FINITE GROUPS THAT NEED MORE GENERATORS THAN ANY PROPER QUOTIENT

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Abstract

A structure theorem is proved for finite groups with the property that, for some integer m with $m \geq 2$, every proper quotient group can be generated by m elements but the group itself cannot.

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Introduction

Let L be a non-cyclic finite group with a unique minimal normal subgroup, M. If M is abelian, assume also that M has a complement in L. Denote by $d(L)$ the minimum of the cardinalities of the generating sets of L.

For each positive integer k, let L^k be the k-fold direct power of L and define the subgroup L_k by

$$L_k = \left\{ (l_1, \ldots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \mod M \right\}.$$

Equivalently, set $\operatorname{diag} L^k = \{(l, \ldots, l) \in L^k \mid l \in L\}$ and $L_k = M^k \operatorname{diag} L^k$. It is easy to see that the socle of L_k is M^k, a direct product of k minimal normal subgroups (each isomorphic to M), and that $L_k/M^k \cong L/M$. The quotient group of L_{k+1} over any minimal normal subgroup is isomorphic to L_k; in particular, the unbounded sequence $d(L_1), \ldots, d(L_k), \ldots$ is non-decreasing, and, by a theorem proved in [10], $d(L_{k+1}) \leq d(L_k) + 1$. Thus if $m \geq d(L)$ then there is a (unique) k such that $d(L_k) = m < d(L_{k+1})$: set $f(m) = k + 1$. We shall comment below on how the function f (which of course depends on L) may be calculated.

In attempting to prove that each finite group with a certain property can be generated by m elements, one frequently considers groups H such that every proper quotient group of H can be generated by m elements but H itself cannot. The main result of this paper is that if $m \geq 2$ then each such H is one of the $L_{f(m)}$ constructed above. By definition, $f(m)$ is never 1, so it is part of the claim that H must have more than one minimal normal subgroup. This part was established in [11]; it depends heavily on the classification of the finite simple groups. There is no need for more work of that kind here. (We do not consider here the analogous problem of non-cyclic groups all of whose proper quotient groups are cyclic.)

This paper was motivated by an attempt at understanding finite groups that are minimal with respect to having non-zero presentation rank (in the sense that the presentation rank of each proper quotient group is 0). It was proved by Gruenberg in [6] that such a group has no abelian minimal normal subgroup and is an H for some $m \geq 2$. In view of this, the main theorem implies that each group which is minimal with respect to having non-zero presentation rank is the $L_{f(d(L))}$ built from an L with non-abelian M, and of course each quotient group of that L/M must have presentation rank 0. It is also shown here that, conversely, if an L has non-abelian M and is such that all quotient groups of L/M have presentation rank 0, then the $L_{f(d(L))}$ formed from that L has presentation rank 1 and all its proper quotient groups have presentation rank 0.

The first example of a group with non-zero presentation rank was given in Cossey, Gruenberg and Kovács [2] as A_5^{20} where A_5 is the alternating group of degree 5. In the present notation, $A_5^{20} = L_{f(2)}$ with $L = A_5$. The result of the previous paragraph and a careful examination of the functions f associated to other choices of L may make it possible to confirm the old feeling that A_5^{20} must be the smallest group with positive presentation rank.

For any finite group G, let $\phi_G(m)$ denote the number of m-bases of G, that is, ordered m-tuplets $(x_1, \ldots, x_m)$ of elements of G that generate G. This function was introduced by Philip Hall [8] with the name of *Eulerian function*.

