



DERIVED HECKE ALGEBRA AND COHOMOLOGY OF ARITHMETIC GROUPS

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Abstract

We describe a graded extension of the usual Hecke algebra: it acts in a graded fashion on the cohomology of an arithmetic group Γ . Under favorable conditions, the cohomology is freely generated in a single degree over this graded Hecke algebra.

From this construction we extract an action of certain p -adic Galois cohomology groups on $H^*(\Gamma, \mathbf{Q}_p)$, and formulate the central conjecture: the motivic \mathbf{Q} -lattice inside these Galois cohomology groups preserves $H^*(\Gamma, \mathbf{Q})$.

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1. Introduction

Let \mathbf{G} be a semisimple \mathbf{Q} -group, and let $Y(K)$ be the associated arithmetic manifold (see (17)). Particularly when $Y(K)$ is not an algebraic variety, it often happens that the same Hecke eigensystem can occur in several different cohomological degrees (see Section 1.2). Our goal is to construct extra endomorphisms of cohomology that partly explain this, and give evidence that these extra endomorphisms are related to certain motivic cohomology groups.

1.1. Derived Hecke algebra. Let v be a prime, $G_v = \mathbf{G}(\mathbf{Q}_v)$ and K_v a maximal compact subgroup. The usual Hecke algebra at v , with coefficients in a (say finite) ring S , can be described as $\mathrm{Hom}_{SG_v}(S[G_v/K_v], S[G_v/K_v])$. If in place of Hom we use Ext (see Section 2 for more details), we get a graded extension, which we may call the ‘local derived Hecke algebra’:

$$\mathcal{H}_{v,S} := \bigoplus_i \mathrm{Ext}_{SG_v}^i(S[G_v/K_v], S[G_v/K_v]). \quad (1)$$

Such a construction has been considered by Schneider [28] in the context of local representation theory in the case when S has characteristic v . In the present paper, however, we are solely interested in the opposite case, when v is invertible on S .

For elementary reasons, the higher exts are ‘almost’ killed by $q_v - 1$, where q_v is the size of the residue field; thus this algebra is of most interest when $q_v = 1$ in S . In that case we have (Section 3, Theorem 3.3) a *Satake isomorphism*: if $q_v = 1$ in S , then $\mathcal{H}_{v,S}$ is isomorphic to the Weyl-invariants on the corresponding algebra for a torus (and is thus graded-commutative).

Now $\mathcal{H}_{v,S}$ acts on the cohomology $H^*(Y(K), S)$ in a graded fashion—the Ext^i component shifts degree by $+i$. (See Section 1.4 for an explicit version, Section 2.6 for the abstract version.) In particular, $\bigotimes_v \mathcal{H}_{v,\mathbf{Z}/p^n}$ acts on $H^*(Y(K), \mathbf{Z}/p^n)$ and then (by passing to the limit as $n \rightarrow \infty$, Section 2.13) we get a ring of endomorphisms

$$\tilde{\mathbb{T}} \subset \mathrm{End} H^*(Y(K), \mathbf{Z}_p),$$

the ‘global derived Hecke algebra.’ The degree zero component of $\tilde{\mathbb{T}}$ is the usual Hecke algebra \mathbb{T} —that is, the subalgebra of $\mathrm{End} H^*$ generated by all Hecke operators. Here, and elsewhere in the introduction, we use in the Hecke algebra only ‘good’ places v relative to K .

It seems to us likely that $\tilde{\mathbb{T}}$ is graded-commutative but we do not know this in general. However, this has limited impact for global applications because, firstly,

different places commute; and secondly our analysis in Section 3 provides a large set of ‘nice primes’ at which the local derived Hecke algebra is graded-commutative. In particular, all of our work in this paper really analyzes the subalgebra generated by these ‘nice primes’ (see after (36) for definition). This subalgebra is what is important for all our applications, as will become clear from the discussion of Section 1.2 onward. In many cases, the existence of this subalgebra is anyway enough to force all of $\tilde{\mathbb{T}}$ to be graded-commutative (see Theorem 5.2 part (ii) and Proposition 8.6).

As we have mentioned, if we decompose $H^j(Y(K), \mathbf{Z}_p)$ into eigencharacters for \mathbb{T} , one finds the same eigencharacters occurring in several different degrees j . See [35] for an elementary introduction to this phenomenon. We want to see that \mathbb{T} is rich enough to account for this.

One way of formalizing ‘rich enough’ is to complete the cohomology at a given character $\chi : \mathbb{T} \rightarrow \mathbf{F}_p$ of the usual Hecke algebra, and ask that $H^*(Y(K), \mathbf{Z}_p)_\chi$ be generated over $\tilde{\mathbb{T}}$ in minimal degree. In other words, we should like to check surjectivity of the map

$$\tilde{\mathbb{T}} \otimes H^{\mathfrak{q}}(Y(K), \mathbf{Z}_p)_\chi \rightarrow H^*(Y(K), \mathbf{Z}_p)_\chi \tag{2}$$

where \mathfrak{q} is the minimal degree where $H^*(Y(K), \mathbf{Z}_p)_\chi$ is nonvanishing. (Note that, if $\tilde{\mathbb{T}}$ is not known to be graded-commutative, it is not *a priori* clear it preserves the χ -eigenspace. However, under very mild assumptions it does, and this will be the situation in the cases we analyze. Alternately one can switch to the graded-commutative subalgebra described above. For the purpose of the introduction, then, the reader may either assume that $\tilde{\mathbb{T}}$ preserves H_χ^* or assume that $\tilde{\mathbb{T}}$ is replaced by the ‘strict global derived Hecke algebra’ defined in Section 2.14.)

