BERNSTEIN'S INEQUALITY FOR LOCALLY COMPACT ABELIAN GROUPS

Dedicated to the memory of Hanna Neumann

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Introduction

This paper is concerned with versions of Bernstein's inequality for Hausdorff locally compact Abelian groups. The ideas used are suggested by Exercise 12, p. 17 of Katznelson's book [4].

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1. Definitions and some general results

Let G be a Hausdorff locally compact Abelian group, Γ its character group, both written additively. The Haar measures on G, Γ are denoted by λ, θ respectively, and are chosen so that Plancherel's theorem holds. We will denote by $C(G)$ (respectively $C_0(G)$, $C_{00}(G)$) the space of bounded continuous functions (respectively continuous functions which vanish at infinity, continuous functions with compact support) on G.

Let $L(G)$ be a translation-invariant linear subspace of $L^p(G)$, $p \in [1, \infty]$, with the following properties:

(a) $L^1 * L(G) \subset L(G)$;

(b) there is a norm $\|\cdot\|_L$ on L such that

$$\|k*f\|_L \leqq \|k\|_1 \|f\|_L$$

for all $k \in L^1(G)$, $f \in L(G)$.

Whenever $g \in L^\infty(G)$, $\Sigma(g)$ denotes the spectrum of g (see [3], (40.21)). It is easily shown that

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$$\Sigma(g) = \bigcup_{\phi \in C_{00}(G)} \Sigma(g * \phi). \tag{1.1}$$

Since for $f \in L(G)$, $\phi \in C_{00}(G)$, it follows that $f * \phi \in L^\infty(G)$, we are guided by (1.1) to extend the definition of spectrum to arbitrary $f \in L(G)$: we retain the same notation, and put

$$\Sigma(f) = \bigcup_{\phi \in C_{00}(G)} \Sigma(f * \phi). \tag{1.2}$$

It follows from (1.2) that

$$\Sigma(\tau_a f) = \Sigma(f) \tag{1.3}$$

for all $f \in L(G)$, $a \in G$, where τ_a is the translation operator defined by

$$\tau_a f(x) = f(x-a).$$

If the Fourier transform of a function $f \in L^p(G)$ is defined as in [2], 1.1, then it is straightforward to show that

$$\Sigma(f) = [\hat{f}], \tag{1.4}$$

where $[\hat{f}]$ denotes the support of the quasimeasure $\hat{f}$. Note also that when $p = \infty$, $\hat{f}$ is actually a pseudomeasure.

Let K be any subset of Γ. We shall write

$$L_K(G) = \{f \in L(G) : \Sigma(f) \subset K\},$$

$$\beta_K^L(a) = \sup \{ \| \tau_a f - f \|_L : f \in L_K(G), \|f\|_L \leqq 1 \}$$

and

$$\omega_K(a) = \sup_{\chi \in K} |\chi(a) - 1|,$$

where ω_K is defined to be zero of K is empty. It follows easily that

$$\omega_{-K} = \omega_K, \omega_{K_1+K_2} \leqq \omega_{K_1} + \omega_{K_2} \text{ and } \omega_{K_1 \cup K_2} \leqq \max\{\omega_{K_1}, \omega_{K_2}\},$$

where $K, K_1, K_2 \subset \Gamma$. Furthermore when K is relatively compact, 1.2.6. of [5] gives immediately that

$$\lim_{a \to 0} \omega_K(a) = 0.$$

LEMMA 1.1. *Let K be a compact subset of Γ and choose $k, l \in L^1(G)$ such that $\hat{k} = 1, \hat{l} = 0$ on a neighbourhood of K. Then*

$$\beta_K^L(a) \leqq \| \tau_a k - k - l \|_1.$$

If K is a set of spectral synthesis (S-set) then we can replace "on a neighbourhood of K" by "on K".

PROOF. We show initially that if k, l satisfy the hypotheses of the lemma then

$$l * f = 0 \text{ and } k * f = f \tag{1.5}$$

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for every $f \in L_K(G)$. For this it suffices to show that (1.5) holds pointwise l.a.e. (since a function in $L^p(G)$, with $p \neq \infty$, which vanishes l.a.e., vanishes a.e.).

