AN AVERAGE DISTANCE RESULT
IN EUCLIDEAN n-SPACE

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The following result has been established by Stadje and Gross:

THEOREM. *If $\langle X, d \rangle$ is a compact connected metric space, then
there is a unique positive real number $a(\langle X, d \rangle)$ with the
property that for each natural number n, and for all
$\{x_1, x_2, \ldots, x_n\} \subset X$, there exists $y \in X$ for which*

$$\frac{1}{n} \sum_{i=1}^{n} d(x_i, y) = a(\langle X, d \rangle) .$$

Stadje also proved that if d is the Euclidean metric on $\mathbb{R}^2$,
and X is a compact convex subset of $\mathbb{R}^2$ then
$a(\langle X, d \rangle) \leq (D/2)\sqrt{5 - 2\sqrt{3}}$, where D is the diameter of X.
Stadje further claimed that his proof of this result generalised
to $\mathbb{R}^n$ giving the same bound, and conjectured that in $\mathbb{R}^2$,
$D/\sqrt{3}$ is an upper bound for $a(\langle X, d \rangle)$.

In this paper we show that if d is the Euclidean metric on
$\mathbb{R}^n$, and X is a compact convex subset of $\mathbb{R}^n$, then
$a(\langle X, d \rangle) \leq D\sqrt{n/(2n+2)}$ where D is the diameter of X. We
also show that if X is the k skeleton of a regular n
simplex J (that is, X is the union of all k simplices whose
vertices are vertices of J), and D is the diameter of X,

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then

$$a(\langle X,\ d\rangle) = \frac{D}{(n+1)\sqrt{2k+2}}\left((k+1)\sqrt{k}+(n-k)\sqrt{k+2}\right)\ .$$

By putting $k = n$ this shows the bound $D\sqrt{n/(2n+2)}$ is optimal.
Further, since for $n \geq 4$ it follows that $\sqrt{n/(2n+2)} \geq \tfrac{1}{2}\sqrt{5-2\sqrt{3}}$,
Stadje's claim that $(D/2)\sqrt{5-2\sqrt{3}}$ is an upper bound for
$a(\langle X,\ d\rangle)$ in $\mathbb{R}^n$ is false for $n \geq 4$.

1. A classical result

THEOREM 0. *If $\langle X, d\rangle$ is a compact connected metric space then
there is a unique positive real number $a(\langle X, d\rangle)$ with the property that
for each natural number n , and for all $\{x_1, x_2, \ldots, x_n\} \subset X$, there*

exists $y \in X$ for which $(1/n)\sum_{i=1}^{n} d(x_i, y) = a(\langle X, d\rangle)$.

For a proof of Theorem 0 see [2] or [5] (where a more general result
is proved with X being a compact connected Hausdorff space and
$d : X \times X \to \mathbb{R}$ being a continuous and symmetric function).

DEFINITION 1. Let $\langle X, d\rangle$ be a compact connected metric space. The
number $a(\langle X, d\rangle)$ defined in Theorem 0 shall be called the *Stadje number*
of $\langle X, d\rangle$. If D is the diameter of X the number $a(\langle X, d\rangle)/D$ is
called the *dispersion constant* of $\langle X, d\rangle$ and is denoted $m(X, d)$.

We now quote a classical result first published in 1901 (see [3])
which leads quickly to an upper bound for Stadje numbers for bounded closed
convex subsets of $\mathbb{R}^n$ (with Euclidean metric) which is the central purpose
of this paper.

THEOREM 1. *Let $\mathbb{R}^n$ be real Euclidean n-space, and X a bounded
subset of $\mathbb{R}^n$ containing at least two points. There exists a unique
closed ball B of smallest radius r such that $X \subseteq B$, and if D is the
diameter of X then $r \leq D\sqrt{n/(2n+2)}$.*

A simple proof of Theorem 1 appears in [1].

