HAMILTONIAN CIRCUITS AND PATHS ON THE n-CUBE

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For positive integral n let C_n denote the n-dimensional unit cube with vertices $(\delta_1, \delta_2, \ldots, \delta_n)$ where $\delta_i = 0$ or 1 for $i = 1, 2, \ldots, n$. Call two vertices of C_n adjacent if the distance between them is 1. By a Hamiltonian circuit in C_n is meant an ordered set $\mathcal{P} = \{P_1, P_2, \ldots, P_{2^n}\}$ of distinct vertices of C_n such that

(i) P_i and P_{i+1} are adjacent for $i = 1, 2, \ldots, 2^n - 1$.

(ii) P_1 and P_{2^n} are adjacent.

By a Hamiltonian path in C_n is meant an ordered set $\mathcal{P} = \{P_1, P_2, \ldots, P_{2^n}\}$ of distinct vertices of C_n satisfying (i).

Since there is no danger of confusion we shall drop the word "Hamiltonian" and use only the words "circuit" and "path". Note that a circuit is a path but the converse is not necessarily true. If a path is not a circuit we shall call it a proper path. Two circuits will be considered equal if it is possible to obtain one from the other by cyclic permutation of the points or by reversing the order of the points or by a combination of these operations. Two proper paths will be considered equal if one can be obtained from the other by reversing the order of the points. The circuits in C_n can be divided into equivalence classes, two circuits being placed in the same class if it is possible to obtain one from the other by applying to C_n a suitable symmetry operation of the group of symmetries of C_n. In the same way the proper paths in C_n can be divided into equivalence classes.

Denote by $h(n)$ the number of circuits in C_n and by $h^*(n)$ the number of equivalence classes of circuits. Let $k(n)$ be the number of proper paths in C_n and $k^*(n)$ the number of equivalence classes of proper paths. It is not difficult to evaluate these functions for $n = 1, 2, 3$. The values are given in the table below.

n	$h(n)$	$h^*(n)$	$k(n)$	$k^*(n)$
1	1	1	0	0
2	1	1	0	0
3	6	1	24	1

However, the evaluation of these functions for $n \geq 4$ appears to be quite difficult. (See Gilbert [1] where it is shown that $h^*(4) = 9$.) In [1] it is also shown that

$$(1) \qquad h^*(n) > C(n-2)!\, n^{-1/2}$$

where C is a constant. We mention also that, since the number of elements of the symmetry group of C_n is $n!\,2^n$,

$$(2) \qquad h(n) \geq h^*(n) \geq \frac{h(n)}{n!2^n}, \quad k(n) \geq k^*(n) \geq \frac{k(n)}{n!2^n}.$$

The main results that we wish to establish in this paper are the following:

THEOREM 1. For all positive integers n

$$(3) \qquad h(n) \geq (7\sqrt{6})^{2^n - 4}$$

and for all $n \geq 3$

$$(4) \qquad k(n) \geq (7\sqrt{6})^{2^n - 20}.$$

THEOREM 2. Let f be any one of the functions defined above. Then $\lim_{n \to \infty} f(n)^{2^{-n}}$ exists and is independent of f.

Note that Theorem 1 and (2) yield a lower bound for $h^*(n)$

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which is substantially larger than that given by (1). We remark also that, in connection with Theorem 2, we cannot decide whether the limit is finite or infinite.

To prove the above theorems we need some lemmas.

LEMMA 1. If T_n denotes the number of circuits which traverse a given edge of C_n, then

$$T_n = \frac{2h(n)}{n}. \tag{5}$$

Proof. Clearly T_n is independent of the edge chosen. Since the number of edges in C_n is $n2^{n-1}$, the number of circuits, counting multiplicities, is nT_n2^{n-1}. Each circuit is counted with 2^n edges. The number of circuits in C_n is therefore $h(n) = \frac{nT_n 2^{n-1}}{2^n} = \frac{nT_n}{2}$. This implies (5).

LEMMA 2. For $r = 1, 2, \ldots, [\frac{n-1}{2}]$ let $T_n(2r+1)$ denote the number of proper paths in C_n with end points P and Q where P and Q differ in $2r+1$ coordinates. Then

$$\sum_{r=1}^{[\frac{n-1}{2}]} 2^{n-1} \binom{n}{2r+1} T_n(2r+1) = k(n). \tag{6}$$

Proof. Consider the set of all unordered pairs (P, Q) where P and Q are points of C_n which differ in $2r+1$ coordinates. P can be chosen in 2^n ways and then Q can be chosen in $\binom{n}{2r+1}$ ways. The total number of such pairs is therefore $2^{n-1}\binom{n}{2r+1}$. Each such pair forms the end points of $T_n(2r+1)$ proper paths. This clearly implies (6).

LEMMA 3. Let $S_n = \max_{1 \le r \le [\frac{n-1}{2}]} T_n(2r+1)$. Then

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(7) $$S_n \geq k(n) / 2^{n-1} \sum_{r=1}^{[\frac{n-1}{2}]} \binom{n}{2r+1} > \frac{k(n)}{4^{n-1}} .$$

Proof. This follows from (6) and the fact that

$$\sum_{r=1}^{[\frac{n-1}{2}]} \binom{n}{2r+1} < 2^{n-1} .$$

In what follows if $P = (\delta_1, \delta_2, \ldots, \delta_a)$ is a vertex of C_a and $Q = (\delta_1', \delta_2', \ldots, \delta_b')$ is a vertex of C_b then (P, Q) denotes the vertex $(\delta_1, \delta_2, \ldots, \delta_a, \delta_1', \ldots, \delta_b')$ of C_{a+b}.