If L is one of the groups described above, let Γ denote the group of those automorphisms of L that act trivially on L/M. Further, if M is abelian, set $q = |\operatorname{End}_{L/M} M|$. We prove that if $m \geq d(L)$ then

$$f(m) = 1 + \begin{cases} \phi_L(m)/|\Gamma|\phi_{L/M}(m) & \text{if } M' = M, \\ \log_q\big(1 + (q-1)\phi_L(m)/|\Gamma|\phi_{L/M}(m)\big) & \text{if } M' = 1. \end{cases}$$

When M is abelian, this and results of Gaschütz [5] lead to

$$f(m) = 1 + \log_q\big(|M|^{m-1} \big/ |H^1(L/M, M)|\big),$$

which could have also been derived from Theorem 6 of Gaschütz [3]. When L is a finite non-abelian simple group, L_k coincides with the kth direct power L^k; the sequence

$\{d(L^k)\}$ is called the *growth sequence* and has been studied in a series of papers by James Wiegold ([13–16], etcetera). In this case $\Gamma = \operatorname{Aut} L$ and $\phi_{L/M}(m) = 1$, so $f(m) = (\phi_L(m)/|\operatorname{Aut} L|) + 1$, a well-known result, proved many years ago by Philip Hall [8].

We are very grateful to Professor Kovács for many useful suggestions and remarks.

Section 1

In this section m is an integer with $m \geq 2$ and we consider finite groups H such that every proper quotient group of H can be generated by m elements but H itself cannot. It will be useful to recall some known results.

THEOREM 1.1 (Gaschütz [4]). *Let N be a normal subgroup of a finite group G and let $g_1, \ldots, g_m \in G$ be such that $G = \langle g_1, \ldots, g_m, N \rangle$. If $d(G) \leq m$, then there exist elements $u_1, \ldots, u_m$ of N such that $G = \langle g_1 u_1, \ldots, g_m u_m \rangle$. Moreover the cardinality of the set $\{(u_1, \ldots, u_m) \in N^m \mid G = \langle g_1 u_1, \ldots, g_m u_m \rangle\}$ is independent of the choice of $g_1, \ldots, g_m$.*

THEOREM 1.2 ([10]). *If G is a finite group and N is a minimal normal subgroup of G, then $d(G) \leq \max\{2, d(G/N) + 1\}$.*

THEOREM 1.3. *If a finite non-cyclic group G contains a unique minimal normal subgroup M, then $d(G) = \max\{2, d(G/M)\}$.*

When M is abelian this was proved by Aschbacher and Guralnick [1, Corollary 1] using the fact that the first cohomology group with coefficients in a faithful simple module is always strictly smaller than the module itself. The case M non-abelian was considered in [11] and depends heavily on the classification of finite simple groups.

Throughout the paper, L will always denote a non-cyclic finite group which has only one minimal normal subgroup, M, and M will be either non-abelian or complemented in L. For each positive integer k, let L^k be the k-fold direct power of L and define the subgroup L_k by

$$L_k = \big\{(l_1, \ldots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \mod M \big\}.$$

Equivalently, $L_k = M^k \operatorname{diag} L^k$. The socle of L_k is M^k, a direct product of k minimal normal subgroups (each isomorphic to M), and $L_k/M^k \cong L/M$. It is easy to see that the quotient group of L_{k+1} over any minimal normal subgroup is isomorphic to L_k; in particular, the sequence $d(L_1), \ldots, d(L_k), \ldots$ is non-decreasing, and, by Theorem 1.2, $d(L_{k+1}) \leq d(L_k) + 1$. Thus if $m \geq d(L)$ then there is a unique k such

that $d(L_k) = m < d(L_{k+1})$: set $f(L, m) = k + 1$. When L can be identified from the context, we write $f(L, m)$ simply as $f(m)$.

The groups L_k play a key role in the study of groups that need more generators than any proper quotient. Indeed we have:

THEOREM 1.4. *Let m be an integer with $m \geq 2$ and H a finite group such that $d(H/N) \leq m$ for every non-trivial normal subgroup N, but $d(H) > m$. Then there is a group L which has a unique minimal normal subgroup M and is such that M is either non-abelian or complemented in L and $H \cong L_{f(L,m)}$.*

PROOF. By Theorem 1.3, H contains at least two different minimal normal subgroups. Suppose that $N_1, \ldots, N_r, \ldots$ are the minimal normal subgroups of H. As $d(H/N_1) \leq m$ by hypothesis, there are m elements $h_1, \ldots, h_m$ of H such that $H = \langle h_1, \ldots, h_m, N_1 \rangle$. Now consider N_r with $r \neq 1$. Of course, $H = \langle h_1, \ldots, h_m, N_1 N_r \rangle$ and, as $H/N_1 N_r$ is isomorphic to the quotient $(H/N_r)/(N_1 N_r/N_r)$ of H/N_r and H/N_r is m-generated, by Theorem 1.1 there exist m elements $x_1, \ldots, x_m \in N_1$ such that $\langle h_1 x_1, \ldots, h_m x_m, N_r \rangle = H$.

