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ORIGINAL RESEARCH PAPER

Unbiased estimator for the ultimate claim prediction error in the chain-ladder model of Mack

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Abstract

We propose a new estimator for the ultimate prediction uncertainty within the famous Mack's distribution-free chain-ladder model, which can be proved to be unbiased (conditionally given the first triangle column) under some additional technical assumptions. A peculiar behaviour of the unbiased estimator is given by its possible negativity. This is a drawback which might be worth trading off for the unbiasedness property, since there is empirical evidence that the likelihood of a negative realisation is extremely low. This offers an alternative to the well-known Mack and BBMW formulas since the latters can be proved to be biased. However, we also show that this novel estimator, as well as the Mack and BBMW formulas, can (with non-negligible probability) materially fail to estimate the true uncertainty.

Keywords: Claims reserving; Distribution-free chain-ladder model; Conditional mean square error of prediction; Ultimate prediction uncertainty

1. Introduction

The chain-ladder algorithm to compute the unpaid claim requirement in an insurance financial statement is the most well-known reservation methodology in actuarial practice. This algorithm has been widely examined by many researchers in the last three decades, such as Kremer (1982), Taylor (1986, 2000), Renshaw (1989), Verrall (1990), Mack (1993), Murphy (1994), Schmidt & Schnaus (1996), Barnett & Zehnwirth (1998), England & Verrall (1999) and Wüthrich & Merz (2008).

In particular, considerable research related to the well-known distribution-free chain-ladder model by Mack has been performed. In terms of the prediction uncertainty estimation within Mack's model, the most relevant contributions include those of Mack (1993), Buchwalder *et al.*, (2006), Mack *et al.*, (2006), Gisler (2006, 2019, 2020), Merz & Wüthrich (2008, 2014), Röhr (2016), England *et al.*, (2019), Diers *et al.*, (2016) and Lindholm *et al.*, (2020).

In 1993, Mack derived an estimator (Mack formula) to quantify the ultimate prediction uncertainty within his model. Later, in 2006, certain controversial discussions occurred (see Mack *et al.*, 2006; Gisler 2006) when Buchwalder *et al.*, (2006) proposed a novel estimator (BBMW formula); however, the question related to which of the two formulas should be preferred went unanswered. Recently, Gisler (2020) stated that the Mack formula should be preferred over the BBMW formula. Notably, Gisler (2020) highlighted several deficiencies related to the BBMW estimator, linked with the conditional resampling approach, which was adopted in Buchwalder *et al.*, (2006) to derive the estimator for the second term of the conditional mean squared error of prediction (MSEP), usually known as the estimation error. Moreover, Gisler (2020) proved that for individual accident years, the Mack estimator of the estimation error does (on average, given partial information) overestimates the true prediction uncertainty, albeit to a smaller extent than the BBMW estimator.

However, Gisler (2019) previously demonstrated that the Mack formula can be derived by applying a certain estimation principle, that is unfortunately, as mentioned in the original article, not fully well-defined and must be applied with caution especially when non-linear functions in the unknown parameters need to be estimated.

Considering these aspects and principally the fact that both Mack and BBMW formulas result to be biased (as demonstrated in Gisler 2020), in this paper, we propose a new estimator for the ultimate prediction uncertainty within the famous Mack's distribution-free chain-ladder model, which under some additional technical assumptions can be proved to be unbiased.

Unluckily, the new unbiased estimator does show some peculiar behaviours, in particular with respect to its possible negativity. This is a drawback which might be worth trading off for the unbiasedness property, since there is empirical evidence that the likelihood of a negative realisation is extremely low.

Organisation of the paper. In section 2, we specify the model assumptions of the Mack chain-ladder model and recall the traditional notation as well as some key results. Moreover, we define the true value of the ultimate claim prediction uncertainty for both single accident years and for the total over all accident years.

In section 3, we propose a novel formula which can be used as an estimator for the true uncertainty and can be demonstrated to be conditionally unbiased given the first triangle column.

In section 4, we consider several numerical examples to compare the results from the different formulas and demonstrate that both the Mack and BBMW formulas as well as the new formula can (with non-negligible probability) materially fail to predict the true uncertainty.

2. Mack Model

As usual in claims reserving, we denote by $C_{i,j} > 0$ the cumulative claim figures from accident years $i \in \{0, \ldots, I\}$ at the end of development years $j \in \{0, \ldots, J\}$, $J \leq I$ and we assume that the claims are fully developed at the end of development year J.

For time $t \in \{0, \ldots, I + J\}$, $\mathcal{D}_t$ indicates the cumulative claims payment data $(C_{i,j})$ up to time t. For instance, at time I we have

$$\mathcal{D}_I = \left\{ C_{i,j} : i = 0, \ldots, I,\ i + j \leq I \right\}. \tag{1}$$

In 1993, the following distribution-free stochastic model underlying the chain-ladder reserving method was introduced by Mack (1993).

Model Assumptions 1 (Mack model).

- *Vectors* $(C_{i,0}, \ldots, C_{i,J})$, $i \in \{0, \ldots, I\}$ *are independent*
- *There exist positive parameters* $f_0, \ldots, f_{J-1}$ *and* $\sigma_0^2, \ldots, \sigma_{J-1}^2$ *such that for all* $i \in \{0, \ldots, I\}$ *and all* $j \in \{0, \ldots, J-1\}$

$$E[C_{i,j+1} | C_{i,0}, \ldots, C_{i,j}] = f_j C_{i,j}, \tag{2}$$

$$Var(C_{i,j+1} | C_{i,0}, \ldots, C_{i,j}) = \sigma_j^2 C_{i,j}. \tag{3}$$

Within Mack's model, the unknown parameters $f_0, \ldots, f_{J-1}$ and $\sigma_0^2, \ldots, \sigma_{J-1}^2$ are estimated at time I by using the following, conditionally given $\mathcal{B}_j = \{C_{i,k}; i + k \leq I, k \leq j\} \subset \mathcal{D}_I$, unbiased estimators

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$$\widehat{f_j} = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}}, \quad j \in \{0, \ldots, J-1\}, \tag{4}$$

$$\widehat{\sigma_j^2} = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{f_j} \right)^2, \quad j \in \{0, \ldots, J-1\}, \tag{5}$$

where, for $J = I$, σ_{J-1}^2 is usually estimated through extrapolation as

$$\widehat{\sigma_{J-1}^2} = \min \left(\widehat{\sigma_{J-3}^2}, \widehat{\sigma_{J-2}^2}, \frac{\left(\widehat{\sigma_{J-2}^2}\right)^2}{\widehat{\sigma_{J-3}^2}} \right). \tag{6}$$

The ultimate claim $C_{i,J}$ for accident year $i \in \{0, \ldots, I\}$ is at time I predicted by

$$\widehat{C}_{i,J} = C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f_j}. \tag{7}$$

Predictor (7) is, conditionally given $\mathcal{B}_{I-i}$, unbiased for $E\left[C_{i,J} \middle| \mathcal{B}_{I-i}\right] = E\left[C_{i,J} \middle| \mathcal{D}_I\right]$, i.e. we have

$$E\left[\widehat{C}_{i,J} \middle| \mathcal{B}_{I-i}\right] = E\left[C_{i,J} \middle| \mathcal{B}_{I-i}\right]. \tag{8}$$

Moreover, the ultimate claim $\sum_{i=0}^{I} C_{i,J}$ for the total over all accident years is at time I predicted by

$$\sum_{i=0}^{I} \widehat{C}_{i,J}. \tag{9}$$

Predictor (9) is, conditionally given $\mathcal{B}_0$, unbiased for $E\left[\sum_{i=0}^{I} C_{i,J} \middle| \mathcal{B}_0\right]$, i.e. we have

$$E\left[\sum_{i=0}^{I} \widehat{C}_{i,J} \middle| \mathcal{B}_0\right] = E\left[\sum_{i=0}^{I} C_{i,J} \middle| \mathcal{B}_0\right]. \tag{10}$$

Furthermore, Mack (1993) also demonstrated that it holds true

$$E\left[\left(\widehat{f_j}\right)^2 \middle| \mathcal{B}_j\right] = f_j^2 + \frac{\sigma_j^2}{\sum_{h=0}^{I-j-1} C_{h,j}}, \quad j \in \{0, \ldots, J-1\}. \tag{11}$$

2.1 Quantification of the ultimate claim prediction error

The objective is to quantify the ultimate claim prediction uncertainty at time I for both single and aggregated accident years. In a distribution-free framework, this is usually done by considering the following quantities of interest which will be also referred to as the true values, because they measure, with respect to the squared loss function, the expected deviation between the unknown ultimate claim amounts ($C_{i,J}$ and $\sum_{i=0}^{I} C_{i,J}$ respectively) and their predicted amounts ($\widehat{C}_{i,J}$ and $\sum_{i=0}^{I} \widehat{C}_{i,J}$, respectively) at time I.

Definition 1 (Conditional MSEP for single accident years). The conditional MSEP of the ultimate claim predictor for single accident year i is defined as

$$\mathrm{msep}_{C_{i,J}|\mathcal{D}_I}\left(\widehat{C}_{i,J}\right) = E\left[\left(C_{i,J} - \widehat{C}_{i,J}\right)^2 \middle| \mathcal{D}_I\right]. \tag{12}$$

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The true value (12) can be decomposed as follows:

$$\begin{aligned}\text{msep}_{C_{i,J}|\mathcal{D}_I}\left(\widehat{C}_{i,J}\right) &= Var\left(C_{i,J}\middle|\mathcal{D}_I\right) + \left(\widehat{C}_{i,J} - E\left[C_{i,J}\middle|\mathcal{D}_I\right]\right)^2 \\ &= \underbrace{Var\left(C_{i,J}\middle|\mathcal{D}_I\right)}_{\text{PV}_i} + \underbrace{C_{i,I-i}^2\left(\prod_{j=I-i}^{J-1}\widehat{f}_j - \prod_{j=I-i}^{J-1} f_j\right)^2}_{\text{EE}_i}, \end{aligned} \tag{13}$$

where PV_i denotes the process variance and EE_i the estimation error of an individual accident year i.