Let $\phi \in C_{00}(G)$ and suppose $l \in L^1(G)$ is such that $\hat{l} = 0$ on a neighbourhood of K (or if K is an S-set, $\hat{l} = 0$ on K). From (1.2) and the assumption that $\Sigma(f) \subset K$, it follows ([3], (40.7)) that

$$l * (\phi * f) = 0,$$

or, equivalently,

$$\phi * (l * f) = 0.$$

Since $\phi \in C_{00}(G)$ was chosen arbitrarily, $l * f = 0$ *l.a.e.*. Furthermore, if $k \in L^1(G)$ is such that $\hat{k} = 1$ on a neighbourhood of K (or if K is an S-set, $\hat{k} = 1$ on K) and $\phi \in C_{00}(G)$ then $(k * \phi - \phi)^\wedge$ vanishes on a neighbourhood of K (or if K is an S-set, $(k * \phi - \phi)^\wedge$ vanishes on K) and by what has already been established,

$$\phi * (k * f - f) = (k * \phi - \phi) * f = 0 \ \textit{l.a.e.},$$

whence it follows that $k * f = f$ *l.a.e.*.

From (1.3) and (1.5),

$$\begin{aligned} \tau_a f - f &= (\tau_a f - f) * k - f * l \\ &= f * (\tau_a k - k - l), \end{aligned}$$

and by (b),

$$\| \tau_a f - f \|_L \leqq \| f \|_L \, \| \tau_a k - k - l \|_1 ,$$

from which the result follows.

LEMMA 1.2. *Let K be a compact subset of Γ and let V be a relatively compact non-void open subset of Γ. Let g, h be the elements of $L^2(G)$ having Fourier transforms ξ_V, ξ_{K+V-V} respectively (where ξ_E denotes the characteristic function of the set E) and put $k = \theta(V)^{-1} gh$. Then $\hat{k} = 1$ on $K + V$, $\hat{k}$ vanishes outside $K + V + V - V$, and*

$$\| \tau_a k - k \|_1 \leqq \theta(V)^{-1} \, \| g \|_2 \, \| h \|_2 \, (\omega_{K+V-V}(a) + \omega_V(a)). \tag{1.6}$$

If K is an S-set, we can replace $K + V$ by K in the statement of the lemma.

PROOF. The first part of Lemma 1.2 is established in Theorem 2.6.1 of [5].

To prove (1.6), consider

$$\begin{aligned} \| \tau_a k - k \|_1 &= \theta(V)^{-1} \, \| (\tau_a h - h) g + (\tau_a g - g) \tau_a h \|_1 \\ &\leqq \theta(V)^{-1} (\| g \|_2 \, \| \tau_a h - h \|_2 + \| h \|_2 \, \| \tau_a g - g \|_2). \end{aligned}$$

By Plancherel's theorem,

$$\begin{aligned}\|\tau_a g - g\|_2^2 &= \int_\Gamma |(\tau_a g - g)\hat{}(\gamma)|^2\, d\theta(\gamma)\\ &= \int_V |\bar{\gamma}(a) - 1|^2\, |\hat{g}(\gamma)|^2\, d\theta(\gamma)\\ &\leqq \omega_V(a)^2 \,\|g\|_2^2,\end{aligned}$$

that is,

$$\|\tau_a g - g\|_2 \leqq \omega_V(a)\,\|g\|_2 .$$

Similarly,

$$\|\tau_a h - h\|_2 \leqq \omega_{K+V-V}(a)\,\|h\|_2 ,$$

giving the desired result.

From Lemmas 1.1, 1.2, we obtain:

THEOREM 1.3. *Suppose the hypotheses of Lemma* 1.2 *are satisfied. Then*

$$\beta_K^L(a) \leqq \left(\frac{\theta(K+V-V)}{\theta(V)}\right)^{\frac{1}{2}} (\omega_V(a) + \omega_{K+V-V}(a)) . \tag{1.7}$$

If, in addition, K is an S-set then

$$\beta_K^L(a) \leqq \left(\frac{\theta(K-V)}{\theta(V)}\right)^{\frac{1}{2}} (\omega_V(a) + \omega_{K-V}(a)).$$

COROLLARY 1.4. *Suppose the hypotheses of Lemma* 1.2 *are satisfied, and* $0 \in V$. *Then*

$$\beta_K^L(a) \leqq 3\left(\frac{\theta(K+V-V)}{\theta(V)}\right)^{\frac{1}{2}} \omega_{K+V-V}(a) .$$