Consider the subgroup $K_r = \langle h_1 x_1, \ldots, h_m x_m \rangle$. We claim that N_1 and N_r are both complements for K_r in H. Obviously $H = K_r N_1 = K_r N_r$; so we have just to prove that $K_r \cap N_i$ is trivial ($i = 1, r$). As $[N_1, N_r] = 1$, the intersection $K_r \cap N_1$ is a normal subgroup of $H = K_r N_r$. Since this normal subgroup is contained in the minimal normal subgroup N_1 of H, if $K_r \cap N_1 \neq 1$ then $N_1 \leq K_r$ and $H = K_r N_1 = K_r$ is m-generated, which contradicts our hypothesis. The claim $K_r \cap N_r = 1$ is proved similarly.

It is now easy to prove that the projections $\pi_r : K_r \cap (N_1 \times N_r) \to N_1$ and $\rho_r : K_r \cap (N_1 \times N_r) \to N_r$ are isomorphisms. Consider π_r first: $\ker \pi_r \leq N_r \cap K_r = 1$, so π_r is injective. Moreover, for any $n_1 \in N_1$ there exist $t \in K_r$ and $n_r \in N_r$ such that $n_1 = t n_r$: then $t = n_1 n_r^{-1} \in (N_1 \times N_r) \cap K_r$ and $t^{\pi_r} = n_1$, so π_r is also surjective. Similar arguments can be applied to ρ_r.

What we shall use from this is that to each $r > 1$ there is a subgroup K_r which complements both N_1 and N_r and an isomorphism $\phi_r : N_1 \to N_r$ (namely $\phi_r = \pi_r^{-1} \rho_r$) such that $K_r \cap (N_1 \times N_r) = \{xx^{\phi_r} \mid x \in N_1\}$. Using that this intersection is normal in K_r, it is easy to see that ϕ_r is a K_r-isomorphism.

When N_1 is abelian, the ϕ_r are in fact H-isomorphisms. We have proved that in this case each minimal normal subgroup of H is abelian and complemented, so the Frattini subgroup of H is trivial, and this implies that soc H admits a complement, K say. We conclude that $H \cong L_k$ where L is the semidirect product $N_1 K$ and k is a suitable integer.

Now assume that N_1 is non-abelian. For this case, we choose k so that the minimal normal subgroups of H are $N_1, \ldots, N_k$. Let $\alpha_1 : H \to \text{Aut } N_1$ be the homomorphism

defined by the conjugation action of H on N_1, so

$$h^{\alpha_1} : x \mapsto x^h \qquad \text{whenever} \quad h \in H, \ x \in N_1,$$

and let L denote the image of α_1. This group has only one minimal normal subgroup, namely the non-abelian group $M = \text{Inn}\, N_1$. For $r > 1$, define $\alpha_r : H \to \text{Aut}\, N_1$ by

$$h^{\alpha_r} : x \mapsto ((x^{\phi_r})^h)^{\phi_r^{-1}} \qquad \text{whenever} \quad h \in H, \ x \in N_1.$$

As $H = K_r N_1$, we can write each h as $h = uv$ with $u \in K_r$, $v \in N_1$, and then

$$((x^{\phi_r})^h)^{\phi_r^{-1}} = ((x^{\phi_r})^u)^{\phi_r^{-1}} = x^u = (x^h)^{v^{-1}}$$

where the first equality holds because N_1 centralizes N_r and the second because ϕ_r is a K_r-map: thus h^α is h^{α_1} followed by the inner automorphism of N_1 induced by v^{-1}. We conclude that

$$h^{\alpha_1} \equiv \cdots \equiv h^{\alpha_k} \mod M.$$

In particular, it follows that each α_r has image L, and the α_r together yield a homomorphism from H onto L_k. The kernel of this homomorphism is the intersection of the kernels of the α_r, that is, of the centralizers of the N_r (with $r = 1, \ldots, k$): that intersection being trivial, our homomorphism $H \to L_k$ is an isomorphism.