In Theorems 5.2 and 7.6, we prove this in two different cases (in both cases, we require the prime p to be large enough):

Theorem 5.2, proved in Section 5, studies the case when \mathbf{G} is (the \mathbf{Q} -group corresponding to) an inner form of SL_n over an imaginary quadratic field, and χ is the character $T \mapsto \deg(T)$ that sends any Hecke operator T to its degree. The main point is that, in this case, $H^*(Y(K), \mathbf{Z}_p)_\chi$ can be described in terms of algebraic K -theory.

Theorem 7.6, proved in Sections 6 and 7, treats the case of χ associated to a tempered cohomological automorphic form, *assuming the existence of Galois representations attached to cohomology classes on $Y(K)$, satisfying the expected properties (see Section 6.2)*. (Our assumptions are similar to Calegari–Geraghty [9]; however, we do not need the assumptions on vanishing of cohomology because we allow ourselves to discard small p . Since the first version of this paper, remarkable progress has been made in analyzing the

Galois representations attached to torsion classes. In particular, the paper [1, Theorem 3.11] of Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor and Thorne establishes, among many other results, the key local–global compatibility needed for Taylor–Wiles patching. Moreover, the paper [11] of Caraiani, Gulotta, Hsu, Johansson, Mocz, Reinecke, and Shih eliminates the use of nilpotent ideals in the Galois representations originally constructed by Scholze. While these do not precisely match with the inputs needed for our setup of the argument, they appear to address the key issues, and so to me it seems very likely that one could produce an unconditional version of our analysis in the near future.) In this analysis we also impose some assumptions on χ for our convenience—for example, ‘minimal level,’ and excluding congruences with other forms—we have not attempted to be general. Here, the main tool of the proof is a very striking interaction between the derived Hecke algebra and the Taylor–Wiles method. We discuss this interaction further in Section 1.4.

The proofs of Sections 5 and 6–7 are quite different, but they have an interesting feature in common. In both cases, we use the derived Hecke algebra at primes q such that restriction to $G_{\mathbf{Q}_q}$ kills certain classes in global Galois cohomology. These classes live inside a certain dual Selmer group (specifically, the right-hand side of (9) below).

That this particular dual Selmer group arises is quite striking, because it seems to be a p -adic avatar of a certain motivic cohomology group; and this same motivic cohomology group is suggested, in [25], to act on the rational cohomology of $Y(K)$. This brings us to the core motivation of this paper: the derived Hecke algebra allows one to construct a p -adic realization of the operations on rational cohomology proposed in [25]. Therefore, we digress to describe the conjectures of [25]. We return to describe the remainder of the current paper in Section 1.3.

1.2. Motivic cohomology. This section is solely motivational, and so we freely assume various standard conjectures without giving complete references. We shall also allow ourselves to be slightly imprecise in the interest of keeping the exposition brief. We refer to the paper [25] for full details.

Let $\chi : \mathbb{T} \rightarrow \mathbf{Q}$ be a character of the usual Hecke algebra, now with \mathbf{Q} values. We suppose that χ is tempered and cuspidal. By this, we mean that there is a collection π_1, \dots, π_r of cuspidal automorphic representations, each tempered at ∞ , such that the generalized χ -eigenspace $H^*(Y(K), \mathbf{C})_\chi$ is exactly equal to the subspace of cohomology associated to the π_i s.

Consider now this generalized eigenspace with rational coefficients

$$H^*(Y(K), \mathbf{Q})_\chi \subset H^*(Y(K), \mathbf{Q}).$$

One can understand its dimension data completely. To do so we introduce some numerical invariants: let δ, \mathfrak{q} be defined such that

$$\delta = \text{rank } \mathbf{G}(\mathbf{R}) - \text{rank } K_\infty. \tag{3}$$

$$2\mathfrak{q} + \delta = \dim Y(K). \tag{4}$$

Then we have (see [5, Theorem III.5.1]; also [4, Corollary 5.5])

$$\dim H^{\mathfrak{q}+i}(Y(K), \mathbf{Q})_\chi = \binom{\delta}{i} \dim H^{\mathfrak{q}}(Y(K), \mathbf{Q})_\chi. \tag{5}$$

In [25], a conjectural explanation for this numerology is proposed. Namely, we construct a δ -dimensional \mathbf{Q} -vector space and suggest that its exterior algebra acts on $H^*(Y(K), \mathbf{Q})_\chi$. To define the vector space requires a discussion first of the motive associated to χ , and then of its motivic cohomology.

