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δ-CONTINUOUS SELECTIONS OF SMALL MULTIFUNCTIONS

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1. Introduction. A *multifunction* $\varphi : X \to Y$ from a topological space X into a topological space Y is a correspondence such that $\varphi(x)$ is a non-empty subset of Y for every $x \in X$. A single-valued function $f : X \to Y$ is called a *selection* of φ if $f(x) \in \varphi(x)$ for all $x \in X$; it is called a continuous selection if f is continuous. It is well-known that not every semi-continuous or even continuous multifunction has a continuous selection (see e.g. [**4**] for a survey on selection theory).

We investigate here some connections between multifunctions which are "almost single-valued" and selections which are "almost continuous". More precisely, let Y be a metric space with metric d, and define that the multifunction φ is *δ-small* for some $\delta > 0$ if the diameter $\operatorname{diam}(\varphi(x)) \leqq \delta$ for all $x \in X$. We further use the terminology of Muenzenberger [**5**] and Smithson [**6**] and say that $f : X \to Y$ is *δ-continuous* if for every $x \in X$ there exists an open neighbourhood $U(x)$ such that $f(U(x)) \subset S_\delta(f(x))$, where

$$S_\delta(f(x)) = \{y \in Y | d(y, f(x)) < \delta\}.$$

(This definition is related to, but not identical with, the definition of δ-continuity originally introduced by Klee [**3**].) The function f is continuous if it is δ-continuous for every $\delta > 0$ [**6**], and clearly φ is single-valued if it is δ-small for all $\delta > 0$. Hence we can think of f as almost continuous and of φ as almost single-valued if δ is small.

We prove in Theorem 2.1 that every selection of an upper or lower semi-continuous δ-small multifunction is 2δ-continuous, and in Theorem 2.2 that for every δ-continuous function f on a compact Hausdorff space X, there exist 2δ-small and either upper or lower semi-continuous multifunctions for which f is a selection. We further show that the multifunctions φ constructed in Theorem 2.2 have "nice" properties for certain spaces. They are convex-valued if Y is a normed linear space, and continuum-valued if Y is a dendrite. Klee's [**3**] and Smithson's [**6**] results that compact convex subsets of finite-dimensional normed linear spaces and dendrites have the proximate fixed point property are obtained as corollaries.

2. δ-continuous selections of small multifunctions. Let X and Y be topological spaces. A multifunction $\varphi : X \to Y$ is called usc (*upper semi-*

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continuous) if for every open set $V \subset Y$ with $\varphi(x) \subset V$ there exists an open neighbourhood U of x such that $\varphi(U) \subset V$. It is called lsc (*lower semi-continuous*) if for every $x \in X$ and for every open set $V \subset Y$ with $\varphi(x) \cap V \neq \emptyset$ there exists an open neighbourhood U of x such that $\varphi(x') \cap V \neq \emptyset$ for all $x' \in U$. If φ is both usc and lsc it is called *continuous*. A *point-closed* (*point-open*) multifunction is a multifunction for which $\varphi(x)$ is closed (open) for each $x \in X$.

We assume from now on that Y is a metric space with a metric d. Let

$$S_\delta(A) = \{y \in Y | d(y, A) < \delta\}$$

and

$$\bar{S}_\delta(A) = \{y \in Y | d(y, A) \leqq \delta\} \quad .$$

The first theorem is an easy consequence of the definitions.

THEOREM 2.1. *If the multifunction $\varphi : X \to Y$ is δ-small and either usc or lsc, then every selection of φ is 2δ-continuous.*

Proof. Let $f : X \to Y$ be a selection of φ.

(i) φ is usc: Take $V(x) = S_{2\delta}(f(x))$. As φ is δ-small we have $\varphi(x) \subset V(x)$, and as φ is usc, there exists an open neighbourhood $U(x)$ such that $\varphi(U(x)) \subset V(x)$. Hence $f(U(x)) \subset S_{2\delta}(f(x))$, so that f is 2δ-continuous.