LEMMA 1. *Let X be a bounded closed convex subset of $\mathbb{R}^n$ containing at least two points, and let B be the closed ball of smallest radius such that $X \subseteq B$. If b is the centre of B then $b \in X$.*

Proof. This can be deduced from the results in [1], specifically Lemma 1, p. 772 and Property 1, p. 773. Alternatively a simple direct proof is as follows. Let A be a closed ball in $\mathbb{R}^n$ of centre a and radius s such that $X \subseteq A$. We will show that if $a \notin X$ and $\delta > 0$ is the distance of a from X then there is a closed ball C of radius $\sqrt{s^2 - \delta^2} < s$ such that $X \subseteq C$. Hence if the centre of a is not in X then A is not the unique ball of smallest radius containing X . Since $a \notin X$ and X is compact there is a point $c \in X$ such that $d(a, c) = \delta$. Let ac be the line in $\mathbb{R}^n$ containing a and c and let $x \in X$. By consideration of the plane containing a, c and x and Euclidean plane geometry we see that if the angle $\langle acx$ is less than $\pi/2$, since $d(a, x) \geq \delta = d(a, c)$, the point y which is the projection of a on the line cx belongs to the segment cx and thus to the convex set X . But y is the point of cx of shortest distance from a , and $y \neq c$ since $\langle acx \neq \pi/2$. Therefore $d(a, y) < d(a, c) = \delta$ which contradicts the fact that δ is the distance of a from X . Hence $\langle acx$ is greater than $\pi/2$ from which it follows, again by Euclidean plane geometry that $d(a, x) \geq \sqrt{d(c, d)^2 + d(a, c)^2} = \sqrt{d(c, x)^2 + \delta^2}$. Since $x \in X \subseteq A$, $d(a, x) \leq s$. Hence $s^2 \geq d(c, x)^2 + \delta^2$ or $d(c, x)^2 \leq s^2 - \delta^2$. Thus if we let C be the closed ball of centre c and radius $\sqrt{s^2 - \delta^2}$ then $x \in C$ or $X \subseteq C$. It is interesting to note that the reduction of the problem to Euclidean plane geometry works in the more general case where instead of $\mathbb{R}^n$ we take any inner product space (thus any real inner product space), and we take X to be a convex compact subset of the inner product space.   //

LEMMA 2. *Let X be a bounded closed convex subset of $\mathbb{R}^n$ and D be the diameter of X . Then $a(\langle X, d \rangle) \leq D\sqrt{n/(2n+2)}$, and if X contains more than a single point $m(X, d) \leq \sqrt{n/(2n+2)}$.*

Proof. If $X = \{x\}$ then trivially $a(\langle X, d \rangle) = 0$ so the result follows. If X contains more than a single point, apply Theorem 1 to find

the ball B of smallest radius r containing X . Let b be the centre of B . By Lemma 1, $b \in X$. Apply Theorem 0 to the subset $\{b\}$ of X to obtain the existence of a point $y \in X$ for which $d(b, y) = a(\langle X, d \rangle)$ But $y \in X \subseteq B$, so $d(b, y) \leq r \leq D\sqrt{n/(2n+2)}$. Hence $a(X, d) \leq D\sqrt{n/(2n+2)}$ and $m(X, d) \leq \sqrt{n/(2n+2)}$, as required.   //

In the next section we demonstrate the upper bound for Stadje numbers so obtained is optimal.

2.   Dispersion constants for skeletons of regular simplices

DEFINITION 2. Let $\{x_1, x_2, \ldots, x_{m+1}\} \subseteq \mathbb{R}^n$, where $m \leq n$ is such that $d(x_i, x_j) = D > 0$, $0 \leq i < j \leq m+1$.

The m-simplex with vertices $x_1, x_2, \ldots, x_{m+1}$ is the convex hull of the set $\{x_1, x_2, \ldots, x_{m+1}\}$, and is denoted $J(x_1, x_2, \ldots, x_{m+1})$

DEFINITION 3. If $J(x_1, x_2, \ldots, x_{m+1})$ is an m simplex in $\mathbb{R}^n$ and $1 \leq k \leq m$ then the *k-skeleton* of $J(x_1, x_2, \ldots, x_{m+1})$ is the union of all the k-simplices whose vertices are in the set $\{x_1, x_2, \ldots, x_{m+1}\}$. It is denoted $J^k(x_1, x_2, \ldots, x_{m+1})$.

For example $J(x_1, x_2, x_3)$ is a closed triangular region with

vertices x_1, x_2 and x_3 and $J^1(x_1, x_2, x_3)$ is the triangle which forms the boundary of $J(x_1, x_2, x_3)$.

DEFINITION 4. The *centroid* of $J(x_1, x_2, \ldots, x_{m+1})$ is the point $(1/(m+1))(x_1 + x_2 + \ldots + x_{m+1})$ of $J(x_1, x_2, \ldots, x_{m+1})$.