1.2.1. The Galois representation and the motive associated to χ . It is conjectured (and in some cases proven [30]) that to such χ there is, for every prime p , a Galois representation $\rho_\chi : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow {}^L\hat{\mathbf{G}}(\mathbf{Q}_p)$, where ${}^L\hat{\mathbf{G}}$ is the Langlands dual group. (*A priori*, this takes $\overline{\mathbf{Q}}_p$ coefficients; we, for simplicity, assume that it can actually be defined over \mathbf{Q}_p . Moreover, in general [7, Section 3.4] one has to replace ${}^L\hat{\mathbf{G}}$ by a slightly different group to define ρ_χ , but the foregoing discussion goes through with no change.)

We shall suppose that p is a good prime, not dividing the level of the original arithmetic manifold $Y(K)$ (for the precise meaning of ‘level,’ see after (19)). In particular, this means that ρ_χ should be crystalline upon restriction to $G_{\mathbf{Q}_p}$.

Now we shall compose ρ with the coadjoint representation ${}^L\hat{\mathbf{G}} \rightarrow \text{Aut}(\hat{\mathfrak{g}}^*)$ of ${}^L\hat{\mathbf{G}}$ on the dual of its own Lie algebra (here $\hat{\mathfrak{g}}$ is the Lie algebra of the dual group to \mathbf{G} , considered as a \mathbf{Q} -group, and $\hat{\mathfrak{g}}^*$ is its \mathbf{Q} -linear dual). The result is

$$\text{Ad}^* \rho_\chi : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \text{Aut}(\hat{\mathfrak{g}}^* \otimes \mathbf{Q}_p).$$

It is also conjectured that $\text{Ad}^* \rho_\chi$ should be motivic. In other words, there should exist a weight zero motive M_{coad} over \mathbf{Q} , the ‘coadjoint motive for χ ,’ whose Galois realization is isomorphic to $\text{Ad}^* \rho_\chi$:

$$H_{\text{et}}^*(M_{\text{coad}} \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p) \simeq \text{Ad}^* \rho_\chi \text{ (in cohomological degree 0)}. \tag{6}$$

For simplicity we shall assume that M_{coad} can be taken to be a Chow motive, and will suppose that the coefficient field of M_{coad} is equal to \mathbf{Q} . (Since (6) only determines the étale realization, it is more natural to consider M_{coad}

as a homological motive. Assuming standard conjectures [23, Section 7.3 Remark 3.bis], this can be promoted (noncanonically) to a Chow motive. The independence of the constructions that follow requires a further conjecture, for example, the existence of the Bloch–Beilinson filtration on K -theory.)

1.2.2. *Motivic cohomology groups associated to χ .* For such an M_{coad} , and indeed for any Chow motive M , we can define (after Voevodsky) a bigraded family of motivic cohomology groups $H_{\text{mot}}^a(M, \mathbf{Q}(q))$; the indexing is chosen so that this admits a comparison map to the corresponding absolute étale cohomology group $H_{\text{ét}}^a(M, \mathbf{Q}_p(q))$.

We will be solely interested in the motivic cohomology group with $a = q = 1$; in this case, with the coadjoint motive, the comparison with étale cohomology gives

$$H_{\text{mot}}^1(M_{\text{coad}}, \mathbf{Q}(1)) \otimes_{\mathbf{Q}} \mathbf{Q}_q \rightarrow H^1(G_{\mathbf{Q}}, \text{Ad}^* \rho_{\chi}(1)). \tag{7}$$

Now Scholl [29, Theorem 1.1.6] has shown that one can define (again for any Chow motive M over \mathbf{Q}) a natural subspace $H_{\text{mot}}^a(M_{\mathbf{Z}}, \mathbf{Q}(q)) \subset H_{\text{mot}}^a(M, \mathbf{Q}(q))$ of its motivic cohomology, informally speaking ‘those classes that extend to a good model over \mathbf{Z} .’ Conjecturally, the analog of the above map should now take values inside the f -cohomology of Bloch and Kato [2]; in the case of interest the analog of (7) is now

$$H_{\text{mot}}^1((M_{\text{coad}})_{\mathbf{Z}}, \mathbf{Q}(1)) \otimes_{\mathbf{Q}} \mathbf{Q}_q \rightarrow H_f^1(G_{\mathbf{Q}}, \text{Ad}^* \rho_{\chi}(1)).$$

Moreover, this map is conjecturally ([2, 5.3(ii)]) an isomorphism.

It may be helpful to note that Beilinson’s conjecture relates this particular motivic cohomology to the value of the L -function for $\text{Ad}^* \rho_{\chi}$ at the edge of the critical strip. In particular, Beilinson’s conjectures imply that

$$\dim_{\mathbf{Q}} H_{\text{mot}}^1((M_{\text{coad}})_{\mathbf{Z}}, \mathbf{Q}(1)) = \text{order of vanishing of } L(s, \text{Ad}^* \rho_{\chi}) \text{ at } s = 0.$$

A routine computation with Γ -factors shows that the right-hand side should indeed be equal to δ .