Definition 2 (Conditional MSEP for aggregated accident years). The conditional MSEP of the ultimate claim predictor for aggregated accident years is defined as

$$\text{msep}_{\sum_{i=0}^{I} C_{i,J}|\mathcal{D}_I}\left(\sum_{i=0}^{I}\widehat{C}_{i,J}\right) = E\left[\left(\sum_{i=0}^{I}\left(C_{i,J} - \widehat{C}_{i,J}\right)\right)^2 \middle| \mathcal{D}_I\right]. \tag{14}$$

The true value (14) can be decomposed as follows:

$$\begin{aligned}\text{msep}_{\sum_{i=0}^{I} C_{i,J}|\mathcal{D}_I}\left(\sum_{i=0}^{I}\widehat{C}_{i,J}\right) &= Var\left(\sum_{i=0}^{I} C_{i,J}\middle|\mathcal{D}_I\right) + \left(\sum_{i=0}^{I}\widehat{C}_{i,J} - E\left[\sum_{i=0}^{I} C_{i,J}\middle|\mathcal{D}_I\right]\right)^2 \\ &= \underbrace{\sum_{i=0}^{I} Var\left(C_{i,J}\middle|\mathcal{D}_I\right)}_{\sum_{i=0}^{I}\text{PV}_i=\text{PV}_{\text{tot}}} + \underbrace{\left(\sum_{i=0}^{I} C_{i,I-i}\left(\prod_{j=I-i}^{J-1}\widehat{f}_j - \prod_{j=I-i}^{J-1} f_j\right)\right)^2}_{\text{EE}_{\text{tot}}}, \end{aligned} \tag{15}$$

where PV_{tot} denotes the process variance for the total over all accident years and EE_{tot} the estimation error for the total over all accident years.

The process variance PV_i can be easily calculated. By iteration, the Model Assumptions 1 indicate that (see Mack, 1993)

$$\text{PV}_i = C_{i,I-i}\sum_{k=I-i}^{J-1}\left(\prod_{m=I-i}^{k-1} f_m\right)\sigma_k^2\left(\prod_{n=k+1}^{J-1} f_n^2\right), \quad i \in \{0, \ldots, I\}. \tag{16}$$

On the other side, the estimation errors (for single accident year i and for the total over all accident years) can be easily expressed as

$$\text{EE}_i = C_{i,I-i}^2\left(\prod_{k=I-i}^{J-1}\left(\widehat{f}_k\right)^2 + \prod_{k=I-i}^{J-1} f_k^2 - 2\prod_{k=I-i}^{J-1} f_k\widehat{f}_k\right), \quad i \in \{0, \ldots, I\}, \tag{17}$$

$$\text{EE}_{\text{tot}} = \sum_{i=1}^{I}\text{EE}_i + 2\sum_{1\le i<j\le I} C_{i,I-i}C_{j,I-j}\left(\prod_{k=I-i}^{J-1}\widehat{f}_k - \prod_{k=I-i}^{J-1} f_k\right)\left(\prod_{k=I-j}^{J-1}\widehat{f}_k - \prod_{k=I-j}^{J-1} f_k\right). \tag{18}$$

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3. The New Formula

Since the model parameter (f_j) and (σ_j^2) are unknown, the true value (12) given by $\mathrm{PV}_i + \mathrm{EE}_i$ cannot be evaluated. For the process variance, the traditional approach in actuarial literature considers the estimator (19) for PV_i based on the data $\mathcal{D}_I$ (see Mack, 1993):

$$\widehat{\mathrm{PV}_i}^{\mathrm{Mack}} = \sum_{k=I-i}^{J-1} C_{i,I-i} \left(\prod_{m=I-i}^{k-1} \widehat{f}_m \right) \widehat{\sigma_k^2} \prod_{n=k+1}^{J-1} \left(\widehat{f}_n\right)^2. \tag{19}$$

But unfortunately, this proposal does not result to be, conditionally given $\mathcal{B}_{I-i}$, unbiased for $E\left[\mathrm{PV}_i \middle| \mathcal{B}_{I-i}\right]$.

On the other hand, for the estimation error, the traditional approach focus on estimating $E\left[\mathrm{EE}_i \middle| \mathcal{B}_{I-i}\right]$ based on the data $\mathcal{D}_I$, since directly estimating the positive term EE_i based on the data $\mathcal{D}_I$ leads to a degenerated estimator (namely 0).

However, the until now known proposals (see Mack, 1993 and Buchwalder *et al.*, 2006) do not result to be, conditionally given $\mathcal{B}_{I-i}$, unbiased for $E\left[\mathrm{EE}_i \middle| \mathcal{B}_{I-i}\right]$ (see Gisler, 2020).

3.1 Single accident years

In this paper, our goal is to derive estimators $\widehat{\mathrm{PV}_i}^{\mathrm{NEW}}$ and $\widehat{\mathrm{EE}_i}^{\mathrm{NEW}}$ that fulfil the properties

$$E\left[\widehat{\mathrm{PV}_i}^{\mathrm{NEW}} \middle| \mathcal{B}_{I-i}\right] = E\left[\mathrm{PV}_i \middle| \mathcal{B}_{I-i}\right] < \infty, \tag{20}$$

and

$$E\left[\widehat{\mathrm{EE}_i}^{\mathrm{NEW}} \middle| \mathcal{B}_{I-i}\right] = E\left[\mathrm{EE}_i \middle| \mathcal{B}_{I-i}\right] < \infty, \tag{21}$$

i.e. estimators which result to be, conditionally given $\mathcal{B}_{I-i}$, unbiased for $E\left[\mathrm{PV}_i \middle| \mathcal{B}_{I-i}\right]$ and $E\left[\mathrm{EE}_i \middle| \mathcal{B}_{I-i}\right]$ respectively.

Let us first present a, conditionally given $\mathcal{B}_{I-i}$, unbiased estimator for $\prod_{k=I-i}^{J-1} f_k^2$.

We have the following Theorem.

Theorem 1. *Provided* $E\left[\left| \prod_{j=I-i}^{J-1} \left(\left(\widehat{f}_j\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}} \right) \right|\right] < \infty$, *under Model Assumptions 1 (with* $J < I$*) we have that the estimator*

$$\prod_{j=I-i}^{J-1} \left(\left(\widehat{f}_j\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}} \right), \qquad i \in \{I-J+1, \ldots, I\}, \tag{22}$$

is, conditionally given $\mathcal{B}_{I-i}$*, unbiased for* $\prod_{j=I-i}^{J-1} f_j^2$.

Proof. From result (11) as well as the conditionally unbiasedness of the parameter estimates $(\widehat{\sigma_j^2})$, we get

$$\infty > E\left[\prod_{j=I-i}^{J-1} \left(\left(\widehat{f}_j\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}} \right) \middle| \mathcal{B}_{I-i} \right]$$

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$$= E\left[E\left[\prod_{j=I-i}^{J-1}\left(\left(\widehat{f_j}\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}}\right)\Bigg|\mathcal{B}_{J-1}\right]\Bigg|\mathcal{B}_{I-i}\right]$$

$$= E\left[\prod_{j=I-i}^{J-2}\left(\left(\widehat{f_j}\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}}\right) E\left[\left(\widehat{f}_{J-1}\right)^2 - \frac{\widehat{\sigma_{J-1}^2}}{\sum_{k=0}^{I-J} C_{k,J-1}}\Bigg|\mathcal{B}_{J-1}\right]\Bigg|\mathcal{B}_{I-i}\right]$$

$$= E\left[\prod_{j=I-i}^{J-2}\left(\left(\widehat{f_j}\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}}\right)\Bigg|\mathcal{B}_{I-i}\right] f_{J-1}^2 = \ldots = \prod_{j=I-i}^{J-1} f_j^2, \quad i \in \{I-J+1, \ldots, I\}.$$

□

In order to guarantee the existence of the involved quantities, in the following we will work under the additional technical assumptions

$$E\left[\prod_{j=I-i}^{J-1}\left(\widehat{f_j}\right)^2\right] < \infty, \quad i \in \{I-J+1, \ldots, I\}, \tag{23}$$

and

$$E\left[\left|\prod_{j=I-i}^{J-1}\left(\left(\widehat{f_j}\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}}\right)\right|\right] < \infty, \quad i \in \{I-J+1, \ldots, I\}. \tag{24}$$

Remark 1. As not uncommonly, the technical assumptions (23) and (24) are unverifiable. However, they can be reasonably assumed to hold true, since they are consistent with data typically encountered in actuarial practice. Moreover, we underline that the core technical requirement for the unbiasedness result stated in Theorem 1 is the technical assumption (24) and not the positivity of estimator (22) along all the possible trajectories.

Under Model Assumptions 1 and the technical assumption (24), there could exist trajectories of the underlying claims payments process for which the realisations of estimator (22) are negative. This is a drawback of the unbiased estimator (22) but not a probabilistic issue, since estimators are random variables, and their realisations may even significantly differ from the quantity they intend to estimate. However, for typical general insurance data, the unbiased estimator (22) results to be positive. This is due to the fact that for typical data, the term $\left(\widehat{f_j}\right)^2$ dominates $\frac{\widehat{\sigma_j^2}}{\sum_{h=0}^{I-j-1} C_{h,j}}$. This is also the argument that justifies the common use of the first-order Taylor approximations in this domain.

Also note that the positivity of estimator (22) follows from the following regularity condition.

Regularity Condition.

$$\sum_{i=0}^{I-j-1} C_{i,j}\left(I-j-1\right) > \sum_{i=0}^{I-j-1} C_{i,j}\left(\frac{C_{i,j+1}/C_{i,j}}{\widehat{f_j}} - 1\right)^2, \quad j \in \{0, \ldots, J-1 < I-1\}. \tag{25}$$

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Display (25) further highlights that for typical data fulfilling Model Assumptions 1, the unbiased estimator (22) results to be positive.

