EXTENSION OF PARTIAL ENDOMORPHISMS OF ABELIAN GROUPS

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It is known [1] that for a partial endomorphism μ of a group G that maps the subgroup $A \subseteq G$ onto $B \subseteq G$ to be extendable to a total endomorphism μ^* of a supergroup $G^* \supseteq G$ such that μ^* is an isomorphism on $G^*(\mu^*)^m$ for some positive integer m, it is necessary and sufficient that there exist in G a sequence of normal subgroups

$$L_1 \subset L_2 \subset \ldots \subset L_m = L_{m+1} = \ldots$$

such that $L_1 \cap A$ is the kernel of μ and

$$(L_{i+1} \cap A)\mu = L_i \cap B,$$

for $i = 1, 2, \ldots, m-1$.

The question then arises whether these conditions could be simplified when the group G is abelian. In this paper it is shown not only that the conditions are simplified when G is abelian but also that the extension group $G^* \supseteq G$ can be chosen as an abelian group.

In §2 the result obtained for a single partial endomorphism is generalised for any finite number of them.

Instead of the free products with one amalgamated subgroup that are used in the general case, we here use direct products with an amalgamated subgroup.

1. Single extension. Let G be a given abelian group and μ be a partial endomorphism of G that maps the subgroup $A \subseteq G$ onto the subgroup $B \subseteq G$. If n is a given positive integer, then we prove the following result.

THEOREM 1. *A necessary and sufficient condition for μ to be extendable to a total endomorphism μ^* of an abelian supergroup $G^* \supseteq G$, such that μ^* is an isomorphism on $G^*(\mu^*)^n$, is that, whenever $x\mu^{n+1}$ is defined and $x\mu^{n+1} = 1$, then $x\mu^n = 1$.*

Proof. The necessity of the condition is obvious, for if the required extension is already established and $x \in A$, where $x\mu^{n+1}$ is defined, then, since μ^* extends μ, we get

$$x\mu^{*n} = x\mu^n \quad \text{and} \quad x\mu^{*n+1} = x\mu^{n+1}.$$

But μ^* is an isomorphism on the group $G^*(\mu^*)^n$; thus

$$x\mu^{*n+1} = 1 \quad \text{implies} \quad x\mu^{*n} = 1,$$

and from this in turn it follows that

$$x\mu^{n+1} = 1 \quad \text{implies} \quad x\mu^n = 1.$$

To prove the sufficiency of the condition, we assume that K is the kernel of the homomorphism μ and put $H=G/K$, denoting by μ_1 the natural mapping of G onto H. The homomorphism μ of A onto B induces an isomorphism θ of A/K onto B. We now form the direct product of G and H amalgamating $B \subseteq G$ with $A/K \subseteq H$ according to θ, denoting this product by

$$G_1 = \{G \times H\,;\, B = A/K\}.$$

μ_1 and μ can now be regarded as partial endomorphisms of G_1, mapping G onto H and A onto $B = A/K$ respectively; clearly μ_1 extends μ.

In G_1 we prove the following lemma.

LEMMA. *If $x\mu_1^{n+1}$ is defined and $x\mu_1^{n+1} = 1$, then $x\mu_1^n = 1$.*

Proof. $x\mu_1 \in H$. Since $x\mu_1^2$ is defined, then $x\mu_1 \in G$ and hence

$$x\mu_1 \in G \cap H = B.$$

Thus there exists $a \in A$ such that

$$x\mu_1 = a\mu. \tag{1.1}$$

Similarly, since $x\mu_1^3$ is defined, then

$$x\mu_1^2 = a_1\mu$$

for some $a_1 \in A$. But also

$$x\mu_1^2 = (a\mu)\mu_1 = a_1\mu = a_1\mu_1,$$

since μ_1 extends μ. Thus

$$(a\mu)a_1^{-1} \in K \subset A\,;$$

hence $a\mu \in A$ and relation (1.1) gives (again because μ_1 extends μ)

$$x\mu_1^2 = a\mu^2.$$

By induction we can show that when $x\mu_1^{n+1}$ is defined, then

$$x\mu_1^n = a\mu^n.$$

If $x\mu_1^{n+1} = 1$, then the map of x by higher powers of μ_1 is defined and

$$x\mu_1^{n+1} = a\mu^{n+1} = 1.$$

Thus $a\mu^n = 1$, which in turn implies that $x\mu_1^n = 1$.

This completes the proof of the lemma.