It is clear that isometric spaces have the same Stadje numbers, and that if $J(x_1, \ldots, x_{m+1})$ is an m simplex in $\mathbb{R}^n$ where $d(x_1, x_2) = D$ and $l_1, l_2, \ldots, l_{n+1}$ is an orthonormal basis for $\mathbb{R}^{n+1}$, then the mapping $\phi : J(x_1, \ldots, x_{m+1}) \to J((D/\sqrt{2})l_1, \ldots, (D/\sqrt{2})l_{m+1})$ given by

$\phi\left(\sum_{i=1}^{n} t_i x_i\right) = (D/\sqrt{2}) \sum_{i=1}^{n} t_i l_i$ is a centroid preserving isometry. Further

by restricting ϕ to $J^k(x_1, \ldots, x_{m+1})$ we see $J^k(x_1, \ldots, x_{m+1})$ and

$J^k((D/\sqrt{2})l_1, \ldots, (D/\sqrt{2})l_{m+1})$ are isometric. It is also evident by

scaling that the Stadje number of $J((D/\sqrt{2})l_1, \ldots, (D/\sqrt{2})l_{m+1})$ is $(D/\sqrt{2})$

multiplied by the Stadje number of $J^k(l_1, \ldots, l_{m+1})$. Thus we restrict

our attention to simplices $J(l_1, \ldots, l_{m+1})$ and their k-skeletons.

LEMMA 3. Let $x \in J(l_1, \ldots, l_{m+1})$ and let $f(x) = \sum_{i=1}^{m+1} d(x, l_i)$.

Then $f(x)$ is a minimum if x is the centroid of $J(l_1, \ldots, l_{m+1})$.

Proof. If $x \in J(l_1, \ldots, l_{m+1})$ then $x = \sum_{i=1}^{m} t_i l_i$ where $t_i \geq 0$

and $\sum_{i=1}^{m+1} t_i = 1$:

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{m+1} d(x, l_i) \\
&= \sum_{i=1}^{m+1} \sqrt{(x-l_i)\cdot(x-l_i)} \\
&= \sum_{i=1}^{m+1} \left[\left(\sum_{j=1}^{m+1} t_j^2\right)+1-2t_i\right]^{\frac{1}{2}}
\end{aligned}
$$

Thus we wish to minimise $\sum_{i=1}^{m} \left[\left(\sum_{j=1}^{m+1} t_j^2\right)+1-2t_i\right]^{\frac{1}{2}}$ subject to $t_i \geq 0$,

$i = 1, 2, \ldots, m+1$ and $\sum_{i=1}^{m+1} t_i = 1$. By standard techniques the minimum

is achieved for $t_1 = t_2 = \ldots = t_{m+1} = 1/(m+1)$, that is when x is the

centroid of $J(l_1, \ldots, l_{m+1})$.   //

LEMMA 4. Let $x = t_1 l_1 + t_2 l_2 + \ldots + t_{k+1} l_{k+1}$, $0 \leq t_j \leq t_1$,

$j = 2, 3, \ldots, k+1$ and $t_1 + t_2 + \ldots + t_{k+1} = 1$. *That is,*

$x \in J\left(l_1, l_2, \ldots, l_{k+1}\right)$ *and the closest vertex to* x *is* l_1. *Let* z

be the centroid of $J\left(l_{i_1}, l_{i_2}, \ldots, l_{i_{k+1}}\right)$ *where* $\left\{l_{i_1}, \ldots, l_{i_{k+1}}\right\}$ *is a*

$k + 1$ *element subset of* $\left\{l_1, \ldots, l_{n+1}\right\}$. *Then* $d\left(l_1, z\right) \geq d(x, z)$.

Proof.

$$d\left(l_1, z\right) = \sqrt{\left(z-l_1\right) \cdot \left(z-l_1\right)}$$

$$= \left[\left(\frac{1}{k+1} \sum_{j=1}^{k+1} l_{i_j} - l_1\right) \cdot \left(\frac{1}{k+1} \sum_{j=1}^{k+1} l_{i_j} - l_1\right)\right]^{\frac{1}{2}}$$

$$= \begin{cases} \sqrt{k/(k+1)}\,, & 1 \in \left\{i_1, \ldots, i_{k+1}\right\}\,, \\ \sqrt{(k+2)/(k+1)}\,, & 1 \notin \left\{i_1, \ldots, i_{k+1}\right\}\,. \end{cases}$$

Suppose $\left\{l_1, \ldots, l_{k+1}\right\} \cap \left\{l_{i_1}, \ldots, l_{i_{k+1}}\right\} = \left\{l_{i_1}, \ldots, l_{i_l}\right\}$. Then

$$d(x, z) = \left[\sum_{i=1}^{k+1} t_i^2 + \frac{1}{k+1} - \frac{2}{k+1}\left(t_{i_1} + \ldots + t_{i_l}\right)\right]^{\frac{1}{2}}.$$

Case 1. $1 \in \left\{i_1, \ldots, i_l\right\}$.