Remark 2. Given data $\mathcal{D}_I$ for which Model Assumptions 1 can be considered appropriate and for sufficiently large data availability (i.e. $I-J$ sufficiently large), the regularity condition (25) is fulfilled. Also note that, when considering a positive set $\mathcal{B}_0$ and when sufficiently restricting the domain of the model parameters, with the help of the time series model described in Merz & Wüthrich (2008) (see Model Assumptions 3.9), all the considered triangles in our simulation studies did fulfil the regularity condition (25). This gives empirical evidence that the likelihood of a negative realisation of the unbiased estimator (22) is extremely low.

Remark 3. For generally excluding the negativity eventuality, we could have considered the positive estimator for $\prod_{j=I-i}^{J-1} f_j^2$ given by

$$\prod_{j=I-i}^{J-1}\left[\left(\left(\widehat{f_j}\right)^2-\frac{\widehat{\sigma_j^2}}{\sum_{h=0}^{I-j-1}C_{h,j}}\right)1_{\left\{\left(\widehat{f_j}\right)^2>\frac{\widehat{\sigma_j^2}}{\sum_{h=0}^{I-j-1}C_{h,j}}\right\}}+\left(\widehat{f_j}\right)^2 1_{\left\{\left(\widehat{f_j}\right)^2\leq\frac{\widehat{\sigma_j^2}}{\sum_{h=0}^{I-j-1}C_{h,j}}\right\}}\right]. \tag{26}$$

Essentially, this estimator coincides, conditional on the data $\mathcal{D}_I$, either with estimator (22) or with the classical upwards biased estimator for $\prod_{j=I-i}^{J-1} f_j^2$ given by $\prod_{j=I-i}^{J-1}\left(\widehat{f_j}\right)^2$. However, for estimator (26) we would not be able to prove the conditional unbiasedness property as stated in Theorem 1 for estimator (22). Moreover, note that estimators (26) and (22) solely differ on an event with an extremely low empirical likelihood.

Consequently, it could be worth trading off the peculiar behaviour of estimator (22) for its unbiasedness property.

Now note that from Theorem 1 by successive conditioning on $\mathcal{B}_{k+1}$, $\mathcal{B}_k$ and $\mathcal{B}_{I-i}$ and looking at (16), we have that the estimator

$$\widehat{\mathrm{PV}}_i^{\mathrm{NEW}}=\sum_{k=I-i}^{J-1}C_{i,I-i}\left(\prod_{m=I-i}^{k-1}\widehat{f_m}\right)\widehat{\sigma_k^2}\prod_{n=k+1}^{J-1}\left[\left(\widehat{f_n}\right)^2-\frac{\widehat{\sigma_n^2}}{\sum_{h=0}^{I-n-1}C_{h,n}}\right], \tag{27}$$

fulfils the desired property (20).

On the other hand, provided $E\left[\prod_{k=I-i}^{J-1}\left(\widehat{f_k}\right)^2\right]<\infty$, from (17) using the conditional unbiasedness and uncorrelatedness of the parameter estimates $\widehat{f_j}$ we have

$$\begin{aligned}\infty > E\left[\mathrm{EE}_i\middle|\mathcal{B}_{I-i}\right]&=C_{i,I-i}^2E\left[\left(\prod_{k=I-i}^{J-1}f_k^2-2\prod_{k=I-i}^{J-1}f_k\widehat{f_k}+\prod_{k=I-i}^{J-1}\left(\widehat{f_k}\right)^2\right)\middle|\mathcal{B}_{I-i}\right]\\&=C_{i,I-i}^2\left(E\left[\prod_{k=I-i}^{J-1}\left(\widehat{f_k}\right)^2\middle|\mathcal{B}_{I-i}\right]-\prod_{k=I-i}^{J-1}f_k^2\right).\end{aligned} \tag{28}$$

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Therefore, from Theorem 1 it easily follows that the estimator

$$\widehat{\mathrm{EE}}_i^{\mathrm{NEW}} = C_{i,I-i}^2 \left(\prod_{k=I-i}^{J-1} \left(\widehat{f}_k\right)^2 - \prod_{k=I-i}^{J-1} \left[\left(\widehat{f}_k\right)^2 - \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] \right), \tag{29}$$

fulfils the desired property (21). Moreover, when for given data $\mathcal{D}_I$ the regularity condition (25) is fulfilled, estimator (29) is positive too.

Combining estimators (27) and (29) yields a novel estimator for (12):

Estimator 1 (Unbiased formula, single accident years). Under Model Assumptions 1, when for given data $\mathcal{D}_I$ the regularity condition (25) is fulfilled, we have the following positive estimator for $PV_i + EE_i$ at time I:

$$\widehat{PV}_i^{NEW} + \widehat{EE}_i^{NEW} = C_{i,I-i} \sum_{k=I-i}^{J-1} \left(\prod_{m=I-i}^{k-1} \widehat{f}_m \right) \widehat{\sigma_k^2} \prod_{n=k+1}^{J-1} \left[\left(\widehat{f}_n\right)^2 - \frac{\widehat{\sigma_n^2}}{\sum_{h=0}^{I-n-1} C_{h,n}} \right]$$
$$+ C_{i,I-i}^2 \left(\prod_{k=I-i}^{J-1} \left(\widehat{f}_k\right)^2 - \prod_{k=I-i}^{J-1} \left[\left(\widehat{f}_k\right)^2 - \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] \right). \tag{30}$$

Remark 4. We underline that (30) is a conditionally given $\mathcal{B}_{I-i}$ unbiased estimator for $E\left[PV_i + EE_i \middle| \mathcal{B}_{I-i}\right]$ under Model Assumptions 1 (with $J < I$) and the additional assumptions $E\left[\prod_{k=I-i}^{J-1} \left(\widehat{f}_k\right)^2\right] < \infty$ and $E\left[\left|\prod_{j=I-i}^{J-1} \left(\left(\widehat{f}_j\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}}\right)\right|\right] < \infty$. Moreover, when for given data $\mathcal{D}_I$ the regularity condition (25) is fulfilled, estimator (30) is positive.

3.2 Aggregated accident years

Our goal is to derive estimators $\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{NEW}}$ and $\widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{NEW}}$ which result to be, conditionally given $\mathcal{B}_0$, unbiased for $E\left[\mathrm{PV}_{\mathrm{tot}} \middle| \mathcal{B}_0\right]$ and $E\left[\mathrm{EE}_{\mathrm{tot}} \middle| \mathcal{B}_0\right]$ respectively.

Let us consider the following proposals:

$$\widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{NEW}} = \sum_{i=1}^{I} \widehat{\mathrm{EE}}_i^{\mathrm{NEW}} + 2 \sum_{1 \le i < j \le I} C_{i,I-i} \left(C_{j,I-j} \prod_{k=I-j}^{I-i-1} \widehat{f}_k \right) \frac{\widehat{\mathrm{EE}}_i^{\mathrm{NEW}}}{C_{i,I-i}^2}. \tag{31}$$

$$\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{NEW}} = \sum_{i=1}^{I} \widehat{\mathrm{PV}}_i^{\mathrm{NEW}}. \tag{32}$$

The following Theorem holds true.

Theorem 2. *Provided* $E\left[\prod_{j=I-i}^{J-1} \left(\widehat{f}_j\right)^2\right] < \infty$ *and* $E\left[\left|\prod_{j=I-i}^{J-1} \left(\left(\widehat{f}_j\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}}\right)\right|\right] < \infty$, $\forall i \in \{I - J + 1, \ldots, I\}$, *under Model Assumptions 1 (with* $J < I$*) we have:*

$$E\left[\widehat{PV_{tot}}^{NEW} \middle| \mathcal{B}_0\right] = E\left[PV_{tot} \middle| \mathcal{B}_0\right] < \infty, \tag{33}$$

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and

$$E\left[\widehat{EE_{tot}}^{NEW}\middle|\mathcal{B}_0\right] = E\left[EE_{tot}\middle|\mathcal{B}_0\right] < \infty. \tag{34}$$

Proof. It holds that

$$E\left[\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{NEW}}\middle|\mathcal{B}_0\right] = E\left[\sum_{i=1}^{I} E\left[\widehat{\mathrm{PV}_i}^{\mathrm{NEW}}\middle|\mathcal{B}_{I-i}\right]\middle|\mathcal{B}_0\right] = E\left[\mathrm{PV}_{\mathrm{tot}}\middle|\mathcal{B}_0\right] < \infty.$$

This proves (33).