If, in addition, K is an S-set then

$$\beta_K^L(a) \leqq 3\left(\frac{\theta(K-V)}{\theta(V)}\right)^{\frac{1}{2}} \omega_{K-V}(a) .$$

PROOF. Let $\chi \in K$. Then $0 \in -\chi + K$ and, since $0 \in V$,

$$\begin{aligned}\omega_V(a) &\leqq \omega_{-\chi+K+V-V}(a)\\ &\leqq \omega_{-\chi}(a) + \omega_{K+V-V}(a)\\ &\leqq 2\,\omega_{K+V-V}(a) .\end{aligned}$$

Hence, from (1.7),

$$\beta_K^L(a) \leqq 3\left(\frac{\theta(K+V-V)}{\theta(V)}\right)^{\frac{1}{2}} \omega_{K+V-V}(a) .$$

If K is an S-set, just replace $K+V$ by K.

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For certain $K \subset \Gamma$, we can obtain estimates of the form

$$\beta_K^L(a) = O(\omega_K(a)) .$$

THEOREM 1.5. *Let K be a compact subset of Γ with the property that there exists a positive integer n such that nK has non-void interior. Then*

$$\beta_K^L(a) \leqq c\, \omega_K(a) ,$$

where $c = c(K)$.

PROOF. Suppose K, n satisfy the hypothesis of the theorem, and choose any $\chi \in \operatorname{int} nK$. Then

$$K \subset K - \chi + \operatorname{int} nK .$$

We can find V, a relatively compact open neighbourhood of zero, such that

$$K + V - V \subset K - \chi + \operatorname{int} nK .$$

Hence

$$\begin{aligned} \omega_{K+V-V}(a) &\leqq \omega_K(a) + \omega_{-\chi}(a) + \omega_{\operatorname{int} nK}(a) \\ &\leqq (2n+1)\, \omega_K(a) . \end{aligned}$$

The result follows from Corollary 1.4.

REMARK 1.6. The hypothesis of Theorem 1.5 is satisfied whenever $\theta(K) > 0$ (see [3], (20.17)).

REMARK 1.7. We can obtain results similar to those obtained in 1.1 – 1.5 by considering a norm ($\| \cdot \|$) on L that satisfies

(b)′ $$\| k * f \| \leqq \| k \|_{1,w} \, \| f \| ,$$

where $k \in L_w^1(G)$, $f \in L(G)$, w is a non-negative locally bounded measurable function satisfying

$$w(x + y) \leqq w(x)\, w(y)$$

for all $x, y \in G$, and

$$L_w^1(G) = \{ k \in L^1(G) : \| k \|_{1,w} = \int_G | k(x) | w(x) dx < \infty \} .$$

However, if we wish to follow the proof of Lemma 1.2, w would be restricted inasmuch as $gw, hw \in L^2(G)$.

2. The Bernstein inequality for bounded functions

We now examine the particular case when $L(G) = L^\infty(G)$, taken with its usual norm. We put

(2.1) $$\beta_K(a) = \sup\{\|\tau_a f - f\|_\infty : f \in L^\infty_K(G), \|f\|_\infty \leqq 1\}.$$

It follows from Lemma 2.1 that the results of §2 apply equally well to $L^p(G)$, $p \in [1, \infty)$.

LEMMA 2.1. *Let $K \subset \Gamma$ and let $L(G)$ be as in §1 with the additional property that there is a set $\Phi \subset C_{00}(G)$ such that for any $f \in L(G)$,*

(2.2) $$\|f\|_L = \sup\{\|f * \phi\|_\infty : \phi \in \Phi\}.$$

Then, for all $a \in G$,

$$\beta^L_K(a) \leqq \beta_K(a).$$

PROOF. Let $\phi \in \Phi$ and $f \in L_K(G)$. Then $\phi * f \in L^\infty_K(G)$ and, by (2.1) and (2.2),

$$\begin{aligned}\|\phi * (\tau_a f - f)\|_\infty &= \|\tau_a \phi * f - \phi * f\|_\infty \\ &\leqq \beta_K(a)\,\|\phi * f\|_\infty \\ &\leqq \beta_K(a)\,\|f\|_L,\end{aligned}$$

whence,

(2.3) $$\sup_{\phi \in \Phi} \|\phi * (\tau_a f - f)\|_\infty \leqq \beta_K(a)\,\|f\|_L.$$

The combination of (2.2) and (2.3) yields the required result.