To keep typography simple, we denote the group $H_{\text{mot}}^1((M_{\text{coad}})_{\mathbf{Z}}, \mathbf{Q}(1))$ simply by L :

$$L := H^1((M_{\text{coad}})_{\mathbf{Z}}, \mathbf{Q}(1)). \tag{8}$$

so that our discussion above says that, granting standard conjectures, L is a \mathbf{Q} -vector space of dimension δ , and it comes with a map

$$L \otimes_{\mathbf{Q}_p} \rightarrow \underbrace{H_f^1(G_{\mathbf{Q}}, \text{Ad}^* \rho_{\chi}(1))}_{:= L_{\mathbf{Q}_p}} \tag{9}$$

1.2.3. *The complex regulator on L and the conjectures of [25].* There is a complex analog to (9): a complex regulator map on L , with target in a certain Deligne cohomology group. Since the details are not important for us, we just call the target of this map $L_{\mathbf{C}}$ and let $L_{\mathbf{C}}^*$ be its \mathbf{C} -linear dual.

In [25], we construct an action of $L_{\mathbf{C}}^*$ on $H^*(Y(K), \mathbf{C})_{\chi}$ by degree 1 endomorphisms, inducing

$$H^q(Y(K), \mathbf{C})_{\chi} \otimes \bigwedge^i L_{\mathbf{C}}^* \xrightarrow{\sim} H^{q+i}(Y(K), \mathbf{C})_{\chi}. \quad (10)$$

The main conjecture of [25] is that this action preserves rational structures, that is, the \mathbf{Q} -linear dual L^* of L carries $H^*(Y(K), \mathbf{Q})_{\chi}$ to itself. In particular, this means that

$$\text{There is a natural graded action of } \bigwedge^* L^* \text{ on } H^*(Y(K), \mathbf{Q})_{\chi}. \quad (11)$$

Therefore, if one accepts the conjecture of (11), and also believes that (9) is an isomorphism, it should be possible to define a ‘natural’ action of

$$\bigwedge^* L^* \otimes \mathbf{Q}_p = \bigwedge^* H_f^1(G_{\mathbf{Q}}, \text{Ad}^* \rho_{\chi}(1))^* \supset H^*(Y(K), \mathbf{Q}_p)_{\chi}. \quad (12)$$

Now there is no explicit mention of motivic cohomology, and this is where the current paper comes into the story: in Section 8, we shall explain how the derived Hecke algebra can be used to produce such an action.

This concludes our review of [25]; we now explain (12) a little bit more.

1.3. The derived Hecke algebra and Galois cohomology. The main result of Section 8 is Theorem 8.5, which constructs an action of $\bigwedge^* H_f^1(G_{\mathbf{Q}}, \text{Ad}^* \rho_{\chi}(1))^*$ on $H^*(Y(K), \mathbf{Q}_p)$. This is characterized in terms of the action of explicit derived Hecke operators. More precisely, we construct in Section 8.24 an isomorphism

$$\tilde{\mathbb{T}} \otimes \mathbf{Q}_p \simeq \mathbb{T} \otimes_{\mathbf{Q}_p} \bigwedge^* H_f^1(G_{\mathbf{Q}}, \text{Ad}^* \rho_{\chi}(1))^* \quad (13)$$

(actually, we do this in a case when $\mathbb{T} = \mathbf{Z}_p$, but in general the argument should yield the above result). Informally, (13) gives an ‘indexing’ of derived Hecke operators by Galois cohomology. We describe it concretely in a moment, see Section 1.5. It can be viewed as a ‘reciprocity law,’ because it relates the action of the (derived) Hecke algebra to the Galois representation in a direct way.

To go further, let us assume that the map (9) is indeed an isomorphism. Denote by L^* the \mathbf{Q} -linear dual of L , by $L_{\mathbf{Q}_p}^*$ the \mathbf{Q}_p -linear dual of $L_{\mathbf{Q}_p}$; we get also an isomorphism $L^* \otimes \mathbf{Q}_p \simeq L_{\mathbf{Q}_p}^*$.

Thus the derived Hecke algebra gives rise to an action of $\wedge^* L_{\mathbf{Q}_p}^*$ on $H^*(Y(K), \mathbf{Q}_p)$. The fundamental conjecture, formulated precisely as Conjecture 8.8, is then the following:

Let $\wedge^ L_{\mathbf{Q}_p}^*$ act on $H^*(Y(K), \mathbf{Q}_p)$ as described above. Then $\wedge^* L^* \subset \wedge^* L_{\mathbf{Q}_p}^*$ preserves rational cohomology $H^*(Y(K), \mathbf{Q}) \subset H^*(Y(K), \mathbf{Q}_p)$.*

The main point of this paper was to get to the point where we can make this conjecture! What it says is that there is a hidden action of L^* on the \mathbf{Q} -cohomology of $Y(K)$, which can be computed, after tensoring with \mathbf{Q}_p , using the derived Hecke algebra.

Here is the current status of evidence for this conjecture:

- (i) The most direct evidence (as of the time of writing) will be given in the paper [17], which is joint work with Michael Harris. There we develop an analog of the derived Hecke algebra in the setting of coherent cohomology, and formulate an analog of the conjecture in this setting. The advantage of this is we are actually able to carry out a numerical test (in the case of classical weight one modular forms) and it indeed works.
- (ii) As we have already mentioned, the conjecture should be seen as a p -adic analog of the conjecture of [25] (which tells the archimedean story). In the archimedean case, we are able to give substantive evidence for the conjecture by other methods (periods of automorphic forms, and analytic torsion).
- (iii) Suitably phrased, the computations of Section 5 can be seen as supporting a modified version of the conjecture. It is also easy to verify that the conjecture holds for tori, as we shall discuss in Section 9 of this paper.