Furthermore, it holds that

$$\begin{aligned}
&E\left[\widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{NEW}}\middle|\mathcal{B}_0\right]\\
&= E\left[\sum_{i=1}^{I} E\left[\widehat{\mathrm{EE}_i}^{\mathrm{NEW}}\middle|\mathcal{B}_{I-i}\right]\middle|\mathcal{B}_0\right] + 2\sum_{1\le i<j\le I} E\left[C_{i,I-i}\left(C_{j,I-j}\prod_{k=I-j}^{I-i-1}\widehat{f}_k\right) E\left[\frac{\widehat{\mathrm{EE}_i}^{\mathrm{NEW}}}{C_{i,I-i}^2}\middle|\mathcal{B}_{I-i}\right]\middle|\mathcal{B}_0\right]\\
&= E\left[\sum_{i=1}^{I} \mathrm{EE}_i\middle|\mathcal{B}_0\right] + 2\sum_{1\le i<j\le I} E\left[C_{i,I-i}\left(C_{j,I-j}\prod_{k=I-j}^{I-i-1}\widehat{f}_k\right)\left(E\left[\prod_{k=I-i}^{J-1}\left(\widehat{f}_k\right)^2\middle|\mathcal{B}_{I-i}\right] - \prod_{k=I-i}^{J-1} f_k^2\right)\middle|\mathcal{B}_0\right]\\
&= E\left[\sum_{i=1}^{I} \mathrm{EE}_i\middle|\mathcal{B}_0\right] + 2\sum_{1\le i<j\le I} E\left[C_{i,I-i}C_{j,I-j}\left(\prod_{k=I-j}^{I-i-1}\widehat{f}_k\right) E\left[\prod_{k=I-i}^{J-1}\left(\widehat{f}_k\right)^2 - \prod_{k=I-i}^{J-1} f_k\prod_{k=I-i}^{J-1}\widehat{f}_k\middle|\mathcal{B}_{I-i}\right]\middle|\mathcal{B}_0\right]\\
&= E\left[\sum_{i=1}^{I} \mathrm{EE}_i\middle|\mathcal{B}_0\right] + 2\sum_{1\le i<j\le I} E\left[C_{i,I-i}C_{j,I-j}E\left[\prod_{k=I-j}^{J-1}\widehat{f}_k\prod_{k=I-i}^{J-1}\widehat{f}_k - \prod_{k=I-i}^{J-1} f_k\prod_{k=I-j}^{J-1}\widehat{f}_k\middle|\mathcal{B}_{I-i}\right]\middle|\mathcal{B}_0\right]\\
&= E\left[\sum_{i=1}^{I} \mathrm{EE}_i\middle|\mathcal{B}_0\right] + 2\sum_{1\le i<j\le I} E\left[C_{i,I-i}C_{j,I-j}E\left[\left(\prod_{k=I-i}^{J-1}\widehat{f}_k - \prod_{k=I-i}^{J-1} f_k\right)\left(\prod_{k=I-j}^{J-1}\widehat{f}_k - \prod_{k=I-j}^{J-1} f_k\right)\middle|\mathcal{B}_{I-i}\right]\middle|\mathcal{B}_0\right]\\
&= E\left[\mathrm{EE}_{\mathrm{tot}}\middle|\mathcal{B}_0\right] < \infty,
\end{aligned}$$

where in the second last step we used the fact that

$$E\left[\left(\prod_{k=I-i}^{J-1}\widehat{f}_k - \prod_{k=I-i}^{J-1} f_k\right)\left(\prod_{k=I-j}^{J-1} f_k\right)\middle|\mathcal{B}_{I-i}\right] = 0.$$

This proves (34). □

Combining estimators (31) and (32) yields a novel estimator for (14):

Estimator 2 (Unbiased formula, aggregated accident years). Under Model Assumptions 1, when for given data $\mathcal{D}_I$ the regularity condition (25) is fulfilled, we have the following positive estimator

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for $\mathrm{PV_{tot}} + \mathrm{EE_{tot}}$ at time I:

$$\widehat{PV}_{tot}^{NEW} + \widehat{EE}_{tot}^{NEW}$$

$$= \sum_{i=1}^{I} C_{i,I-i} \sum_{k=I-i}^{J-1} \left(\prod_{m=I-i}^{k-1} \widehat{f}_m \right) \widehat{\sigma_k^2} \prod_{n=k+1}^{J-1} \left[\left(\widehat{f}_n\right)^2 - \frac{\widehat{\sigma_n^2}}{\sum_{h=0}^{I-n-1} C_{h,n}} \right]$$

$$+ \sum_{i=1}^{I} C_{i,I-i}^2 \left(\prod_{k=I-i}^{J-1} \left(\widehat{f}_k\right)^2 - \prod_{k=I-i}^{J-1} \left[\left(\widehat{f}_k\right)^2 - \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] \right)$$

$$+ 2 \sum_{1 \le i < j \le I} C_{i,I-i} \left(C_{j,I-j} \prod_{k=I-j}^{I-i-1} \widehat{f}_k \right) \left(\prod_{k=I-i}^{J-1} \left(\widehat{f}_k\right)^2 - \prod_{k=I-i}^{J-1} \left[\left(\widehat{f}_k\right)^2 - \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] \right). \tag{35}$$

Remark 5. We underline that (35) is a conditionally given $\mathcal{B}_0$ unbiased estimator for $E\left[PV_{tot} + EE_{tot} \middle| \mathcal{B}_0\right]$ under Model Assumptions 1 (with $J < I$) and the additional assumptions $E\left[\prod_{j=I-i}^{J-1} \left(\widehat{f}_j\right)^2\right] < \infty$ and $E\left[\left| \prod_{j=I-i}^{J-1} \left(\left(\widehat{f}_j\right)^2 - \frac{\widehat{\sigma_j^2}}{\sum_{k=0}^{I-j-1} C_{k,j}} \right) \right|\right] < \infty, \quad \forall i \in \{I-J+1, \ldots, I\}$. Moreover, when for given data $\mathcal{D}_I$ the regularity condition (25) is fulfilled, estimator (35) is positive.

Remark 6. Using the telescope formula

$$\left(\prod_{k=I-i}^{J-1} \left(\widehat{f}_k\right)^2 - \prod_{k=I-i}^{J-1} \left[\left(\widehat{f}_k\right)^2 - \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] \right)$$

$$= \sum_{k=I-i}^{J-1} \prod_{n=I-i}^{k-1} \left(\widehat{f}_n\right)^2 \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \prod_{m=k+1}^{J-1} \left[\left(\widehat{f}_m\right)^2 - \frac{\widehat{\sigma_m^2}}{\sum_{h=0}^{I-m-1} C_{h,m}} \right], \tag{36}$$

and rearranging the summation indices leads to the following more simple display of formula (35)

$$\widehat{PV}_{tot}^{NEW} + \widehat{EE}_{tot}^{NEW}$$

$$= \sum_{i=1}^{I} \sum_{k=I-i}^{J-1} \left(C_{i,I-i} \prod_{m=I-i}^{k-1} \widehat{f}_m \right) \widehat{\sigma_k^2} \prod_{n=k+1}^{J-1} \left[\left(\widehat{f}_n\right)^2 - \frac{\widehat{\sigma_n^2}}{\sum_{h=0}^{I-n-1} C_{h,n}} \right]$$

$$+ \sum_{k=0}^{J-1} \left(\sum_{i=I-k}^{I} C_{i,I-i} \prod_{m=I-i}^{k-1} \widehat{f}_m \right)^2 \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \prod_{m=k+1}^{J-1} \left[\left(\widehat{f}_m\right)^2 - \frac{\widehat{\sigma_m^2}}{\sum_{h=0}^{I-m-1} C_{h,m}} \right]. \tag{37}$$

For comparison reasons in section 4 (Numerical examples), we recall the Mack and BBMW formulas (see, e.g. Wüthrich & Merz, 2008).

3.3 Mack formula

Mack's formulas for individual and for aggregated accident years are reported in the following sections.

3.3.1 Single accident years

For individual accident years, the Mack formula reads:

$$\widehat{\mathrm{PV}}_i^{\mathrm{Mack}} + \widehat{\mathrm{EE}}_i^{\mathrm{Mack}} = \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{C_{i,I-i}\prod_{j=I-i}^{k-1}\widehat{f_j}} + \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{\sum_{h=0}^{I-k-1} C_{h,k}}. \tag{38}$$

3.3.2 Aggregated accident years

For aggregated accident years, the Mack formula reads:

$$\begin{aligned}\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{Mack}} + \widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{Mack}} &= \sum_{i=1}^{I} \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{C_{i,I-i}\prod_{j=I-i}^{k-1}\widehat{f_j}} + \sum_{i=1}^{I} \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{\sum_{h=0}^{I-k-1} C_{h,k}} \\ &\quad + 2 \sum_{1\le i<j\le I} \widehat{C}_{i,J}\widehat{C}_{j,J} \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{\sum_{h=0}^{I-k-1} C_{h,k}}.\end{aligned} \tag{39}$$

Remark 7. Merging the second and third term in (39) by rearranging the summation indices leads to the following more simple display

$$\widehat{PV_{tot}}^{Mack} + \widehat{EE_{tot}}^{Mack} = \sum_{i=1}^{I} \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{C_{i,I-i}\prod_{j=I-i}^{k-1}\widehat{f_j}} + \sum_{k=0}^{J-1} \left(\sum_{i=I-k}^{I} \widehat{C}_{i,J}\right)^2 \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{\sum_{h=0}^{I-k-1} C_{h,k}}, \tag{40}$$

which corresponds to the form of the Mack formula derived in Gisler (2019).

3.4 BBMW formula

BBMW's formulas for individual and for aggregated accident years are reported in the following sections.

3.4.1 Single accident years

For individual accident years, the BBMW formula reads:

$$\begin{aligned}&\widehat{\mathrm{PV}}_i^{\mathrm{Mack}} + \widehat{\mathrm{EE}}_i^{\mathrm{BBMW}} \\ &= \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2}/\left(\widehat{f_k}\right)^2}{C_{i,I-i}\prod_{j=I-i}^{k-1}\widehat{f_j}} + C_{i,I-i}^2 \left(\prod_{k=I-i}^{J-1}\left[\left(\widehat{f_k}\right)^2 + \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}}\right] - \prod_{k=I-i}^{J-1}\left(\widehat{f_k}\right)^2\right).\end{aligned} \tag{41}$$

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3.4.2 Aggregated accident years

For aggregated accident years, the BBMW formula reads:

$$\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{Mack}} + \widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{BBMW}}$$

$$= \sum_{i=1}^{I} \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2} / \left(\widehat{f_k}\right)^2}{C_{i,\,I-i} \prod_{j=I-i}^{k-1} \widehat{f_j}} + \sum_{i=1}^{I} C_{i,\,I-i}^2 \left(\prod_{k=I-i}^{J-1} \left[\left(\widehat{f_k}\right)^2 + \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] - \prod_{k=I-i}^{J-1} \left(\widehat{f_k}\right)^2 \right)$$

$$+ 2 \sum_{1 \le i < j \le I} C_{i,\,I-i} \left(C_{j,I-j} \prod_{k=I-j}^{I-i-1} \widehat{f_k} \right) \left(\prod_{k=I-i}^{J-1} \left[\left(\widehat{f_k}\right)^2 + \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] - \prod_{k=I-i}^{J-1} \left(\widehat{f_k}\right)^2 \right). \tag{42}$$