Without loss of generality put $i_1 = 1$;

$$d(x, z)^2 = \sum_{i=2}^{k+1} t_i^2 + \frac{1}{k+1} - \frac{2t_1}{k+1} + t_1^2 - \frac{2}{k+1}\left(t_{i_2} + \ldots + t_{i_l}\right)$$

$$\leq \left(\sum_{i=2}^{k+1} t_i\right)^2 + \left(t_1 - \frac{1}{k+1}\right)^2 + \frac{k}{(k+1)^2}$$

$$= \left(1-t_1\right)^2 + \left(t_1 - \frac{1}{k+1}\right)^2 + \frac{k}{(k+1)^2}$$

$$= \left(1 - \frac{1}{k+1}\right)^2 - 2\left(1-t_1\right)\left(t_1 - \frac{1}{k+1}\right) + \frac{k}{(k+1)^2}$$

$$\leq \left(1 - \frac{1}{k+1}\right)^2 + \frac{k}{k+1} \quad \left(\text{since } \frac{1}{k+1} \leq t_1 \leq 1\right)$$

$$= \frac{k}{k+1} = d\left(l_1, z\right)^2.$$

Case 2. $1 \notin \{i_1, \ldots, i_l\}$ whence $1 \notin \{i_1, \ldots, i_{k+1}\}$.

$$d(x, z)^2 = \sum_{i=1}^{k+1} t_i^2 + \frac{1}{k+1} - \frac{2}{k+1} \left(t_{i_1} + \ldots + t_{i_l}\right)$$

$$\leq \left(\sum_{i=1}^{k+1} t_i\right)^2 + \frac{1}{k+1}$$

$$= \frac{k+2}{k+1}$$

$$= d(l_1, z)^2 . \qquad //$$

LEMMA 5. $a\left(\left\langle J^k(l_1, \ldots, l_{m+1}), d\right\rangle\right) = \frac{1}{(m+1)\sqrt{k+1}} \left[(k+1)\sqrt{k}+(m-k)\sqrt{k+2}\right]$.

Proof. Write X for $J^k(l_1, \ldots, l_{m+1})$.

By Theorem 0 there is a $y \in X$ for which

$$\frac{1}{m+1} \sum_{i=1}^{m+1} d(l_i, y) = a(\langle X, d\rangle) .$$

By symmetry we may suppose $y \in J(l_1, \ldots, l_{k+1})$. Let z be the centroid of $J(l_1, \ldots, l_{k+1})$.

By Lemma 3,

(i) $$\sum_{i=1}^{k+1} d(l_i, y) \geq \sum_{i=1}^{k+1} d(l_i, z) .$$

If $k+1 < i \leq m+1$, $J(l_1, \ldots, l_{k+1}, l_i)$ is a $(k+1)$-simplex and the line zl_i is perpendicular to the space S determined by the points $\{l_1, l_2, \ldots, l_{k+1}\}$ (because

$$(l_i - z) \cdot (l_t - l_1) = \left(l_i - \frac{1}{k+1} \sum_{j=1}^{k+1} l_j\right) \cdot (l_t - l_1) = 0$$

if $0 \leq t \leq k+1$).

Thus as y is in the space S ,

(ii) $$d(l_i, z) \leq d(l_i, y) \quad \text{for} \quad k+1 < i \leq n+1 .$$

By (i) and (ii),

$$\sum_{i=1}^{m+1} d(l_i, y) \geq \sum_{i=1}^{m+1} d(l_i, z) .$$

Thus

(iii) $\qquad a(\langle X, d\rangle) \geq \dfrac{1}{m+1} \sum_{i=1}^{m+1} d(l_i, z)$

$$= \frac{1}{m+1} \left[\sum_{i=1}^{k+1} d(l_i, z) + \sum_{i=k+2}^{m+1} d(l_i, z) \right] .$$