REMARK. Note that, because of our fairly strong assumptions, (13) is even true *integrally* in the setting of Section 8, that is, the global derived Hecke algebra is an exterior algebra over \mathbf{Z}_p . I do not expect this to be true in general; however, the rational statement (13) should remain valid. One might imagine that the derived deformation ring of [14] will have better integral properties than the derived Hecke algebra.

1.4. Explication, Koszul duality, Taylor–Wiles. We now explain the action of the derived Hecke algebra, and its relationship to Galois cohomology, as explicitly as possible, in the case when $Y(K)$ is an arithmetic hyperbolic 3-manifold. Besides explicating the foregoing abstract discussion, this will also

have the advantage that it allows us to explain the relationship between the derived Hecke algebra and the Taylor–Wiles method.

Suppose \mathbf{G} arises from PGL_2 over an imaginary quadratic field F , that is, $\mathbf{G} = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{PGL}_2$. Let \mathcal{O} be the ring of integers of F . Therefore the associated manifold $Y(K)$ (see (17)) is a finite union of hyperbolic 3-orbifolds. Let us suppose, for simplicity, that the class number of F is odd; then, at full level, the associated arithmetic manifold is simply the quotient of hyperbolic 3-space \mathbb{H}^3 by $\mathrm{PGL}_2(\mathcal{O})$.

In what follows, we fix a prime p and will work always with cohomology with \mathbf{Z}/p^n coefficients.

Let \mathfrak{q} be a prime ideal of \mathcal{O} , relatively prime to p , and let $\mathbf{F}_{\mathfrak{q}} = \mathcal{O}/\mathfrak{q}$ the residue field. Let

$$\alpha : \mathbf{F}_{\mathfrak{q}}^* \longrightarrow \mathbf{Z}/p^n$$

be a homomorphism. By means of the natural homomorphism

$$\Gamma_0(\mathfrak{q}) \rightarrow \mathbf{F}_{\mathfrak{q}}^*$$

sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a/d$, we may regard α as a cohomology class $\langle \alpha \rangle \in H^1(\Gamma_0(\mathfrak{q}), \mathbf{Z}/p^n)$. Here, $\Gamma_0(\mathfrak{q})$ is as usual defined by the condition that $c \in \mathfrak{q}$.

Then a typical ‘derived Hecke operator’ of degree +1 is the following (see Section 2.10 for more):

$$T_{\mathfrak{q},\alpha} : H^1\mathrm{PGL}_2(\mathcal{O}) \xleftarrow{\pi_1^*} H^1\Gamma_0(\mathfrak{q}) \xrightarrow{\cup\langle\alpha\rangle} H^2\Gamma_0(\mathfrak{q}) \xrightarrow{\pi_2^*} H^2\mathrm{PGL}_2(\mathcal{O}). \quad (14)$$

Here π_1, π_2 are the two natural maps $\Gamma_0(\mathfrak{q}) \rightarrow \mathrm{PGL}_2(\mathcal{O})$.

In words, we pull back to level $\Gamma_0(\mathfrak{q})$, cup with $\langle \alpha \rangle$, and push back (the ‘other way’) to level 1. If we omitted the cup product, we would have the usual Hecke operator $T_{\mathfrak{q}}$. The class α itself is rather uninteresting—it is a ‘congruence class’ in the terminology of [10], that is, it becomes trivial on a congruence subgroup—but nonetheless this operation seems to be new even in this case.

The role of torsion coefficients is vital: If we took the coefficient ring above to be \mathbf{Z} , there are no homomorphisms $\mathbf{F}_{\mathfrak{q}}^* \rightarrow \mathbf{Z}$; more generally, $\mathbf{F}_{\mathfrak{q}}^*$ has only torsion cohomology in positive degree. In fact, even to obtain ‘interesting’ operations with \mathbf{Z}/p coefficients, we need at least that p divide $N(\mathfrak{q}) - 1$ (that is, that $N(\mathfrak{q}) = 1$ in the coefficient ring \mathbf{Z}/p , as we mentioned in Section 1.1).

What that means is that elements of $\tilde{\mathbb{T}}$ in characteristic zero necessarily arise in a very indirect way: as a limit of operations from $\mathcal{H}_{\mathfrak{q}}$ for *larger and larger primes* \mathfrak{q} . This situation is perhaps reminiscent of the Taylor–Wiles method, and indeed one miracle of the story is that, although the definition of $\tilde{\mathbb{T}}$ is completely natural, it interacts in a rich way with the Taylor–Wiles method (not merely with

its output, for example, $R = T$ theorems, but with the internal structure of the method itself).

To see why this is so let us examine (14): the Taylor–Wiles method studies the action of \mathbf{F}_q^* on the cohomology of $\Gamma_1(q)$ (these are the ‘diamond operators’). On the other hand, when we study $T_{q,\alpha}$, we are studying the action of $H^*(\mathbf{F}_p^*)$ on $H^*(\Gamma_0(q))$. In both settings it is vital that $Nq - 1$ be divisible by high powers of p .