We now consider estimates for $\beta_K(a)$ in three special cases:

(a) K supports no true pseudomeasure;

(b) K is an S-set which is the closure of its interior;

(c) Γ has a compactly generated open subgroup.

THEOREM 2.2. *If $K \subset \Gamma$ supports no true pseudomeasure then*

$$\beta_K(a) \leqq c\,\omega_K(a),$$

where $c = c(K)$.

PROOF. Let $f \in L^\infty_K(G)$. We can use (1.4) and the assumption that K supports no true pseudomeasure to deduce the existence of a bounded measure μ on Γ, supported by K, such that

$$\hat{f} = \mu.$$

Consider $g \in C(G)$ defined by

(2.4) $$g(x) = \int_\Gamma \chi(x)\,d\mu(\chi).$$

We show that $g = f$ *l.a.e.*.

We can find a μ-measurable function h such that

$$h\,d|\mu| = d\mu \text{ and } |h(\chi)| = 1$$

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for all $\chi \in \Gamma$. Let $t \in L^1(G)$. Then, using the definition of the Fourier transform of a bounded function, (2.4) gives

$$\hat{g}(\bar{t}) = g(\bar{t})$$

$$= \int_G g(x)\bar{t}(x)\,d\lambda(x)$$

$$(2.5) \qquad = \int_G \left(\int_\Gamma \chi(x)h(\chi)\,d|\mu|(\chi) \right) \bar{t}(x)\,d\lambda(x).$$

Now $\lambda, |\mu|$ are positive measures, the function v on $G \times \Gamma$ defined by

$$v : (x, \chi) \to \chi(x)\,h(\chi)\,\bar{t}(x)$$

is $\lambda \times |\mu|$ - measurable, and v vanishes outside a $\lambda \times |\mu| - \sigma -$ finite set. Furthermore,

$$\int_\Gamma \left(\int_G |\chi(x)\,h(\chi)\,\bar{t}(x)|\,d\lambda(x) \right) d|\mu|(\chi) \leqq \|t\|_1 \,\|\mu\|_M < \infty\,,$$

where $\|\mu\|_M = |\mu|(\Gamma)$. Hence we can apply the Fubini-Tonelli theorem to (2.5), to obtain

$$\hat{g}(\bar{t}) = \int_\Gamma \left(\int_G \chi(x)\,\bar{t}(x)\,d\lambda(x) \right) d\mu(\chi)\,,$$

and thus,

$$\hat{g}(\bar{t}) = \int_\Gamma \hat{\bar{t}}(\chi)\,d\mu(\chi)$$

$$= \hat{f}(\bar{t}).$$

As $t \in L^1(G)$ was chosen arbitrarily, and the Fourier transform is one-to-one, $g = f$ *l.a.e.*.

Since μ is supported by K, we now see that

$$|\tau_a f(x) - f(x)| = \left| \int_K (\chi(-a) - 1)\,\chi(x)\,d\mu(\chi) \right| \ l.a.e.$$

$$\leqq \omega_K(a)\,\|\mu\|_M.$$

But as K supports no true pseudomeasure, it must be Helson set (see [1], (3.2)) and hence there exists $c > 0$ such that

$$\|\mu\|_M \leqq c\,\|f\|_\infty$$

(see [3], (41.12)). As c is independent of the choice of f, the result follows.

THEOREM 2.3. *Let K be a compact S-set which is the closure of its interior. Then*

$$\beta_K(a) = \inf\{\|\tau_a k - k - l\|_1 : k, l \in L^1(G), \hat{k} = 1, \hat{l} = 0 \text{ on } K\}.$$

PROOF. Choose integrable functions k, l such that $\hat{k} = 1$, $\hat{l} = 0$ on K. From Lemma 1.1, we have

$$\beta_K(a) \leqq \|\tau_a k - k - l\|_1$$

and hence

$$\beta_K(a) \leqq \inf\{\|\tau_a k - k - l\|_1 : k, l \in L^1(G), \hat{k} = 1, \hat{l} = 0 \text{ on } K\}. \tag{2.6}$$

To prove the reverse inequality, we consider the complex-valued map

$$A : C_{0\,K}(G) \to \boldsymbol{C},$$

defined by

$$Af = f(-a) - f(0), \tag{2.7}$$

where $a \in G$ is given.