Regardless of whether N_1 is abelian, we have proved that $H \cong L_k$ with $k \geq 2$, whence it follows that $H/N_1 \cong L_{k-1}$. Thus $d(L_{k-1}) \leq m < d(L_k)$. By the definition of the function f, this means that $k = f(L, m)$, and the proof of the theorem is complete.

Section 2

The aim of this section is to describe how the function f can be evaluated.

LEMMA 2.5. *Given a homomorphism from an L_k onto L/M, consider the set $\mathscr{S}$ of normal subgroups N of L_k arising as kernels of those homomorphisms of L_k onto L which composed with the natural $L \to L/M$ yield the given $L_k \to L/M$. The cardinality of the set $\mathscr{S}$ is k when M is non-abelian; it is $(q^k - 1)/(q - 1)$ when M is abelian and q is the number of (L/M)-endomorphisms of M.*

PROOF. If $\beta : L_k \to L/M$ is a surjective homomorphism, then $\ker \beta = \text{soc}\, L_k = M^k$. So the normal subgroups N we have to count are precisely the normal subgroups of L_k contained in $\text{soc}\, L$ and such that $L_k/N \cong L$. If M is non-abelian then the k

direct factors of M^k are the unique minimal normal subgroups of L_k and the normal subgroups N we are considering are precisely the direct products of $k - 1$ of them, so we have exactly k possibilities. If M is abelian we have to count the kernels of surjective (L/M)-homomorphisms from M^k to M and there are $(q^k - 1)/(q - 1)$ of these where q is the number of (L/M)-endomorphisms of M.

For any finite group G, let $\phi_G(m)$ denote the number of m-bases of G, that is, ordered m-tuplets $(x_1, \ldots, x_m)$ of elements of G that generate G. This function was introduced by Philip Hall [8] with the name of *Eulerian function*.

In [5] Gaschütz studied Eulerian functions for solvable groups. Here we generalize some of the ideas contained in [5] to the non-solvable case.

Let Γ denote the group of those automorphisms of L that act trivially on L/M.

LEMMA 2.6. *Let F be a free group of rank $m \geq d(L)$. Given a homomorphism from F onto L/M, consider the set $\mathcal{R}$ of normal subgroups N of F arising as kernels of those homomorphisms of F onto L which composed with the natural $L \to L/M$ yield the given $F \to L/M$. The cardinality of the set $\mathcal{R}$ is $\phi_L(m)/|\Gamma|\phi_{L/M}(m)$.*

PROOF. Let $x_1, \ldots, x_m$ be a basis of F. A surjective homomorphism $\beta : F \to L/M$ is uniquely determined by $\beta(x_1) = l_1 M, \ldots, \beta(x_m) = l_m M$, where $L = \langle l_1, \ldots, l_m, M \rangle$. Now let $\gamma : F \to L$ be a surjective homomorphism which composed with the natural $L \to L/M$ yields β; we must have $\gamma(x_1) = l_1 z_1, \ldots, \gamma(x_m) = l_m z_m$ with $z_1, \ldots, z_m \in M$ and $L = \langle l_1 z_1, \ldots, l_m z_m \rangle$. By Theorem 1.1 the number of possible choices for $(z_1, \ldots, z_m)$ is $\phi_L(m)/\phi_{L/M}(m)$, independently of the choice of $(l_1, \ldots, l_m)$; so the number of possibilities for γ is $\phi_L(m)/\phi_{L/M}(m)$. Now let γ_1, γ_2 be two of these homomorphisms; $\ker \gamma_1 = \ker \gamma_2 = N$ if and only if there exists an automorphism α of L which acts trivially on L/M such that γ_2 is equal to γ_1 composed with α. We conclude that the cardinality of $\mathcal{R}$ is $\phi_L(m)/|\Gamma|\phi_{L/M}(m)$.