(ii) φ is lsc: This time take $V(x) = S_\delta(f(x))$. As φ is lsc and $\varphi(x) \cap V(x) \neq \emptyset$, there exists an open neighbourhood $U(x)$ such that $\varphi(x') \cap V(x) \neq \emptyset$ for all $x' \in U(x)$. As φ is δ-small we have $d(f(x), f(x')) < 2\delta$ for all $x' \in U(x)$, hence $f(U(x)) \subset S_{2\delta}(f(x))$ and f is 2δ-continuous.

Note that the constant 2δ in the conclusion of the theorem cannot be replaced by δ. As an example we can take the multifunction $\varphi : [0, 1] \to [0, 1]$ defined by

$$\begin{aligned} \varphi(x) &= \tfrac{1}{4} \quad \text{for} \quad 0 \leqq x < \tfrac{1}{2}, \\ \varphi(\tfrac{1}{2}) &= \{\tfrac{1}{4}, \tfrac{3}{4}\}, \\ \varphi(x) &= \tfrac{3}{4} \quad \text{for} \quad \tfrac{1}{2} < x \leqq 1 \end{aligned}$$

which has no $\frac{1}{2}$-continuous selection although it is a $\frac{1}{2}$-small usc multifunction. But the constant 2δ can be replaced by $\delta + \epsilon$, for any $\epsilon > 0$.

We now show that on the other hand every δ-continuous function can be a selection of a suitable small usc or lsc multifunction, at least if X is compact Hausdorff. The proof is more complicated, but constructive.

THEOREM 2.2. *Let X be compact Hausdorff and $f : X \to Y$ be δ-continuous. Then there exist both a point-closed usc and a point-open lsc 2δ-small multifunction for which f is a selection.*

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Proof. (i) Construction of a point-closed usc multifunction. Choose for every $x \in X$ an open neighbourhood $U(x)$ such that $f(U(x)) \subset S_\delta(f(x))$. As X is compact, the cover $\{U(x)|x \in X\}$ has a finite subcover

$$\{U(x_i)|i \in I \text{ and } I \text{ finite}\}.$$

For every $x \in X$, denote by $J(x)$ the subset of I given by $i \in J(x)$ if and only if $x \in U(x_i)$. Define a correspondence $\varphi : X \to Y$ by

$$\varphi(x) = \bigcap [\bar{S}_\delta(f(x_i))|i \in J(x)]. \tag{1}$$

We show that φ has the desired properties.

For all $x \in X$ we have $x \in \bigcap [U(x_i)|i \in J(x)]$, and hence

$$\begin{aligned} f(x) \in f[\bigcap (U(x_i)|i \in J(x))] &\subset \bigcap [f(U(x_i))|i \in J(x)] \\ &\subset \bigcap [\bar{S}_\delta(f(x_i))|i \in J(x)] = \varphi(x). \end{aligned}$$

So $\varphi(x) \neq \emptyset$, and f is a selection of the multifunction φ.

To show that φ is usc, choose any $x \in X$, and let V be an open set with $\varphi(x) \subset V$. As $J(x)$ is finite, the set $N(x) = \bigcap [U(x_i)|i \in J(x)]$ is an open neighbourhood of x. If $x' \in N(x)$, then the definition (1) of φ implies that $\varphi(x') \subseteq \varphi(x)$. Therefore we have $\varphi(N(x)) \subseteq \varphi(x) \subset V$. As $\varphi(x)$ is closed and diam $\varphi(x) \leqq 2\delta$ by construction, φ is the desired point-closed usc and 2δ-small multifunction.