But these two actions just mentioned are very closely related. More generally, if a group G acts on a space X , the action of G on homology of X and the action of its group cohomology $H^*(G)$ on the equivariant cohomology $H_G^*(X)$ are closely related: when G is a compact torus, for example, this relationship is just Koszul duality [16]. This is just the situation we are in, with $G = \mathbf{F}_q^*$, and X the classifying space of $\Gamma_1(q)$.

1.5. Relationship to Galois cohomology: the ‘reciprocity law’. Continuing our discussion from Section 1.4, let us describe explicitly how the operator $T_{q,\alpha}$ is related to Galois cohomology. Said differently, we are explicating the indexing of derived Hecke operators by Galois cohomology that is implicit in (13). The result could be considered to be a reciprocity law, in the same sense as the usual relationship between Hecke operators and Frobenius eigenvalues.

This discussion is (probably inevitably) a bit more technical. We must again localize our story to a given Hecke eigenclass and also make some further assumptions on the prime q . For a more precise discussion and proofs, see Section 8.28 of the main text.

Fix now a character $\chi : \mathbb{T} \rightarrow \mathbf{Z}_p$ of the usual Hecke algebra at level $Y(K)$. Let

$$\rho : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_2(\mathbf{Z}_p)$$

be the Galois representation conjecturally associated to χ , and let ρ_m be its reduction modulo p^m . We shall assume that ρ is crystalline at all primes above p , and also that $p > 2$. Let S be the set of finite primes at which ρ is ramified (necessarily including all primes above p).

Let $\text{Ad}\rho$ be the composite of ρ with the adjoint representation of PGL_2 ; we think of the underlying space of $\text{Ad}\rho$ as the space of 2×2 matrices with trace zero and entries in \mathbf{Z}_p . Also, let $\text{Ad}^*\rho$ be the \mathbf{Z}_p -linear dual to $\text{Ad}\rho$ (this is identified with $\text{Ad}\rho$ as a Galois module, so long as $p \neq 2$, but we prefer to try to keep them conceptually separate). Finally, $\text{Ad}^*\rho(1)$ will be the Tate-twist of $\text{Ad}^*\rho$.

Let $\mathfrak{q} \notin S$ be a prime of F and let $F_{\mathfrak{q}}$ be the completion of F at \mathfrak{q} . Embed

$$\mathbf{Z}_p \text{ with trivial } \text{Gal}(\overline{F_{\mathfrak{q}}}/F_{\mathfrak{q}}) \text{ action} \hookrightarrow \text{Ad } \rho|_{\text{Gal}(\overline{F_{\mathfrak{q}}}/F_{\mathfrak{q}})} \tag{15}$$

$$1 \mapsto 2\rho(\text{Frob}_{\mathfrak{q}}) - \text{trace} \rho(\text{Frob}_{\mathfrak{q}}).$$

Explicitly, $\rho(\text{Frob}_{\mathfrak{q}})$ is a 2×2 matrix over \mathbf{Z}_p , and the right-hand side above is a 2×2 matrix over \mathbf{Z}_p with trace zero. This eccentric looking formula is a special case of a construction that makes sense for all groups, see Section 8.28.

The map (15) gives rise to a similar embedding $\mathbf{Z}/p^m \rightarrow \text{Ad } \rho_m|_{G_{F_{\mathfrak{q}}}}$, and thus a pairing of $G_{F_{\mathfrak{q}}}$ -modules:

$$\mathbf{Z}/p^m \times \text{Ad}^* \rho(1) \rightarrow \mu_{p^m}.$$

Thus by local reciprocity we get a pairing

$$H^1(F_{\mathfrak{q}}, \mathbf{Z}/p^m) \times H^1(F_{\mathfrak{q}}, \text{Ad}^* \rho(1)) \rightarrow \mathbf{Z}/p^m \mathbf{Z},$$

and then (by restricting the second argument to $F_{\mathfrak{q}}$)

$$H^1(F_{\mathfrak{q}}, \mathbf{Z}/p^m) \times H_f^1\left(\mathcal{O}\left[\frac{1}{S}\right], \text{Ad}^* \rho(1)\right) \rightarrow \mathbf{Z}/p^m \mathbf{Z},$$

Here $H^1(\mathcal{O}[\frac{1}{S}], -)$ denotes the subspace of classes in $H^1(F, -)$ that are unramified outside S , and the f subscript means that we restrict further to classes that are crystalline at p .