We can now prove the main result of this section:

THEOREM 2.7. *If $m \geq d(L)$ then, with $q = |\text{End}_{L/M} M|$,*

$$f(m) = 1 + \begin{cases} \phi_L(m)/|\Gamma|\phi_{L/M}(m) & \text{if } M' = M, \\ \log_q \left(1 + (q-1)\phi_L(m)/|\Gamma|\phi_{L/M}(m)\right) & \text{if } M' = 1. \end{cases}$$

PROOF. Let F be a free group of rank m. Given a surjective homomorphism $\beta : F \to L$, consider the set $\mathcal{R}$ defined in Lemma 2.6 and let $R = \bigcap_{N \in \mathcal{R}} N$. It is easy to see that $F/R \cong L_k$ for some k depending on the rank m of F, and that F/R is the largest quotient of F isomorphic to L_i for some i; this means that $f(m) = 1 + k$. Now β induces a surjective homomorphism $\bar\beta : F/R \cong L_k \to L/M$ and the map

$N \mapsto N/R$ is a bijection between $\mathscr{R}$ and the set $\mathscr{S}$ defined in Lemma 2.5. So we
have, by Lemma 2.5 and Lemma 2.6,

$$\frac{\phi_L(m)}{|\Gamma|\phi_{L/M}(m)} = |\mathscr{R}| = |\mathscr{S}| = \begin{cases} k & \text{if } M' = M, \\ (q^k - 1)/(q - 1) & \text{if } M' = 1. \end{cases}$$

Since $k = f(m) - 1$, this completes the proof.

For the case of abelian M, Gaschütz proved ([5, Theorem 2]) that

$$\phi_L(m)/\phi_{L/M}(m) = |M|^m - a$$

where a is the number of complements of M in L. Since $a = |M||H^1(L/M, M)|$
while $|\Gamma| = (q - 1)a$, one concludes that

$$f(m) = 1 + \log_q\left(|M|^{m-1} \big/ |H^1(L/M, M)|\right).$$

This form of our result could be deduced more directly from Theorem 6 of Gaschütz
[3].

When L is a non-abelian simple group, what Theorem 2.7 gives is that

$$f(m) = (\phi_L(m)/|\operatorname{Aut} L|) + 1,$$

a well-known result, proved many years ago by Philip Hall [8]. In any case, $\operatorname{Aut} L$
permutes the m-bases regularly and $\phi_L(m)/|\operatorname{Aut} L|$ is the number of orbits. A similar
interpretation can be given for the number $\phi_L(m)/\phi_{L/M}(m)|\Gamma|$. Fix $(l_1, \ldots, l_m) \in L^m$
with the property that $\langle l_1, \ldots, l_m, M \rangle = L$ and consider the set

$$\Omega = \{(\bar{l}_1, \ldots, \bar{l}_m) \in L^m \mid L = \langle \bar{l}_1, \ldots, \bar{l}_m \rangle, \ l_i \equiv \bar{l}_i \mod M \text{ for each } 1 \leq i \leq m\}.$$

The subgroup Γ of $\operatorname{Aut} L$ stabilizes Ω and $\phi_L(m)/\phi_{L/M}(m)|\Gamma|$ is the number of orbits
for the action of Γ on Ω (and is independent, by Theorem 1.1, of the choice of
$l_1, \ldots, l_m$).