Thus we have embedded G in the abelian group G_1 which possesses a partial endomorphism μ_1 extending μ such that whenever $x\mu_1^{n+1}$ is defined and $x\mu_1^{n+1} = 1$, then $x\mu_1^n = 1$.

We define a sequence of such groups inductively as follows. When, for a positive integer j, the abelian group G_j is defined with a partial endomorphism μ_j mapping the subgroup $G_{j-1} \subseteq G_j$ onto $H_{j-1} \subseteq G_j$ such that μ_j extends each μ_i, $i<j$, and $x\mu_j^{n+1} = 1$ (when the left-hand side is defined) implies $x\mu_j^n = 1$, we define G_{j+1} to be the direct product

$$G_{j+1} = \{G_j \times H_j \,;\, H_{j-1} = G_{j-1}/K\},$$

where $H_j = G_j/K$.

If we define μ_{j+1} to be the natural mapping of G_j onto H_j, then, as shown in the previous lemma, when $x\mu_{j+1}^{n+1}$ is defined and equals 1, we have $x\mu_{j+1}^{n} = 1$.

We form the group

$$G^* = \bigcup_{j=1}^{\infty} G_j.$$

G^* is evidently abelian. Define μ^* as follows. For any $g^* \in G^*$, that is to say, $g^* \in G_k$ for some suitable positive integer k, we put

$$g^*\mu^* = g^*\mu_k.$$

μ^* thus defined is an endomorphism of G^* which extends μ. Moreover, for any $g^* \in G^*$,

$$g^*\mu^{*n+1} = g\mu_j^{\,n+1}$$

for some integer j. Thus, if $g^*\mu^{*n+1} = 1$, then $g^*\mu_j^{n+1} = 1$, which implies $g^*\mu_j^n = 1$ and hence $g^*\mu^{*n} = 1$.

This completes the proof that μ^* is an isomorphism on the abelian group $G^*(\mu^*)^n$.

2. Simultaneous extensions. In this section we derive necessary and sufficient conditions for the simultaneous extension of two partial endomorphisms of an abelian group to total endomorphisms of one and the same abelian group with similar conditions imposed on them. The result of Theorem 2 generalises in an obvious manner to any finite number of partial endomorphisms.

THEOREM 2. *Let μ and ν be two partial endomorphisms of an abelian group G which map A onto B and C onto D respectively, A, B, C, D being subgroups of G. For μ and ν to be extendable to total endomorphisms μ^* and ν^* of one and the same abelian group $G^* \supseteq G$, such that, for given positive integers m and n, μ^* is an isomorphism on $G^*(\mu^*)^m$ and ν^* is an isomorphism on $G^*(\nu^*)^n$, it is necessary and sufficient that*

(2.1) *whenever* $x\mu^{m+1}$ *is defined, then* $x\mu^{m+1} = 1$ *implies* $x\mu^m = 1$;

(2.2) *whenever* $y\nu^{n+1}$ *is defined, then* $y\nu^{n+1} = 1$ *implies* $y\nu^n = 1$.

Proof. The necessity of the conditions is immediate. Assume that K_μ, K_ν are the kernels of μ, ν respectively. Let μ_1 be the natural mapping of G onto G/K_μ and form the direct product G_1 of G and G/K_μ amalgamating B with A/K_μ according to μ_1:

$$G_1 = \{G \times G/K_\mu \,;\, B = A/K_\mu\}.$$

G_1 is abelian and contains

(1) the subgroup G mapped by μ_1 onto G/K_μ with μ_1 satisfying the condition corresponding to (2.1),

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(2) the subgroup C mapped by $\nu_1 = \nu$ onto D.

We repeat the process embedding G_1 in G_2, this time leaving $\mu_1 = \mu_2$ as it is and extending ν_1 to ν_2, which satisfies the condition corresponding to (2.2).

We carry on inductively and form the group

$$G^* = \bigcup_{j=1}^{\infty} G_j ,$$

which is abelian. Define μ^* and ν^* in the following manner. For any $g^* \in G^*$, that is, $g^* \in G_k$ for some suitable k, we put

$$g^*\mu^* = g^*\mu_k,$$
$$g^*\nu^* = g^*\nu_k.$$

As in Theorem 1 we can show that μ^* and ν^* are total endomorphisms of G^* which satisfy the required conditions.

REFERENCE

1. C. G. Chehata, An embedding theorem for groups, *Proc. Glasgow Math. Assoc.* **4** (1960), 140–143.

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