Now take, as in Section 1.4, an element $\alpha : \mathbf{F}_{\mathfrak{q}}^* \rightarrow \mathbf{Z}/p^m$ indexing the derived Hecke operator $T_{\mathfrak{q}, \alpha}$, and make an arbitrary extension to a homomorphism $\tilde{\alpha} : F_{\mathfrak{q}}^*/(1 + \mathfrak{q}) \rightarrow \mathbf{Z}/p^m$. This defines a class $\tilde{\alpha} \in H^1(F_{\mathfrak{q}}, \mathbf{Z}/p^m)$, well defined up to unramified classes. The pairing of $\tilde{\alpha}$ with $H_f^1(\mathcal{O}[\frac{1}{S}], \text{Ad}^* \rho(1))$, as above, is easily seen to be independent of choice; thus from a prime ideal \mathfrak{q} and a homomorphism $\mathbf{F}_{\mathfrak{q}}^* \rightarrow \mathbf{Z}/p^m$ we have obtained a homomorphism:

$$[\mathfrak{q}, \alpha] : H_f^1\left(\mathcal{O}\left[\frac{1}{S}\right], \text{Ad}^* \rho(1)\right) \longrightarrow \mathbf{Z}/p^m. \tag{16}$$

In the main text of this paper (Lemma 8.29), we prove a variant of the following statement under some further local hypotheses on the representation ρ . (The meaning of ‘variant’ is this: we only work with simply connected groups—that is, we prove an analogous result for SL_2 rather than PGL_2 —and we impose various local conditions on the residual representations. In the introduction, we have stuck with PGL_2 because it is more familiar. For example, for SL_2 , we would need to use only the squares of the usual Hecke operators.)

Claim: There exists $N_0(m)$ such that for each pair of prime ideals $\mathfrak{q}, \mathfrak{q}'$ satisfying

- (a) $\text{Norm}(\mathfrak{q}) \equiv \text{Norm}(\mathfrak{q}') \equiv 1$ modulo $p^{N_0(m)}$
- (b) the eigenvalues of $\rho(\text{Frob}_{\mathfrak{q}})$, mod p , are distinct elements of $\mathbf{Z}/p\mathbf{Z}$, and the same for \mathfrak{q}' ;
- (c) $[\mathfrak{q}, \alpha] = [\mathfrak{q}', \alpha']$ in the notation of (16)

the actions of $T_{\mathfrak{q}, \alpha}$ and $T_{\mathfrak{q}', \alpha'}$ on $H^*(Y(K), \mathbf{Z}/p^m)$ coincide.

This is a ‘reciprocity law,’ of the same nature as the reciprocity law relating Frobenius and Hecke eigenvalues. It is the basis for (13).

It is not as precise as one would like, because of the annoying extra conditions on $\mathfrak{q}, \mathfrak{q}'$ but it is good enough to get (13). It is certainly natural to believe that

$$[\mathfrak{q}, \alpha] = [\mathfrak{q}', \alpha'] \stackrel{?}{\implies} T_{\mathfrak{q}, \alpha} = T_{\mathfrak{q}', \alpha'}$$

(where the equality on the right is an equality of endomorphisms of \mathbf{Z}/p^m -cohomology), without imposing condition (a) or (b) above. It would be good to prove not only this but a version that gives information at bad places. Such a formulation is presumably related to a derived deformation ring, as we describe in the next Section.

1.6. Further discussion and problems. It is not really surprising in retrospect that such cohomology operations should exist. It took me a long time to find them because of their subtle feature of being patched together from torsion levels. There is a relatively simple archimedean analog made via differential forms, see [25].

Here are some metaphors and problems:

- (a) In the ‘Shimura’ context a corresponding structure is provided by ‘Lefschetz operators’ (although these act nontrivially only for nontempered representations). But the derived Hecke algebra operators do not recover this structure. Indeed, for weight reasons, one expects that the higher degrees of the derived Hecke algebra act trivially in the Shimura case. The example of GL_2 over a field with both real and complex places shows a mixture of features, which would be interesting to study further.
- (b) The theory of completed cohomology of Calegari–Emerton [8] already predicts that, if we pass up a congruence tower, cohomology becomes (under certain conditions) concentrated in a single degree. Said another way,

all the degrees of cohomology have ‘the same source,’ and thus one expects to be able to pass from one to another.

For this reason, it will be interesting to study the action of the *mod p derived Hecke algebra of a p-adic group*; but we stay away from this in the current paper. (Our results and a global-to-local argument suggest that this derived Hecke algebra might have a nice structure theory. As mentioned this is studied in [24, 28]; there is also recent work of Ronchetti [27].)

- (c) There is also a story of ‘derived deformation rings,’ developed in [14]; there is a pro-simplicial ring \tilde{R} that represents deformations of Galois representations with coefficients in simplicial rings. The precise definition of \tilde{R} , and—assuming similar conjectures to those assumed here—a construction of its action on integral homology, are given in the paper [14].

However, the relationship between $\tilde{\mathbb{T}}$ and \tilde{R} is not one of equality: the former acts on cohomology, raising cohomological degree, and the latter naturally acts on homology, raising homological degree. See the final section of [14] for a formulation of the relationship between the two actions.

Our expectation is that \tilde{R} will have better integral properties than $\tilde{\mathbb{T}}$, in general.

- (d) Numerical invariants: We can use $\tilde{\mathbb{T}}$ to shift a class from degree q to the complementary degree $\dim Y(K) - q$ and then cup the resulting classes. This gives an analog of the ‘Petersson norm’ which makes sense for a torsion class (or a p -adic class). What is the meaning of the resulting numerical invariants?

1.7. Notation. We try to adhere to using ℓ or p for the characteristic of coefficient rings, and using q or v for the residue field size of nonarchimedean fields. Thus we may talk about the ‘ ℓ -adic Hecke algebra of a v -adic group.’