Since A is clearly linear and $\|\cdot\|_\infty-$ continuous, the Hahn-Banach theorem ensures that it can extended to a continuous linear functional A' on $C_0(G)$ such that

$$\|A'\| \leqq \|A\|.$$

Now by the Riesz representation theorem, there is a bounded measure μ such that

$$A'f = \int_G \check{f}\, d\mu = \mu * f(0)$$

for all $f \in C_0(G)$, where

$$\check{f} : x \to f(-x).$$

Combining (1.3) and (2.7) yields

$$(\tau_{-x} f)(-a) - (\tau_{-x} f)(0) = A(\tau_{-x} f) = \mu * (\tau_{-x} f)(0) = (\tau_{-x}(\mu * f))(0),$$

or equivalently,

$$f(x - a) - f(x) = \mu * f(x)$$

for all $x \in G$. Hence for every $f \in C_{0\,K}(G)$ and $a \in G$,

$$\tau_a f - f = \mu * f$$

and we have

$$\begin{aligned} \|\mu\|_M = \|A'\| \leqq \|A\| &= \sup\{|f(-a) - f(0)| : f \in C_{0\,K}(G), \|f\|_\infty \leqq 1\} \\ &\leqq \sup\{\|\tau_a f - f\|_\infty : f \in C_{0\,K}(G), \|f\|_\infty \leqq 1\} \\ &\leqq \sup\{\|\tau_a f - f\|_\infty : f \in L^\infty_K(G), \|f\|_\infty \leqq 1\}, \end{aligned}$$

that is,

$$\|\mu\|_M \leqq \beta_K(a). \tag{2.8}$$

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Choose $\varepsilon > 0$. Now there exists $g \in L^1(G)$ such that $\hat{g} = 1$ on K, $\hat{g}$ has compact support and $\| g \|_1 < 1 + \varepsilon$ (see [5], 2.6.8). Put $h = \mu * g$. Then $\hat{h} = \hat{\mu}$ on K. Since K is an S-set, we have for any $f \in C_{0,K}(G)$,

$$(2.9) \qquad \begin{aligned} h * f &= \mu * f \\ &= \tau_a f - f \end{aligned}$$

and, by (2.8) and the choice of g,

$$(2.10) \qquad \| h \|_1 \leqq \beta_K(a)(1 + \varepsilon).$$

Let $k \in L^1(G)$ be such that $\hat{k} = 1$ on K. We want to show that $(h - \tau_a k + k)\hat{}$ vanishes on K.

Let $f \in C_{0,K}(G)$. Then we have, once more using the fact that K is an S-set,

$$\begin{aligned} (h - \tau_a k + k) * f &= \tau_a f - f - \tau_a k * f + k * f \\ &= 0 \end{aligned}$$

by (2.9), whence it follows that

$$(2.11) \qquad (h - \tau_a k + k)\hat{} \text{ vanishes on } \Sigma(f).$$

Let $\chi \in \operatorname{int} K$. We can find $f_\chi \in L^1 \cap C_{0,\operatorname{int} K}(G)$ such that $\hat{f}_\chi(\chi) = 1$ (see [5], 2.6.2). By (2.11), $(h - \tau_a k + k)\hat{}$ vanishes on $\Sigma(f_\chi)$, and hence $(h - \tau_a k + k)\hat{}$ vanishes on $\bigcup_{\chi \in \operatorname{int} K} \Sigma(f_\chi) = \operatorname{int} K$. But $h - \tau_a k + k \in L^1(G)$ and so we appeal to the continuity of $(h - \tau_a k + k)\hat{}$ to deduce that it vanishes on $\overline{\operatorname{int} K} = K$.

Put $-l = h - \tau_a k + k$. Then $l \in L^1(G)$ and $\hat{l} = 0$ on K. Also

$$\| \tau_a k - k - l \|_1 = \| h \|_1 \leqq \beta_K(a)(1 + \varepsilon)$$

by (2.10), and hence

$$(2.12) \qquad \inf \{ \| \tau_a k - k - l \|_1 : k, l \in L^1(G), \hat{k} = 1, \hat{l} = 0 \text{ on } K \} \leqq \beta_K(a)(1 + \varepsilon).$$

But $\varepsilon > 0$ was chosen arbitrarily, so (2.6) and (2.12) give the desired result.