Remark 8. Using the telescope formula

$$\left(\prod_{k=I-i}^{J-1} \left[\left(\widehat{f_k}\right)^2 + \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \right] - \prod_{k=I-i}^{J-1} \left(\widehat{f_k}\right)^2 \right)$$

$$= \sum_{k=I-i}^{J-1} \prod_{n=I-i}^{k-1} \left[\left(\widehat{f_n}\right)^2 + \frac{\widehat{\sigma_n^2}}{\sum_{h=0}^{I-n-1} C_{h,n}} \right] \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \prod_{m=k+1}^{J-1} \left(\widehat{f_m}\right)^2, \tag{43}$$

and rearranging the summation indices leads to the following more simple display of formula (42)

$$\widehat{PV_{tot}}^{Mack} + \widehat{EE_{tot}}^{BBMW}$$

$$= \sum_{i=1}^{I} \left(\widehat{C}_{i,J}\right)^2 \sum_{k=I-i}^{J-1} \frac{\widehat{\sigma_k^2} / \left(\widehat{f_k}\right)^2}{C_{i,\,I-i} \prod_{j=I-i}^{k-1} \widehat{f_j}}$$

$$+ \sum_{k=0}^{J-1} \left(\sum_{i=I-k}^{I} C_{i,\,I-i} \prod_{m=I-i}^{k-1} \widehat{f_m} \right)^2 \frac{\widehat{\sigma_k^2}}{\sum_{h=0}^{I-k-1} C_{h,k}} \prod_{m=k+1}^{J-1} \left[\left(\widehat{f_m}\right)^2 + \frac{\widehat{\sigma_m^2}}{\sum_{h=0}^{I-m-1} C_{h,m}} \right]. \tag{44}$$

3.5 Peculiar behaviours of the Unbiased formula (35)

We have seen that strictly under Model Assumptions 1 (or even including the technical assumptions (23) and (24)), the Unbiased estimator (35) does show a theoretical peculiar behaviour with respect to its possible negativity. However, for given data $\mathcal{D}_I$ which satisfies the regularity condition (25), the Unbiased estimator (35) results to be positive.

Strictly under Model Assumptions 1, another theoretical peculiar behaviour of the Unbiased estimator $\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{NEW}} + \widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{NEW}}$ relates to the fact that, when looking it as a multivariable function in the estimated parameters $(\widehat{\sigma_j^2})$, it may not increase for increasing values of its arguments. This is curious, since the true value $\mathrm{PV}_{\mathrm{tot}} + \mathrm{EE}_{\mathrm{tot}}$ does not show this behaviour in the sense that data with underlying higher volatility, i.e. with larger (σ_j^2) parameters, do lead to a larger mean squared error of prediction.

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3.6 Final comments

The Unbiased formula (35) was derived focusing on a traditional well desired property for estimators, namely unbiasedness. In our view, this approach is much more straightforward and easy to explain (and to remember) compared to the approaches underlying the derivation of both Mack and BBMW formulas.

We highlight that, when for given data $\mathcal{D}_I$ the regularity condition (25) is fulfilled, it holds true

$$\underbrace{\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{NEW}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{NEW}}}_{\text{Unbiased formula}}<\underbrace{\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{Mack}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{Mack}}}_{\text{Mack formula}}<\underbrace{\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{Mack}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{BBMW}}}_{\text{BBMW formula}}, \tag{45}$$

i.e. the Unbiased formula is smaller than Mack and BBMW formulas. However, as we will see in the numerical section 4, the three formulas generally produce results which are very close.

Moreover, note that given data $\mathcal{D}_I$, the true value $\mathrm{PV}_{\mathrm{tot}}+\mathrm{EE}_{\mathrm{tot}}$ is a fix number which quantifies the true prediction uncertainty. Therefore, we would appreciate if the three formulas were to deliver results close to this quantity. However, the magnitude and the frequency of the possible deviations we observed in our numerical analysis (see section 4) is remarkable, even when considering average-sized triangles often available in actuarial practice.

As a consequence, we would like to make the actuarial community aware that, even when Mack's model perfectly fits the available data, the ultimate claim prediction uncertainty estimated according to the three considered formulas (Mack, BBMW and Unbiased) may with non-negligible probability, perhaps materially (especially for small-sized triangles), deviate from the true value. In particular, the latter can result to be bigger than the BBMW formula or even smaller than the Unbiased formula. This fact can be demonstrated by simulations, by considering generated triangles that exactly fulfil the claims payments evolution described through Mack's model.

As usually done in statistics, MSEP given by

$$E\left[\left(\text{estimator}-(\mathrm{PV}_{\mathrm{tot}}+\mathrm{EE}_{\mathrm{tot}})\right)^2\middle|\mathcal{B}_0\right], \tag{46}$$

can be considered for assessing the performance of the different estimators.

In that respect, when hypothetically assuming

- Model Assumptions 1 (with $J<I$)
- the additional technical assumptions (23) and (24)
- the positivity of estimator (22) along all the (finite) trajectories of the claims payments process

it would be possible to show that the Unbiased formula (35) outperforms the Mack and BBMW formulas in terms of (46). It namely holds true:

$$\begin{aligned}
&E\left[\left(\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{Mack}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{Mack}}-\mathrm{PV}_{\mathrm{tot}}-\mathrm{EE}_{\mathrm{tot}}\right)^2\middle|\mathcal{B}_0\right]\\
&\quad-E\left[\left(\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{NEW}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{NEW}}-\mathrm{PV}_{\mathrm{tot}}-\mathrm{EE}_{\mathrm{tot}}\right)^2\middle|\mathcal{B}_0\right]\\
&=E\Bigg[\left(\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{Mack}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{Mack}}+\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{NEW}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{NEW}}-2\mathrm{PV}_{\mathrm{tot}}-2\mathrm{EE}_{\mathrm{tot}}\right)\\
&\qquad\cdot\left(\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{Mack}}+\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{Mack}}-\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{NEW}}-\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{NEW}}\right)\Bigg|\mathcal{B}_0\Bigg]\\
&>\delta\underbrace{E\left[\widehat{\mathrm{PV}}_{\mathrm{tot}}^{\mathrm{Mack}}-\mathrm{PV}_{\mathrm{tot}}\middle|\mathcal{B}_0\right]}_{>0}+\delta\underbrace{E\left[\widehat{\mathrm{EE}}_{\mathrm{tot}}^{\mathrm{Mack}}-\mathrm{EE}_{\mathrm{tot}}\middle|\mathcal{B}_0\right]}_{>0}>0,
\end{aligned} \tag{47}$$

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provided it exists $\delta > 0$ with

$$P\left[\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{Mack}} + \widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{Mack}} - \left(\widehat{\mathrm{PV}_{\mathrm{tot}}}^{\mathrm{NEW}} + \widehat{\mathrm{EE}_{\mathrm{tot}}}^{\mathrm{NEW}}\right) > \delta \middle| \mathcal{B}_0\right] = 1. \tag{48}$$

Also, under the assumptions of the time series model described in Merz & Wüthrich (2008) (with $J < I$) our simulation studies indicate (when ensuring the regularity condition (25) to be fulfilled for all the considered triangles) an empirical superiority of the Unbiased formula (35) in terms of (46) (see section 4).

However, since the quantity (46) might not be the most appropriate and the three above-mentioned assumptions might not be consistent, from a theoretical point of view the assessment of the "goodness" of the three estimators might still be a subject for future research.

4. Numerical Examples

Let us first analyse a simulated example.

4.1 A simulated example

Consider the cumulative payments $(C_{i,j})$ showed in Table 1 which have been simulated (for a given set $\mathcal{B}_0$ and independently for each accident year i) by using the true parameters (f_j), (σ_j^2) specified in Table 2 according to the time series model

$$C_{i,j+1} = f_j C_{i,j} + \sigma_j \sqrt{C_{i,j}}\varepsilon_{i,j+1}, \tag{49}$$

with $(\varepsilon_{i,j+1})$ i.i.d., uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$. Consequently, having additionally ensured positiveness of the data (see Model Assumptions 3.9 (time series model) and the related Remarks 3.10 in Merz & Wüthrich, 2008), these cumulative payments fulfil Model Assumptions 1.

Also recall that when the true parameters are known, the true value $\mathrm{PV}_{\mathrm{tot}} + \mathrm{EE}_{\mathrm{tot}}$ for the ultimate prediction uncertainty can be computed analytically through formulas (16) and (18).

These data yield the parameter estimates related to Mack's model, as indicated in Table 3, as well as the ultimate estimates $(\widehat{C}_{i,J})$ and reserves $(\widehat{C}_{i,J} - C_{i,I-i})$ at time I, as indicated in Table 4. Moreover, using estimators (39), (42), and (35), we can obtain the prediction uncertainties, as indicated in Table 5.

Given $\mathcal{D}_I$, the empirical true value $\mathrm{PV}_{\mathrm{tot}} + \mathrm{EE}_{\mathrm{tot}}$ is calculated by simulating the future claims payments (30'000 simulations using the true parameters (f_j), (σ_j^2)) and computing the average of the squared differences between simulated ultimate value and the ultimate estimate at time I given by $\sum_{i=0}^{I} \widehat{C}_{i,J}$. As expected, this numerical simulation yields a result that is well aligned with the true value.

Table 5 indicates that the three formulas (Mack, BBMW and Unbiased) to quantify the ultimate prediction uncertainty yield similar results. However, the true value $\mathrm{PV}_{\mathrm{tot}} + \mathrm{EE}_{\mathrm{tot}}$ is considerably smaller than the estimated values, and, in this case, the Unbiased formula (35) achieves a result that is most similar to the true value.

However, such a scenario does not usually occur. In fact, if another simulated triangle (see Table 6) based on the same true parameters and same set $\mathcal{B}_0$ is considered, the results presented in Table 7 are achieved. It can be noted that in this case, the BBMW formula yields the result closest to the true ultimate uncertainty.

