**, §4.3, Exercise 5] the hypothesis on S forces S to be a semilattice P of semigroups S_α, where each S_α contains a group ideal G_α such that S_α/G_α is a nil semigroup.

Let $R[S]$ be semiprime. Suppose that there exists $\sigma \in P$ such that $R[S_\sigma]$ is not semiprime. Then $R[S_\sigma]$ contains a nonzero nilpotent ideal A such that $A^2=0$. If $A \cap R[G_\sigma]=0$, then $R[S]$ has a nonzero nilpotent ideal B by Lemmas 1 and 2; if $A \cap R[G_\sigma] \neq 0$, then $R[S]$ has a nonzero nilpotent ideal C by Lemma 3. Consequently, each $R[S_\sigma]$ must be semiprime to avoid a contradiction. It now follows from [**4**, Lemma 4] that each S_α is a group.

The converse follows from [**3**, Theorem 1] or [**4**, Corollary 1].

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