REMARK 2.4. We consider the circle group $\boldsymbol{T}$ with $K = [-N, N]$. Noticing that K is a compact S-set, we can use Theorem 1.3 with $V = K$ to obtain

$$\beta_K(a) \leqq 3\sqrt{2}\, \omega_K(a).$$

It can be shown that if $N > 1$ and

$$\beta_K(a) \leqq \alpha\, \omega_K(a)$$

for all $a \in \boldsymbol{T}$, then $\alpha > 1$; compare the 'classical' Bernstein inequality.

THEOREM 2.5. *Let K be a compact subset of Γ and let Ω be a compactly generated open subgroup of Γ. Then there exists a compact set $K_0 \subset \Gamma$ and a finite set $F \subset K \backslash \Omega$ such that*

$$\omega_K(a) \leqq N\, \omega_{K_0}(a) + \omega_F(a),$$

where $N = N(K, K_0)$.

PROOF. We can assume without loss of generality that $0 \in K$. Since $\{\chi + \Omega : \chi \in K\}$ is an open cover of K, the compactness of K implies the existence of $\chi_1, \cdots, \chi_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} (\chi_i + \Omega)$$

where, without loss of generality, we can assume that $\chi_1 = 0$ and $\chi_i \notin \Omega$ for $i > 1$. Now $K_i = K \cap (\chi_i + \Omega)$ is closed (as Ω is closed) and since $K_i \subset K$, K_i is compact.

As Ω is compactly generated, there is an open neighbourhood W of zero such that $\overline{W}$ is compact and

$$\Omega = \bigcup_{m=1}^{\infty} mW.$$

Since for each $i \in \{1, 2, \cdots, n\}$,

$$K_i \subset \chi_i + \Omega$$

and $-\chi_i + K_i$ is compact, there is an m_i such that

$$-\chi_i + K_i \subset \bigcup_{m=1}^{m_i} mW = m_i W.$$

Hence

$$\omega_{K_i}(a) \leqq |\chi_i(a) - 1| + m_i \omega_W(a).$$

Finally, since $K = \bigcup_{i=1}^{n} K_i$ and $\chi_1 = 0$, it follows that

$$\begin{aligned} \omega_K(a) &\leqq \max_{1 \leqq i \leqq n} |\chi_i(a) - 1| + \omega_W(a) \max_{1 \leqq i \leqq n} m_i \\ &\leqq \omega_F(a) + N\, \omega_{K_0}(a), \end{aligned}$$

where $F = \{\chi_2, \chi_3, \cdots, \chi_n\}$, $N = \max_{1 \leqq i \leqq n} m_i$ and $K_0 = \overline{W}$.

COROLLARY 2.6. *If Γ is compactly generated then there exists a compact set $K_0 \subset \Gamma$ and a positive integer $N = N(K, K_0)$ such that*

$$\omega_K(a) \leqq N\, \omega_{K_0}(a).$$

3. Differentiation along a one-parameter subgroup.

The type of estimate obtained for $\beta_K(a)$ in §1 can be linked with the 'classical' Bernstein inequality by considering differentiation along a one-parameter subgroup of G.

Let H be a one-parameter subgroup of G, that is, $H = \rho(\boldsymbol{R})$ where ρ is a continuous homomorphism from $\boldsymbol{R}$ into G. We put

$$D_\rho f(x) = \lim_{r \to 0} r^{-1}(f(x + \rho(r)) - f(x)).$$

If the limit exists finitely for all $x \in G$ then f is said to be differentiable along ρ. It will appear in Theorem 3.3 that every bounded continuous function with compact spectrum is differentiable along ρ, and Corollary 3.5 gives an estimate for $\| D_\rho f \|_\infty$. It is not much of a restriction to consider only bounded continuous functions with compact spectra since if $f \in L^\infty_K(G)$, where K is a compact subset of Γ, then f is equal l.a.e. to a (uniformly) continuous function (see (1.5)).

Let ρ be a continuous homomorphism from $\boldsymbol{R}$ into G. For $\chi \in \Gamma$, consider the map

$$\eta_\chi : \boldsymbol{R} \to \boldsymbol{C},$$

defined by

$$\eta_\chi(r) = \chi(\rho(r)).$$

η_χ is clearly a continuous homomorphism of $\boldsymbol{R}$ into the circle group, that is, η_χ is a continuous character of $\boldsymbol{R}$, and we can deduce the existence of a unique $\lambda_\chi \in \boldsymbol{R}$ such that for every $r \in \boldsymbol{R}$,

$$\eta_\chi(r) = \exp(i\lambda_\chi r).$$

We require two technical lemmas.