Finally, Table 8 presents the results of the performance simulation study (herein 50'000 triangles based on the same true parameters and same set $\mathcal{B}_0$ are considered) related to the expected

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Table 1. Example 1: Cumulative payments ($C_{i,j}$).

i/*j*	0	1	2	3	4	5	6	7	8	9	10	11	12
0	65’971	93’977	153’900	179’785	235’807	267’996	281’408	316’691	326’726	337’556	359’336	372’286	376’973
1	64’913	157’347	202’827	283’647	360’745	415’762	429’018	468’920	534’776	582’498	605’717	628’852	
2	64’019	174’946	330’174	507’279	635’517	803’765	944’734	1’065’840	1’164’170	1’234’940	1’291’107		
3	60’412	160’551	281’175	452’177	664’043	847’500	908’333	942’100	983’815	1’014’437			
4	60’994	172’012	224’271	310’410	421’458	502’431	597’008	628’692	700’158				
5	82’391	224’355	277’463	441’493	570’113	641’125	792’838	871’763					
6	75’977	155’179	299’669	445’931	505’293	590’708	734’989						
7	74’212	87’136	135’433	189’653	283’374	365’485							
8	65’557	109’086	203’401	343’608	456’945								
9	66’116	89’469	117’546	118’576									
10	66’782	98’407	95’662										
11	71’205	117’434											
12	72’624												

Table 2. The true model parameters.

j	0	1	2	3	4	5	6	7	8	9	10	11
f_j	2.000	1.500	1.400	1.300	1.200	1.150	1.100	1.070	1.060	1.050	1.030	1.020
σ_j^2	16’900	10’000	6’400	4’900	3’600	2’500	1’600	900	400	100	25	9

Table 3. Example 1: Parameter estimates.

j	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{f_j}$	2.003	1.525	1.470	1.311	1.206	1.152	1.086	1.084	1.053	1.047	1.037	1.013
$\widehat{\sigma_j^2}$	26’471	14’052	7’017	4’353	2’026	4’160	1’019	1’118	457	68	1.05	0.02

Table 4. Example 1: Ultimate estimates and reserves at time I.

i	$\widehat{C}_{i,J}$	$(\widehat{C}_{i,J} - C_{i,I-i})$
0	376’973	0
1	636’769	7’917
2	1’356’246	65’139
3	1’115’643	101’206
4	810’933	110’775
5	1’094’483	222’720
6	1’002’282	267’293
7	574’220	208’735
8	866’018	409’073
9	294’508	175’932
10	349’325	253’663
11	653’897	536’463
12	810’155	737’531
Total	9’941’452	3’096’447

Table 5. Example 1: The ultimate prediction error for aggregated accident years (as a percentage of the reserves).

	Prediction error$^{1/2}$	Process variance$^{1/2}$	Estimation error$^{1/2}$
Mack formula (39)	490’627 (15.8%)	429’735	236’735
BBMW formula (42)	490’741 (15.8%)	429’735	236’970
Unbiased formula (35)	489’713 (15.8%)	428’820	236’500
True value	384’351 (12.4%)	372’481	94’785
True value (empirical)	384’865 (12.4%)		

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Table 6. Example 2: Cumulative payments ($C_{i,j}$).

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	65’971	187’958	344’405	439’786	614’193	662’285	786’108	911’763	938’338	965’090	1’023’516	1’054’659	1’072’869
1	64’913	93’936	103’214	136’486	208’890	241’909	253’084	293’980	312’383	342’935	365’469	380’311	
2	64’019	140’629	186’520	205’785	283’690	335’372	395’181	446’817	469’567	501’198	520’948		
3	60’412	100’735	106’167	188’834	231’957	296’436	316’313	342’334	368’969	381’722			
4	60’994	71’286	71’699	113’281	106’591	113’371	143’006	146’710	176’703				
5	82’391	213’956	279’304	346’391	484’429	632’496	752’848	779’297					
6	75’977	205’051	358’828	467’630	555’140	618’271	683’078						
7	74’212	105’289	108’207	142’900	199’517	268’919							
8	65’557	98’112	184’062	199’007	229’821								
9	66’116	76’987	152’162	241’746									
10	66’782	172’000	190’652										
11	71’205	125’926											
12	72’624												

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Table 7. Example 2: The ultimate prediction error for aggregated accident years (as a percentage of the reserves).

	Prediction error$^{1/2}$	Process variance$^{1/2}$	Estimation error$^{1/2}$
Mack formula (39)	475’458 (18.2%)	399’960	257’083
BBMW formula (42)	475’631 (18.2%)	399’960	257’404
Unbiased formula (35)	474’335 (18.2%)	398’831	256’763
True value	514’190 (19.7%)	386’880	338’697
True value (empirical)	513’869 (19.7%)		

Table 8. Expected squared deviation between estimator$^{1/2}$ and true value$^{1/2}$, given $\mathcal{B}_0$.

Estimator	$E\left[\left(\text{estimator}^{1/2} - (\text{PV}_{\text{tot}} + \text{EE}_{\text{tot}})^{1/2}\right)^2 \middle\vert \mathcal{B}_0\right]^{1/2}$ (empirical)
Mack formula	111’284
BBMW formula	111’307
Unbiased formula	111’171

squared deviation between estimator and true value, given $\mathcal{B}_0$. Here, it can be empirically observed that the Unbiased formula (35) slightly outperforms the other formulas.

Increasing the number of available accident years related to the above considered examples, we get the data shown in Tables 9 and 10. Computing the prediction uncertainties at different points in time (i.e. for different triangles sizes), we get the results displayed in Tables 11 and 12.

From Tables 11 and 12, we can observe that, at each evaluation point in time, the three formulas do produce very close results. Moreover with increasing data availability, the possible gaps between estimators and true values reduce, since the estimators for the model parameters tend to get closer to the true parameters values.

In particular, as we can see from Table 13, at each evaluation point in time the Unbiased formula is performing slightly better than Mack and BBMW formulas. Moreover, the expected squared deviation between estimator and true value, given $\mathcal{B}_0$, does decrease with increasing triangle size.

Remark 9. We highlight that all the considered triangles in this simulation study do fulfil the regularity condition (25). Also, we would like to further underline that deviations from the true value are ordinary and simply due to the random variable nature of estimators. Nevertheless, the large deviations we observed in our simulation study, as well as their high frequency (see Table 14), is noteworthy. Moreover, it is important to underline the key role played by the largeness of the data availability given by $I - J$. Indeed, a large data availability helps to reduce the probability that estimator (35) does significantly differ from the true value. In other words, $I - J$ should be sufficiently large (in fact larger than usually considered in the examples presented in actuarial literature or usually available in actuarial practice) in order to get an estimator which is well close distributed around the true value. A large data availability also helps to reduce the variance of the estimated sigma parameters which generally tends to be very high.

4.2 Performance simulation study for different model parameter choices

Similarly as in the previous section, we simulate triangles according the time series model (49) with $(\varepsilon_{i,j+1})$ independent and shifted Gamma distributed with some given shape parameter α, with variance equal to 1 and expected value equal to 0. Then we run the performance analysis

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Table 9. Example 1 extended: Cumulative payments ($C_{i,j}$).

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	65’971	93’977	153’900	179’785	235’807	267’996	281’408	316’691	326’726	337’556	359’336	372’286	376’973
1	64’913	157’347	202’827	283’647	360’745	415’762	429’018	468’920	534’776	582’498	605’717	628’852	641’664
2	64’019	174’946	330’174	507’279	635’517	803’765	944’734	1’065’840	1’164’170	1’234’940	1’291’107	1’320’394	1’344’345
3	60’412	160’551	281’175	452’177	664’043	847’500	908’333	942’100	983’815	1’014’437	1’078’913	1’110’290	1’133’825
4	60’994	172’012	224’271	310’410	421’458	502’431	597’008	628’692	700’158	751’541	797’209	822’004	839’913
5	82’391	224’355	277’463	441’493	570’113	641’125	792’838	871’763	894’636	966’834	1’002’860	1’024’748	1’042’032
6	75’977	155’179	299’669	445’931	505’293	590’708	734’989	805’892	865’876	885’785	917’890	944’797	960’668
7	74’212	87’136	135’433	189’653	283’374	365’485	446’963	461’470	468’728	514’249	545’536	566’204	577’834
8	65’557	109’086	203’401	343’608	456’945	504’607	549’682	632’797	655’584	672’495	719’988	744’545	757’119
9	66’116	89’469	117’546	118’576	185’650	197’592	213’174	256’860	280’429	303’948	317’219	329’044	
10	66’782	98’407	95’662	151’965	150’900	203’406	194’874	199’719	231’999	230’908	239’918		
11	71’205	117’434	151’816	248’361	303’508	387’811	417’993	481’994	523’657	557’415			
12	72’624	84’808	102’663	129’466	175’186	175’105	184’382	231’315	235’838				
13	69’416	87’661	164’054	248’275	351’393	475’289	581’022	598’347					
14	68’616	168’115	214’701	308’513	423’020	558’593	663’451						
15	69’127	126’411	183’177	304’000	443’417	482’717							
16	69’628	181’068	222’080	275’285	297’546								
17	70’102	151’222	219’712	313’410									
18	69’925	155’371	196’698										
19	70’045	93’749											
20	72’103												

Table 10. Example 2 extended: Cumulative payments ($C_{i,j}$).