(ii) Construction of a point-open lsc multifunction. For every $x \in X$ there exists an open neighbourhood $U'(x)$ with $f(U'(x)) \subset S_\delta(f(x))$, and as X is regular, we can select an open neighbourhood $U(x)$ with

$$U(x) \subset \mathrm{Cl}(U(x)) \subset U'(x),$$

where Cl denotes the closure. Let $\{U(x_i)|i \in I\}$ again be a finite subcover of the cover $\{U(x)|x \in X\}$ of the compact space X. For any $x \in X$ let this time $J(x)$ be the subset of I given by $i \in J(x)$ if and only if $x \in \mathrm{Cl}(U(x_i))$ and define $\psi : X \to Y$ by

$$\psi(x) = \bigcap [S_\delta(f(x_i))|i \in J(x)]. \tag{2}$$

In a similar way as in (i) we see that $f(x) \in \psi(x)$ for every $x \in X$, so that f is a selection of the multifunction ψ.

To show that ψ is lsc, choose for any $x \in X$ an open set V with $V \cap \psi(x) \neq \emptyset$. As $I \backslash J(x)$ is finite, the set $\bigcup [\mathrm{Cl}(U(x_i)|i \in I\backslash J(x)]$ is closed, and we can find an open neighbourhood $N(x)$ such that

$$N(x) \cap (\bigcup [\mathrm{Cl}(U(x_i)|i \in I\backslash J(x)]] = \emptyset.$$

Therefore $\psi(x') \supseteq \psi(x)$ for all $x' \in N(x)$, and hence $\psi(x') \cap V \neq \emptyset$. So ψ is lsc, and it is point-open and 2δ-small by construction.

This completes the proof of Theorem 2.2.

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We notice that in general $\varphi(x) \neq \psi(x)$ and that φ is usually not lsc and ψ not usc. It would be interesting to know whether one can also construct a continuous 2δ-small multifunction for which f is a selection.

3. Application to the proximate fixed point property. The space X with metric d has the p.f.p.p. (*proximate fixed point property*) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every δ-continuous function $f : X \to X$ there is a point $x \in X$ with $d(x, f(x)) < \epsilon$. (The concept of the p.f.p.p. was introduced by Klee [**3**], but we use the slightly different definitions of Smithson [**6**] and Muenzenberger [**5**].) Theorem 2.2 can be used to obtain results on the p.f.p.p. The method consists in showing that for suitable spaces the function φ defined in (1) has additional properties which ensure a fixed point of φ.

PROPOSITION 3.1. *If Y is a convex subset of a normed linear space, then the multifunction φ defined in* (1) *is convex-valued.*

Proof. The set $\bar{S}_\delta(f(x_i))$ is convex, and the intersection of convex sets is convex also.

COROLLARY 3.2. *A compact convex subset C of a finite-dimensional normed linear space has the p.f.p.p.* (See Klee [**3**, Theorem 3].)

Proof. For any $\epsilon > 0$, take $\delta = \epsilon/3$, and let $f : C \to C$ be δ-continuous. The function $\varphi : C \to C$ defined in (1) is usc and convex-valued, and hence has a fixed point x_0; i.e. a point with $x_0 \in \varphi(x_0)$ [**1**, p. 251]. As φ is 2δ-small and $f(x_0) \in \varphi(x_0)$, we have $d(x_0, f(x_0)) \leqq 2\delta < \epsilon$, so that C has the p.f.p.p.

A *dendrite* is a metrizable continuum in which every pair of points can be separated by a third one. It has a convex metric [**2**].

PROPOSITION 3.3. *If Y is a dendrite with a convex metric, then the multifunction φ defined in* (1) *is continuum-valued.*

Proof. Each set $\bar{S}_\delta(f(x_i))$ used in the definition of φ is arc-connected, as it contains with every point $z \neq f(x_i)$ also the unique arc from z to $f(x_i)$. Hence it is a continuum, and the intersection of finitely many continua in a dendrite is a continuum [**8**, p. 88].

COROLLARY 3.4. *A dendrite has the p.f.p.p.* (See Smithson [**6**].)

Proof. The proof is analogous to that of Corollary 3.2, as dendrites have the fixed point property for usc and continuum-valued multifunctions [**7**].

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