LEMMA 3.1. *The map*

$$F : \Gamma \to \boldsymbol{R},$$

defined by

$$F(\chi) = \lambda_\chi,$$

is continuous.

PROOF. As F is a homomorphism of Γ into $\boldsymbol{R}$, it suffices to prove that F is continuous at zero. In view of 1.2.6 of [5], it suffices to show that, given a compact set $D \subset R$ and $\varepsilon > 0$,

$$\sup_{r \in D} \left| \exp(iF(\chi)r) - 1 \right| < \varepsilon \tag{3.1}$$

for all χ in some neighbourhood of zero.

Now (3.1) is equivalent to

$$\sup_{r \in D} |\chi(\rho(r)) - 1| < \varepsilon,$$

which is implied by

$$\sup_{x \in \rho(D)} |\chi(x) - 1| < \varepsilon. \tag{3.2}$$

Since ρ is continuous and $D \subset \boldsymbol{R}$ is compact, $\rho(D)$ is compact in G; hence, by [5], 1.2.6 again,

$$V = \{\chi \in \Gamma : \sup_{x \in \rho(D)} |\chi(x) - 1| < \varepsilon\}$$

is a neighbourhood of zero. Using (3.2), we see that (3.1) holds for all $\chi \in V$.

LEMMA 3.2. *Let K be a compact subset of Γ. Then there exist $k, j \in L^1(G)$ such that*

(a) $\hat{k} = 1$ *on a neighbourhood of K*, $\hat{k} \in C_{00}(\Gamma)$;

(b) $\lim_{r \to 0} \| r^{-1}(\tau_{-\rho(r)}k - k) - j \|_1 = 0$.

PROOF. Let W be a relatively compact neighbourhood of K. Then $W + V - V$ is relatively compact, where V is a relatively compact non-void open set. Let g, h be the elements of $L^2(G)$ having Fourier transforms ξ_V, ξ_{W-V} respectively, and put $k = \theta(V)^{-1}gh$. Consider the functions s, t on Γ defined by

$$s = F\xi_V; \quad t = F\xi_{W-V}.$$

As F is continuous and V, $W - V$ are relatively compact, $s, t \in L^2(\Gamma)$. Let $p, q \in L^2(G)$ be chosen so that $\hat{p} = s$ and $\hat{q} = t$. Put

$$j = i\theta(V)^{-1}(ph + qg).$$

Then $j \in L^1(G)$.

Now consider the difference

$$\begin{aligned}
&\| r^{-1}(\tau_{-\rho(r)}k - k) - j \|_1 \\
&= \theta(V)^{-1} \| r^{-1}(\tau_{-\rho(r)}g - g)h + r^{-1}(\tau_{-\rho(r)}h - h)\tau_{-\rho(r)}g - iph \\
&\quad - iq\tau_{-\rho(r)}g + iq(\tau_{-\rho(r)}g - g) \|_1 \\
&\leqq \theta(V)^{-1}(\| r^{-1}(\tau_{-\rho(r)}g - g) - ip \|_2 \| h \|_2 \\
&\quad + \| g \|_2 \| r^{-1}(\tau_{-\rho(r)}h - h) - iq \|_2 + \| q \|_2 \| \tau_{-\rho(r)}g - g \|_2).
\end{aligned} \tag{3.3}$$

We will show that each of the terms in (3.3) tends to zero in the limit as $r \to 0$.

By Plancherel's theorem,

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$$\| r^{-1}(\tau_{-\rho(r)}g - g) - ip \|_2 = \| (r^{-1}(\tau_{-\rho(r)}g - g) - ip)^{\wedge} \|_2$$
$$= \left(\int_{\Gamma} |\hat{g}(\chi)|^2 \, |r^{-1}(\chi(\rho(r)) - 1) - i\hat{p}(\chi)|^2 \, d\theta(\chi) \right)^{\frac{1}{2}}$$
$$\leqq \| g \|_2 \sup_{\chi \in V} |r^{-1}(\chi(\rho(r)) - 1) - i\hat{p}(\chi)|$$
$$\leqq \| g \|_2 \sup_{\lambda\chi \in Q_V} |r^{-1}(\exp(i\lambda_\chi r) - 1) - i\lambda_\chi|,$$