But, as has been seen in the proof of Lemma 3,

$$d(l_i, z) = \begin{cases} \sqrt{k/(k+1)} , & 1 \leq i \leq k+1 , \\ \\ \sqrt{(k+2)/(k+1)} , & k+1 < i \leq m+1 . \end{cases}$$

Thus

(iv) $\qquad a(\langle X, d\rangle) \geq \dfrac{1}{(m+1)\sqrt{k+1}} [(k+1)\sqrt{k}+(m-k)\sqrt{k+2}] .$

Let $z_1, z_2, \ldots, z_s$ be those points of X which are centroids of k-subsimplices of X. There are ${}^{m+1}C_{k+1}$ such k-subsimplices, so $s = {}^{m+1}C_{k+1}$.

By Theorem 0, there is a $y \in X$ for which

$$\frac{1}{s} \sum_{i=1}^{s} d(z_i, y) = a(\langle X, d\rangle) .$$

By symmetry we may suppose $y \in J(l_1, \ldots, l_{k+1})$, and that y is closest to l_1 (that is, $y = t_1 l_1 + t_2 l_2 + \ldots + t_{k+1} l_{k+1}$, where $0 \leq t_j \leq t_1$ for $j = 2, 3, \ldots, k+1$ and $t_1 + t_2 + \ldots + t_{k+1} = 1$).

Now if z is the centroid of any k subsimplex of X, by Lemma 4, $d(l_1, z) \geq d(y, z)$.

Thus

$$a(\langle X, d \rangle) = \frac{1}{s} \sum_{i=1}^{s} d(z_i, y) \leq \frac{1}{s} \sum_{i=1}^{s} d(z_i, l_1) .$$

Again if z_i is the centroid of a k-subsimplex with vertex l_1, $d(z_i, l_1) = \sqrt{k/(k+1)}$, and if z_i is the centroid of a k-subsimplex which does not have vertex l_1, $d(z_i, l_1) = \sqrt{(k+2)/(k+1)}$. Further, as there are ${}^mC_k$ k-subsimplices with vertex l_1, and ${}^{m+1}C_{k+1} - {}^mC_k$ other subsimplices, it follows that

$$(v) \qquad a(\langle X, d \rangle) \leq \frac{1}{{}^{m+1}C_{k+1}} \left[{}^mC_k\sqrt{k/(k+1)} + \left({}^{m+1}C_{k+1} - {}^mC_k \right)\sqrt{(k+2)/(k+1)} \right]$$

$$= \frac{1}{(m+1)\sqrt{k+1}} \left[(k+1)\sqrt{k} + (m-k)\sqrt{k+2} \right] .$$

The lemma is established by (iv) and (v).   //

This leads to the two main results in this paper, namely:

THEOREM 2.

$$a\left[\left\langle J^k(x_1, \ldots, x_{m+1}), d \right\rangle \right] = \frac{D}{(m+1)\sqrt{2k+2}} \left[(k+1)\sqrt{k} + (m-k)\sqrt{k+2} \right]$$

and

$$m\left[J^k(x_1, \ldots, x_{m+1}), d \right] = \frac{(k+1)\sqrt{k} + (m-k)\sqrt{k+2}}{(m+1)\sqrt{2k+2}}$$

Proof. This is a trivial consequence of Lemma 5 using congruence and scaling.

THEOREM 3. *If X is a bounded closed convex subset of $\mathbb{R}^n$ and D is the diameter of X, then*

$$a(\langle X, d \rangle) \leq D\sqrt{n/(2n+2)} ,$$

and the bound is optimal.

Proof. The bound is established in Lemma 2. To see it is optimal, note that $J(x_1, \ldots, x_{n+1})$ is a bounded closed convex subset of $\mathbb{R}^n$ and $J(x_1, \ldots, x_{n+1}) = J^n(x_1, \ldots, x_{n+1})$. Thus

$$a\left(\langle J\left(x_1, \ldots, x_{n+1}\right), d\rangle\right) = D\sqrt{n/(2n+2)} \ .$$

3. Some related results

Since this paper was written, Szekeres and Szekeres [6] have independently shown that Stadje's bound is incorrect, and have found Stadje numbers for all compact convex subsets of a normed vector space with real or complex scalars. Their paper "The average distance theorem for compact convex regions" will also appear in the *Bulletin of the Australian Mathematical Society*, as will a related paper "Average distances in compact connected spaces" by Yost [8].

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