Section 3

The presentation rank $\operatorname{pr}(G)$ of a finite group G is an invariant whose definition
comes from the study of relation modules (see [7] for more details). It also plays a
role in the study of the minimal number of generators $d(G)$ of G. Let I_G denote the
augmentation ideal of $\mathbb{Z}G$, and $d(I_G)$ the minimal number of elements of I_G needed to
generate I_G as a G-module. Roggenkamp [12] proved that $d(G) = d(I_G) + \operatorname{pr}(G)$. It is

known that $\mathrm{pr}(G) = 0$ for many groups G, including all solvable groups, all Frobenius groups and all 2-generator groups, but examples of groups with presentation rank n can be constructed for every positive integer n. On the other hand, the only known examples of groups with non-zero presentation rank are sufficiently high direct powers of perfect groups and groups related to them. The smallest example which is known is the direct product A_5^{20} of 20 copies of the alternating group of degree 5. It is an open problem whether it is possible to construct examples of different kinds or whether there exists a group G with $\mathrm{pr}(G) \neq 0$ and $|G| < |A_5^{20}|$.

In this section we study groups that are minimal with respect to having non-zero presentation rank (in the sense that the presentation rank of each proper quotient group is 0).

Gruenberg proved ([6, (2.4)] and [7, Proposition 6.2]):

THEOREM 3.8. *If H is minimal with respect to having non-zero presentation rank, then H contains no non-trivial solvable normal subgroups, $d(H) > 2$, and $d(H/N) < d(H)$ for every non-trivial normal subgroup N of H.*

To continue the study of the structure of these groups the following result is useful. It can be considered as a particular case of a theorem proved by Kimmerle and Williams [9, Theorem 4.3], but it is also implicit in earlier papers, for example in [2] and in [6].

LEMMA 3.9. *If H contains a non-trivial normal subgroup N all of whose chief factors are non-abelian, then $d(I_H) = \max\{2, d(I_{H/N})\}$.*

We can now prove

THEOREM 3.10. *Each group which is minimal with respect to having non-zero presentation rank is the $L_{f(L,d(L))}$ built from an L with non-abelian M, and of course each quotient group of that L/M must have presentation rank 0. Conversely, if an L has non-abelian M and is such that all quotient groups of L/M have presentation rank 0, then the $L_{f(d(L))}$ formed from that L has presentation rank 1 and all its proper quotient groups have presentation rank 0.*

PROOF. By Theorem 3.8 and Theorem 1.4, a group which is minimal with respect to having non-zero presentation rank is the $L_{f(m)}$ built from an L with non-abelian M. Of course each quotient group of L/M, being a proper quotient group of $L_{f(m)}$, must have presentation rank 0. As $L_{f(m)}/M^{f(m)} \cong L/M$, we have that

$$d(I_{L_{f(m)}}) = \max\{2, d(I_{L/M})\} = \max\{2, d(L/M)\} = d(L)$$

(by Lemma 3.9 and Theorem 1.3), while $d(L_{f(m)}) = m + 1$ (by Theorem 1.2), so $\mathrm{pr}(L_{f(m)}) = m + 1 - d(L) \geq 1$, with equality if and only if $m = d(L)$. Since $L_{f(d(L))}$ is a quotient group of $L_{f(m)}$ whenever $m \geq d(L)$, the first claim now follows.

Conversely, suppose that M is non-abelian and that all quotient groups of L/M have presentation rank 0. We noted above that $\text{pr}(L_{f(d(L))}) = 1$. A proper quotient group of $L_{f(d(L))}$ is isomorphic either to a quotient group of L/M or to an L_k with $k < f(d(L))$. In the former case it has presentation rank 0 by hypothesis, while in the latter case it has presentation rank 0 because

$$\text{pr}(L_k) = d(L_k) - d(I_{L_k}) = d(L) - \max\{2, d(L/M)\} = 0.$$

This completes the proof of the theorem.

To conclude we note that, analyzing in more detail the proof of Theorem 1.3, one notices that the number $\phi_L(m)/\phi_{L/M}(m)$ is in general quite large (for example $\phi_L(m)/\phi_{L/M}(m) \geq |M^{m-2}|$), so it should be possible to prove that $f(m)$ is also large. So we have the following informal interpretation of Theorem 3.10: a finite group with non-zero presentation rank should contain a section isomorphic to the direct product of 'many' copies of a non-abelian simple group.

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