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	65’971	187’958	344’405	439’786	614’193	662’285	786’108	911’763	938’338	965’090	1’023’516	1’054’659	1’072’869
1	64’913	93’936	103’214	136’486	208’890	241’909	253’084	293’980	312’383	342’935	365’469	380’311	390’817
2	64’019	140’629	186’520	205’785	283’690	335’372	395’181	446’817	469’567	501’198	520’948	540’248	549’444
3	60’412	100’735	106’167	188’834	231’957	296’436	316’313	342’334	368’969	381’722	392’552	399’797	406’906
4	60’994	71’286	71’699	113’281	106’591	113’371	143’006	146’710	176’703	187’566	198’276	204’282	209’203
5	82’391	213’956	279’304	346’391	484’429	632’496	752’848	779’297	862’031	891’441	920’643	944’292	966’753
6	75’977	205’051	358’828	467’630	555’140	618’271	683’078	742’642	757’184	800’687	831’751	860’802	875’118
7	74’212	105’289	108’207	142’900	199’517	268’919	309’143	322’390	364’680	388’305	409’149	422’618	433’242
8	65’557	98’112	184’062	199’007	229’821	323’821	417’562	455’690	452’664	473’328	508’394	523’539	536’821
9	66’116	76’987	152’162	241’746	369’210	470’862	551’091	569’581	605’399	618’641	651’011	676’509	
10	66’782	172’000	190’652	255’302	387’277	468’843	587’566	597’084	620’082	670’313	690’828		
11	71’205	125’926	242’809	285’221	365’531	438’043	494’447	567’400	584’375	625’980			
12	72’624	143’573	231’559	389’133	507’921	608’124	736’357	860’309	883’606				
13	69’416	155’153	244’999	299’583	399’807	536’649	591’982	616’025					
14	68’616	167’893	299’318	470’271	568’365	662’990	824’927						
15	69’127	89’066	83’966	115’503	112’301	129’632							
16	69’628	143’005	196’379	261’363	325’647								
17	70’102	178’166	277’851	369’514									
18	69’925	149’867	183’251										
19	70’045	193’588											
20	72’103												

Table 11. Example 1: The ultimate prediction error for aggregated accident years (as a percentage of the reserves) for different triangles sizes.

	$I=9$ $J=9$	$I=12$ $J=12$	$I=16$ $J=12$	$I=20$ $J=12$
Mack formula (39)	579'474 (19.2%)	490'627 (15.8%)	458'046 (16.3%)	447'210 (14.7%)
BBMW formula (42)	579'733 (19.2%)	490'741 (15.8%)	458'112 (16.3%)	447'248 (14.7%)
Unbiased formula (35)	578'395 (19.1%)	489'713 (15.8%)	457'424 (16.3%)	446'771 (14.6%)
True value	673'590 (22.3%)	384'351 (12.4%)	383'673 (13.7%)	384'772 (12.6%)
True value (empirical)	671'765 (22.2%)	384'865 (12.4%)	383'214 (13.7%)	386'215 (12.7%)
Chain-ladder reserves	3'021'352	3'096'447	2'803'458	3'051'423
Mean unpaid losses (empirical)	3'568'807	3'003'186	2'825'695	3'076'597

Table 12. Example 2: The ultimate prediction error for aggregated accident years (as a percentage of the reserves) for different triangles sizes.

	$I=9$ $J=9$	$I=12$ $J=12$	$I=16$ $J=12$	$I=20$ $J=12$
Mack formula (39)	385'816 (20.1%)	475'458 (18.2%)	480'883 (14.7%)	478'842 (14.2%)
BBMW formula (42)	386'005 (20.1%)	475'631 (18.2%)	480'963 (14.7%)	478'895 (14.2%)
Unbiased formula (35)	384'695 (20.0%)	474'335 (18.2%)	480'213 (14.7%)	478'348 (14.1%)
True value	925'734 (48.2%)	514'190 (19.7%)	438'029 (13.4%)	458'861 (13.6%)
True value (empirical)	928'872 (48.3%)	513'869 (19.7%)	440'187 (13.5%)	459'576 (13.6%)
Chain-ladder reserves	1'921'321	2'611'709	3'268'351	3'383'968
Mean unpaid losses (empirical)	2'772'488	2'951'016	3'453'913	3'575'031

Table 13. Expected squared deviation between $\text{estimator}^{1/2}$ and $\text{true value}^{1/2}$, given $\mathcal{B}_0$, for different triangles sizes.

Estimator	$I=9$ $J=9$	$I=12$ $J=12$	$I=16$ $J=12$	$I=20$ $J=12$
Mack formula	195'358	111'284	74'765	59'651
BBMW formula	195'466	111'307	74'773	59'655
Unbiased formula	195'125	111'171	74'705	59'616

Table 14. Probabilities of a high deviation from the true value, given $\mathcal{B}_0$, for different triangles sizes.

	$I=9$ $J=9$	$I=12$ $J=12$	$I=16$ $J=12$	$I=20$ $J=12$
$P\left[\left\lvert\frac{\left(\widehat{\text{PV}_{\text{tot}}}^{\text{NEW}}+\widehat{\text{EE}_{\text{tot}}}^{\text{NEW}}\right)^{1/2}-(\text{PV}_{\text{tot}}+\text{EE}_{\text{tot}})^{1/2}}{(\text{PV}_{\text{tot}}+\text{EE}_{\text{tot}})^{1/2}}\right\rvert\geq 10\%\,\middle\vert\,\mathcal{B}_0\right]$ (empirical)	79%	69%	53%	40%
$P\left[\left\lvert\frac{\left(\widehat{\text{PV}_{\text{tot}}}^{\text{NEW}}+\widehat{\text{EE}_{\text{tot}}}^{\text{NEW}}\right)^{1/2}-(\text{PV}_{\text{tot}}+\text{EE}_{\text{tot}})^{1/2}}{\text{Reserves at time I}}\right\rvert\geq 2\%\,\middle\vert\,\mathcal{B}_0\right]$ (empirical)	73%	54%	27%	15%

for different model parameter choices (f_j) and (σ_j^2), which have been inspired by some real data triangles in actuarial practice. The studied data $\mathcal{B}_0$ are shown in Table 15, and the chosen model parameters for the different classes of business are shown in Table 16.

The results of the performance analysis (40'000 triangles considered for each model parameter choice) related to the expected squared deviation between estimator and true value, given $\mathcal{B}_0$,

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Table 15. The set $\mathcal{B}_0$ by class of business.

i	Motor Liability	Motor Hull	Property	Commercial Liability
0	15'043	7'980	10'441	19'583
1	14'899	9'847	11'229	14'991
2	15'254	7'984	12'805	15'686
3	14'066	8'609	10'705	12'374
4	14'137	9'441	11'986	10'128
5	14'472	10'890	10'206	9'743
6	15'762	12'871	25'048	8'128
7	14'376	15'149	12'573	4'003
8	14'328	13'471	10'795	3'552
9	13'899	9'171	11'038	4'203
10	13'583	5'354	12'541	3'854
11	13'229	3'389	10'124	2'866
12	14'249	3'758	10'027	2'770
13	14'253	3'480	13'815	2'887
14	15'001	4'864	19'456	2'866
15	14'589	4'650	10'587	2'770
16	15'546	4'903	14'935	2'887
17	20'299	19'399	161'960	5'070
18	26'937	21'987	133'164	3'163
19	28'702	21'103	130'833	3'230
20	30'486	21'525	191'698	3'452

are presented in Table 17. Again, it can be empirically observed that the Unbiased formula (35) slightly outperforms the other formulas.

4.3 Two real data examples

In this section, we analyse two real data examples. Note that in this case, there is also a model risk involved since we do not know, whether the data do exactly fulfil Model Assumptions 1 or not. Therefore, adding a loading on the three formulas is generally appropriate. This might also be the main reason why, based on various empirical studies on real claims data, there seems to hold a conjecture suggesting that the classical Mack estimator may be rather on the optimistic side.

4.3.1 The Merz-Wüthrich triangle

We consider the Private Liability triangle presented in Merz & Wüthrich (2014) (see Table 18) and first observe that the regularity condition (25) is fulfilled.

Evaluating the three estimators for the ultimate prediction uncertainty, we get the results displayed in Table 19 and remark that the formulas lead to very close outcomes.

Since we do not know the true model parameters (f_j) and (σ_j^2) we cannot compute the true value $\mathrm{PV}_{\mathrm{tot}} + \mathrm{EE}_{\mathrm{tot}}$. However, based on the estimated parameters at time 16 we can make some rational guesses for the true model parameters and then compute the true value for the ultimate prediction uncertainty. By doing this, we observe that $\mathrm{PV}_{\mathrm{tot}} + \mathrm{EE}_{\mathrm{tot}}$ could materially deviate from the estimated values. This is remarkable, because it means that if the cumulative payments will effectively evolve according to Model Assumptions 1 with model parameters as given by our guesses, then the true prediction uncertainty would materially deviate from the three available formulas. Moreover

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Table 16. Performance study by class of business: The true model parameters and distribution assumptions.

(Motor Liability) Shifted Gamma distribution with shape parameter $\alpha = 1.5$, scale parameter $\theta = \sqrt{(1/1.5)}$ and expected value 0																
j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_j	1.530	1.124	1.093	1.070	1.051	1.036	1.041	1.024	1.022	1.024	1.023	1.007	1.006	1.013	1.014	1.033
σ_j^2	227.4	25.4	19.7	18.6	18.3	13.9	10.6	8.1	6.1	4.7	3.6	2.7	2.1	1.6	1.2	38.6
(Motor Hull) Shifted Gamma distribution with shape parameter $\alpha = 8$, scale parameter $\theta = \sqrt{(1/8)}$ and expected value 0																
j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_j	1.309	1.003	1.004	1.000	1.001											
σ_j^2	206.3	0.9	0.5	0.1	0.3											
(Property) Shifted Gamma distribution with shape parameter $\alpha = 5$, scale parameter $\theta = \sqrt{(1/5)}$ and expected value 0																
j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_j	2.326	1.192	1.050	1.064	1.009	1.003	0.997	0.999	0.999	0.999						
σ_j^2	10748.9	666.6	122.1	181.2	93.1	6.2	2.2	2.7	0.5	0.4						
(Commercial Liability) Shifted Gamma distribution with shape parameter $\alpha = 1.5$, scale parameter $\theta = \sqrt{(1/1.5)}$ and expected value 0																
j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_j	3.072	1.692	1.374	1.204	1.195	1.075	1.078	1.069	1.065	1.024	1.017	1.018	1.014	1.008	1.007	1.054
σ_j^2	8002.2	2302.7	897.5	624.2	356.2	203.2	116.0	66.2	37.8	17.2	14.0	7.0	6.0	1.8	1.0	151.6

Table 17. Expected squared deviation between $\text{estimator}^{1/2}$ and $\text{true value}^{1/2}$, given $\mathcal{B}_0$, by class of business.