LEMMA 4. For all positive integers m and n we have

(8) $$h(n+m) \geq n\, 2^{n-1}\, T_n^{2^m}\, h(m)$$

(9) $$h(n+m) \geq S_n^{2^m}\, h(m)$$

(10) $$k(n+m) \geq S_n^{2^m}\, k(m)$$

(11) $$k(n+m) \geq n\, 2^{n-1}\, T_n^{2^m}\, k(m) .$$

Proof. To prove (8) let e be an edge of C_n with end points P and Q and let $\mathcal{P}_i = \{P, P_{i1}, \ldots, P_{is}, Q\}$, $i = 1, 2, \ldots, T_n$, be the circuits in C_n which traverse e. $(s = 2^n-2)$. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ be a circuit in C_m. $(r = 2^m)$. Then for $1 \leq i_1, i_2, \ldots, i_r \leq T_n$ the following sequence of points is a circuit in C_{n+m}:

$$\{ (P_1, P), (P_1, P_{i_1 1}), (P_1, P_{i_1 2}), \ldots, (P_1, P_{i_1 s}), (P_1, Q),$$

$$(P_2, Q), (P_2, P_{i_2 s}), (P_2, P_{i_2 s-1}), \ldots, (P_2, P_{i_2 1}), (P_2, P),$$

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$$(P_3, P),\ (P_3, P_{i_3 1}),\ (P_3, P_{i_3 2}),\ \ldots,\ (P_3, P_{i_3 s}),\ (P_3, Q),$$

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$(P_r, Q),\ (P_r, P_{i_r s}),\ (P_r, P_{i_r s-1}),\ \ldots,\ (P_r, P_{i_r 1}),\ (P_r, P)\ \}$$

The numbers $i_1, i_2, \ldots, i_r$ can be chosen in $T_n^{2^m}$ ways, e can be chosen in $n\,2^{n-1}$ ways and ρ can be chosen in $h(m)$ ways. The total number of circuits constructed in this manner is therefore $n\,2^{n-1}\,T_n^{2^m}\,h(m)$. This proves (8).

The proofs of (9), (10) and (11) are similar to the proof of (8) and we omit the details.

We are now in a position to prove Theorems 1 and 2. From (9) and the fact that $S_3 = 6$ we have

$$(12) \qquad h(m+3) \geq 6^{2^m}\, h(m)$$

and it is easy to verify that (12) implies (3). Similarly (10) implies (4). Thus Theorem 1 is proved.

Let $\underline{\alpha} = \liminf_{n\to\infty} h(n)^{2^{-n}} \leq \limsup_{n\to\infty} h(n)^{2^{-n}} = \overline{\alpha}$. We show that $\underline{\alpha} = \overline{\alpha}$. From (8) and (5) it follows that for all positive integers a and b,

$$h(ab) > \left(\frac{2h(b)}{b}\right)^{2^{ab-b}} h(ab-b) > \left(\frac{2h(b)}{b}\right)^{2^{ab-b}} .$$

Thus

$$(13) \qquad h(ab)^{2^{-ab}} > \left(\frac{2}{b}\right)^{2^{-b}} h(b)^{2^{-b}} .$$

Suppose first that $\overline{\alpha} < \infty$ and let $\epsilon > 0$ be given. Let b be the least integer for which $h(b)^{2^{-b}} > \overline{\alpha} - \epsilon$ and $\left(\frac{2}{b}\right)^{2^{-b}} > 1 - \epsilon$.

Then (13) implies

$$(14) \qquad h(ab)^{2^{-ab}} > (\bar{\alpha} - \epsilon)(1 - \epsilon).$$

Let $n = ab + r$ where $1 \leq r \leq b-1$. Then by (8) we have

$$h(n) = h(ab + r) > \left(\frac{2h(ab)}{ab}\right)^{2^r} h(r) \geq \left(\frac{2h(ab)}{ab}\right)^{2^r}.$$

Thus

$$(15) \qquad h(n)^{2^{-n}} > \left(\frac{2}{ab}\right)^{2^{-ab}} h(ab)^{2^{-ab}} > (\bar{\alpha} - \epsilon)(1 - \epsilon)^2.$$

(14) and (15) imply that $\underline{\alpha} = \bar{\alpha}$.

Suppose now that $\bar{\alpha} = \infty$ and let A be a positive number. Let b be the least integer for which $h(b)^{2^{-b}} > 4A$. Then (13) and some straightforward calculations show that $h(n)^{2^{-n}} > A$ for all sufficiently large n. It follows that $\underline{\alpha} = \infty$ and hence

$$\alpha = \lim_{n \to \infty} h(n)^{2^{-n}} \quad \text{exists.}$$

That $\beta = \lim_{n \to \infty} k(n)^{2^{-n}}$ exists can be established in a similar manner, using (10) and (7) instead of (8) and (5). The same type of argument starting with (11) and (5) can be used to show that $\beta \geq \alpha$. Finally the fact that $\beta \leq \alpha$ can be proved using (9) and (7). It follows that $\alpha = \beta$. This, with (2), completes the proof of Theorem 2.

In conclusion we remark that the best known upper bounds for $h(n)$ and $k(n)$ are $h(n) < n^{2^n}$ and $k(n) < n^{2^n}$.

REFERENCES

1. E.N. Gilbert, Gray codes and paths on the n-cube. Bell Syst. Tech. J. 37, (1958), pages 815-826.

2. See also W.H. Mills, Some complete cycles on the n-cube. Proc. Amer. Math. Soc. 14, (1963), pages 640-643.

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