where $Q_V = F(\bar{V})$. If $\lambda_\chi \neq 0$,

$$|r^{-1}(\exp(i\lambda_\chi r) - 1) - i\lambda_\chi| = |r^{-1} \exp(i\tfrac{1}{2}\lambda_\chi r)(\exp(i\tfrac{1}{2}\lambda_\chi r) - \exp(-i\tfrac{1}{2}\lambda_\chi r)) - i\lambda_\chi|$$
$$= |r^{-1} 2 \sin\tfrac{1}{2}\lambda_\chi r - \lambda_\chi \exp(-i\tfrac{1}{2}\lambda_\chi r)| \leqq |\lambda_\chi| (|(\tfrac{1}{2}\lambda_\chi r)^{-1} \sin\tfrac{1}{2}\lambda_\chi r - 1| + |1 - \exp(-i\tfrac{1}{2}\lambda_\chi) r|).$$

The final inequality holds trivially if $\lambda_\chi = 0$.

Now Q_V is compact, and hence we can find $\lambda > 0$ such that

$$Q_V \subset [-\lambda, \lambda]. \tag{3.4}$$

Let $\lambda_\chi \in Q_V$. Since $1 - (\sin x/x)$ increases with x on $[0, \pi]$, reference to (3.4) yields

$$|(\tfrac{1}{2}\lambda_\chi r)^{-1} \sin\tfrac{1}{2}\lambda_\chi r - 1| \leqq |(\tfrac{1}{2}\lambda r)^{-1} \sin\tfrac{1}{2}\lambda r - 1|$$

for all $r \in [-2\pi/\lambda, 2\pi/\lambda]$. As $\sin x$ increases with x on $[0, \frac{1}{2}\pi]$, appealing to (3.4) again gives

$$|1 - \exp(-i\tfrac{1}{2}\lambda_\chi r)| = 2|\sin\tfrac{1}{4}\lambda_\chi r| \leqq 2|\sin\tfrac{1}{4}\lambda r|$$

for all $r \in [-2\pi/\lambda, 2\pi/\lambda]$. Hence

$$\sup_{\lambda\chi \in Q_V} |r^{-1}(\exp(i\lambda_\chi r) - 1) - i\lambda_\chi| \leqq \lambda(|(\tfrac{1}{2}\lambda r)^{-1} \sin\tfrac{1}{2}\lambda r - 1| + 2|\sin\tfrac{1}{4}\lambda r|)$$

for all $r \in [-2\pi/\lambda, 2\pi/\lambda]$, and it follows that

$$\lim_{r \to 0} \left(\sup_{\lambda\chi \in Q_V} |r^{-1}(\exp(i\lambda_\chi r) - 1) - i\lambda_\chi| \right) = 0.$$

Thus the first term in (3.3) tends to zero as $r \to 0$. The second term in (3.3) is treated similarly. For the third term in (3.3), see [3], (20.4) and use the continuity of ρ.

Finally, we notice that k satisfies hypothesis (a) of the lemma.

THEOREM 3.3. *Let K be a compact subset of Γ and let f be a bounded continuous function with spectrum contained in K. Then $D_\rho f(x)$ exists finitely for all $x \in G$.*

PROOF. We use the functions k, j obtained in Lemma 3.2. Consider

$$\begin{aligned} \left| r^{-1}(\tau_{-\rho(r)}f(x) - f(x)) - j * f(x) \right| &\leqq \left\| r^{-1}(\tau_{-\rho(r)}f - f) - j * f \right\|_\infty \\ = \left\| r^{-1}(\tau_{-\rho(r)}k - k) * f - j * f \right\|_\infty &\leqq \left\| r^{-1}(\tau_{-\rho(r)}k - k) - j \right\|_1 \left\| f \right\|_\infty \\ &\to 0 \text{ as } r \to 0. \end{aligned} \tag{3.5}$$

Hence $\lim_{r \to 0} r^{-1}(\tau_{-\rho(r)}f(x) - f(x))$ exists finitely for all $x \in G$.

REMARK 3.4. We notice that the limit (3.5) is attained uniformly with respect to x in G.

COROLLARY 3.5. *Suppose the hypotheses of Theorem 3.3 are satisfied. Then*

$$\left\| D_\rho f \right\|_\infty \leqq d \left\| f \right\|_\infty ,$$

where d is independent of the choice of f.

PROOF. $\left\| D_\rho f \right\|_\infty = \left\| j * f \right\|_\infty \leqq \left\| j \right\|_1 \left\| f \right\|_\infty ,$

and j depends only on K, W and V.

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