Estimator	Motor Liability	Motor Hull	Property	Commercial Liability
Mack formula	6'044.5	428.062	23'963.1	7'718.1
BBMW formula	6'044.6	428.063	23'964.5	7'718.4
Unbiased formula	6'044.1	428.061	23'957.6	7'709.1

note that, when believing in the above mentioned empirical conjecture, we might conclude that our first two guesses are more realistic, because for these choices the true value $\text{PV}_{\text{tot}} + \text{EE}_{\text{tot}}$ is bigger than the three estimated values, meaning that the real uncertainty is higher than indicated by the classical Mack formula. Also note that the difficulty in making a rational guess is highly driven by the sigma's parameters, since the variances of $(\widehat{\sigma_j^2})$ are very high for the considered triangle size. In that respect, we remark that when having more accident years data at our disposal, the guessing procedure would likely be facilitated and the related true values will tend to be closer to the three formulas, as also indicated by the simulated example in section 4.1 (see Table 13).

4.3.2 The Taylor-Ashe triangle

We consider the triangle presented in Taylor & Ashe (1983) (see Table 20) and first observe that the regularity condition (25) is fulfilled.

Evaluating the three estimators for the ultimate prediction uncertainty, we get the results displayed in Table 21 and again remark that the formulas lead to very close outcomes.

Similarly to the previous example, we observe that the true value $\text{PV}_{\text{tot}} + \text{EE}_{\text{tot}}$ computed based on some rational guesses for the true model parameters could materially deviate from the three formulas. Moreover, the fact that the true value can be bigger than the estimated values for several model parameter guesses might also be a driver for the empirical conjecture suggesting that the classical Mack estimator may be on the optimistic side.

5. Conclusion

Within the chain-ladder model of Mack and under some additional technical assumptions, we derived a new estimator for the ultimate prediction uncertainty that is unbiased (conditionally given the first triangle column). Unluckily, from a theoretical point of view, the unbiased estimator does also show some peculiar behaviours, in particular with respect to its possible negativity. However, from a practice point of view, these behaviours do not create difficulties when considering typical insurance data. Also, the required additional technical assumptions can be reasonably assumed to hold true, since they are consistent with data typically encountered in actuarial practice, and should not be confused with the strong (and possibly inconsistent) assumption of positivity along all the (finite) trajectories of the claims payments process.

Furthermore, we underline that the three considered formulas (Mack, BBMW and Unbiased) produce extremely similar results, and the differences may not be significant for applications in actuarial practice.

Therefore, using the straightforward explained Unbiased formula (which deliver the smallest result) does not lead to any relevant misestimation of reserve risk compared to the Mack and BBMW formulas, also in case the true value occurs to be bigger than its estimators.

Indeed, we demonstrated that all three estimators can, with non-negligible probability, materially fail to quantify the true uncertainty, especially when considering small or average-sized triangles often available in practice.

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Table 18. The Merz-Wüthrich triangle: Cumulative payments ($C_{i,j}$), estimated and guessed parameters.

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	13’109	20’355	21’337	22’043	22’401	22’658	22’997	23’158	23’492	23’664	23’699	23’904	23’960	23’992	23’994	24’001	24’002
1	14’457	22’038	22’627	23’114	23’238	23’312	23’440	23’490	23’964	23’976	24’048	24’111	24’252	24’538	24’540	24’550	
2	16’075	22’672	23’753	24’052	24’206	24’757	24’786	24’807	24’823	24’888	24’986	25’401	25’681	25’705	25’732		
3	15’682	23’464	24’465	25’052	25’529	25’708	25’752	25’770	25’835	26’075	26’082	26’146	26’150	26’167			
4	16’551	23’706	24’627	25’573	26’046	26’115	26’283	26’481	26’701	26’718	26’724	26’728	26’735				
5	15’439	23’796	24’866	25’317	26’139	26’154	26’175	26’205	26’764	26’818	26’836	26’959					
6	14’629	21’645	22’826	23’599	24’992	25’434	25’476	25’549	25’604	25’709	25’723						
7	17’585	26’288	27’623	27’939	28’335	28’638	28’715	28’759	29’525	30’302							
8	17’419	25’941	27’066	27’761	28’043	28’477	28’721	28’878	28’948								
9	16’665	25’370	26’909	27’611	27’729	27’861	29’830	29’844									
10	15’471	23’745	25’117	26’378	26’971	27’396	27’480										
11	15’103	23’393	26’809	27’691	28’061	29’183											
12	14’540	22’642	23’571	24’127	24’210												
13	14’590	22’336	23’440	24’029													
14	13’967	21’515	22’603														
15	12’930	20’111															
16	12’539																
(Parameter estimates)																	
$\widehat{f_j}$	1.511	1.054	1.027	1.017	1.013	1.011	1.003	1.011	1.007	1.001	1.006	1.004	1.0036	1.0004	1.0004	1.00004	
$\widehat{\sigma_j^2}$	29.499	16.833	2.648	5.444	3.248	11.424	0.188	2.613	2.086	0.054	0.908	0.540	0.720	0.008	0.0002	0.000003	
(Guess 1)																	
f_j	1.530	1.055	1.030	1.018	1.013	1.011	1.011	1.011	1.007	1.006	1.005	1.004	1.003	1.0005	1.0004	1.00005	
σ_j^2	25.000	12.000	4.500	4.000	3.500	2.000	2.000	2.000	1.000	0.900	0.800	0.400	0.300	0.010	0.0001	0.000001	
(Guess 2)																	
f_j	1.500	1.060	1.030	1.025	1.010	1.010	1.0045	1.0044	1.002	1.0015	1.001	1.001	1.0007	1.0006	1.0004	1.00004	
σ_j^2	35.000	20.000	3.000	3.000	2.000	2.000	1.500	1.500	1.000	0.900	0.800	0.500	0.500	0.020	0.0002	0.0000035	
(Guess 3)																	
f_j	1.532	1.047	1.027	1.017	1.013	1.005	1.003	1.011	1.004	1.0024	1.004	1.004	1.004	1.0005	1.0004	1.00005	
σ_j^2	36.644	19.308	2.863	5.536	1.530	7.000	0.188	2.683	0.288	0.083	1.247	0.596	0.942	0.008	0.0001	0.000003	

Table 19. The Merz-Wüthrich triangle: The ultimate prediction error for aggregated accident years (as a percentage of the reserves).

	Prediction error$^{1/2}$	Process variance$^{1/2}$	Estimation error$^{1/2}$
Mack formula (39)	3'233.681 (13.4%)	2'467.086	2'090.497
BBMW formula (42)	3'233.698 (13.4%)	2'467.086	2'090.524
Unbiased formula (35)	3'233.606 (13.4%)	2'467.011	2'090.470
True value (Guess 1)	3'717.576 (15.4%)	2'091.983	3'073.105
True value (Guess 2)	5'326.065 (22.1%)	2'053.842	4'914.133
True value (Guess 3)	2'756.582 (11.4%)	2'272.219	1'560.694

Table 20. The Taylor-Ashe triangle: Cumulative payments ($C_{i,j}$), estimated and guessed parameters.

i/j	0	1	2	3	4	5	6	7	8	9
0	357'848	1'124'788	1'735'330	2'218'270	2'745'596	3'319'994	3'466'336	3'606'286	3'833'515	3'901'463
1	352'118	1'236'139	2'170'033	3'353'322	3'799'067	4'120'063	4'647'867	4'914'039	5'339'085	
2	290'507	1'292'306	2'218'525	3'235'179	3'985'995	4'132'918	4'628'910	4'909'315		
3	310'608	1'418'858	2'195'047	3'757'447	4'029'929	4'381'982	4'588'268			
4	443'160	1'136'350	2'128'333	2'897'821	3'402'672	3'873'311				
5	396'132	1'333'217	2'180'715	2'985'752	3'691'712					
6	440'832	1'288'463	2'419'861	3'483'130						
7	359'480	1'421'128	2'864'498							
8	376'686	1'363'294								
9	344'014									
(Parameter estimates)										
$\widehat{f_j}$	3.491	1.747	1.457	1.174	1.104	1.086	1.054	1.077	1.018	
$\widehat{\sigma_j^2}$	160'280	37'737	41'965	15'183	13'731	8'186	447	1'147	447	
(Guess 1)										
f_j	3.300	1.740	1.500	1.200	1.110	1.090	1.065	1.060	1.020	
σ_j^2	160'000	42'000	38'000	15'000	13'000	9'000	700	600	500	
(Guess 2)										
f_j	3.500	1.700	1.550	1.220	1.150	1.100	1.065	1.055	1.010	
σ_j^2	150'000	55'000	43'000	16'000	14'000	8'000	1'000	900	700	
(Guess 3)										
f_j	3.700	1.750	1.500	1.180	1.130	1.100	1.050	1.080	1.020	
σ_j^2	120'000	36'000	33'000	12'000	10'000	5'000	600	400	200	

Table 21. The Taylor-Ashe triangle: The ultimate prediction error for aggregated accident years (as a percentage of the reserves).

	Prediction error$^{1/2}$	Process variance$^{1/2}$	Estimation error$^{1/2}$
Mack formula (39)	2'447'095 (13.1%)	1'878'292	1'568'532
BBMW formula (42)	2'447'618 (13.1%)	1'878'292	1'569'349
Unbiased formula (35)	2'444'848 (13.1%)	1'876'045	1'567'717
True value (Guess 1)	2'092'493 (11.2%)	1'928'143	812'891
True value (Guess 2)	3'312'339 (17.7%)	2'112'207	2'551'504
True value (Guess 3)	2'814'234 (15.1%)	1'756'879	2'198'474

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