\mathbf{G} will denote a reductive algebraic group over a number field F . In the local part of our paper—Sections 2, 3, 4—we shall work over the completion of such an F at an arbitrary finite place. In our global applications we will be more specific (just for ease of notation, for example, not worrying about multiple primes above the residue characteristic): Section 5 we take F quadratic imaginary, and in Section 6 onward we take $F = \mathbf{Q}$.

It will be convenient at many points to assume that \mathbf{G} is split, and then to fix a maximal split torus \mathbf{A} inside \mathbf{G} , and also a Borel subgroup \mathbf{B} containing \mathbf{A} . This endows the cocharacter lattice $X_*(\mathbf{A}) = \text{Hom}(\mathbb{G}_m, \mathbf{A})$ with a positive cone

$X_*(\mathbf{A})^+ \subset X_*(\mathbf{A})$, the dual to the cone spanned by the roots of \mathbf{A} on \mathbf{B} . We denote by $r = \dim X_*(\mathbf{A})$ the rank of \mathbf{G} .

For v a place of F we let F_v be the completion of F at v , $\mathcal{O}_v \subset F_v$ the integer ring, \mathbf{F}_v the residue field and write q_v for the cardinality of \mathbf{F}_v . We also put

$$G_v = \mathbf{G}(F_v).$$

Attached to \mathbf{G} and a choice of open compact subgroup $K \subset \mathbf{G}(\mathbf{A}_{F,f})$ (the finite adèle points of \mathbf{G}) there is attached an ‘arithmetic manifold’ $Y(K)$, which is a finite union of locally symmetric spaces:

$$Y(K) = \mathbf{G}(F) \backslash (S_\infty \times \mathbf{G}(\mathbf{A}_f)) / K, \tag{17}$$

where S_∞ is the ‘disconnected symmetric space’ for $\mathbf{G}(F \otimes \mathbf{R})$ —the quotient of $\mathbf{G}(F \otimes \mathbf{R})$ by a maximal compact connected subgroup. Although it is a minor point, we take $Y(K)$ as an orbifold, not a manifold, and always compute its cohomology in this sense.

As before, we introduce the integer invariants \mathfrak{q}, δ :

$$\delta = \text{rank}(\mathbf{G}(F \otimes \mathbf{R})) - \text{rank}(\text{maximal compact of } \mathbf{G}(F \otimes \mathbf{R})), \tag{18}$$

and define \mathfrak{q} so that $2\mathfrak{q} + \delta = \dim Y(K)$. These have the same significance as described in (5), at least if we suppose that the center of \mathbf{G} is anisotropic over F .

We work only with open compact subgroups with a product structure, that is,

$$K = \prod_v K_v \tag{19}$$

where $K_v \subset \mathbf{G}(F_v)$ is an open compact subgroup, and K_v is a hyperspecial maximal compact of G_v for all but finitely many primes v . A prime v will be ‘good’ for K when K_v is hyperspecial and the ambient group \mathbf{G} is quasisplit at v . The ‘level of K ’ will be, by definition, the (finite) set of all primes v which are not good.

\mathbf{G} has a dual group G^\vee , which we regard as a *split Chevalley group over \mathbf{Z}* ; in particular, its Lie algebra is defined over \mathbf{Z} , and its points are defined over any ring R . We regard it as equipped with a maximal torus T^\vee inside a Borel subgroup B^\vee .

In the discussion of the Taylor–Wiles method, which takes place in Sections 6 and 8, it is convenient to additionally assume:

\mathbf{G} is simply connected and G^\vee is adjoint.

This is a minor issue, to avoid the usual difficulties of ‘square roots.’ One could (better) replace G^\vee by some version of the c -group of [7].

When we discuss Galois cohomology, we follow the usual convention that, for a module M under the absolute Galois group $\text{Gal}(\bar{L}/L)$ of a field M , we denote by $H^*(L, M)$ the continuous cohomology of the profinite group $\text{Gal}(\bar{L}/L)$ with coefficients in M . For L a number field, with ring of integers $\mathcal{O} \subset L$, we denote by $H^1(\mathcal{O}[\frac{1}{S}], M) \subset H^1(L, M)$ the subset of classes that are unramified outside S and $H_f^1(\mathcal{O}[\frac{1}{S}], M) \subset H^1(\mathcal{O}[\frac{1}{S}], M)$ the classes that are, moreover, crystalline at p .

Warning: In the literature, the subscript f is often used to mean classes that are crystalline at p and unramified at *all* other places, not merely at places outside S . Indeed, we implicitly used this notation in Sections 1.2 and 1.3 when we wrote $H_f^1(G_{\mathbb{Q}}, -)$. However, in the remainder of the text, we will not use this convention. To avoid any confusion, on the one occasion (in Section 9) we wish to refer to classes that are crystalline at p and unramified at all other places, we use the notation $H_{f,\text{ur}}^1$. This notation will be reprised when it is used so the reader need not remember it now.

2. Derived Hecke algebra

We introduce the derived Hecke algebra (Definition 2.2) and then give two equivalent descriptions in Sections 2.3 and 2.4. The model given in Section 2.3 is by far the most useful. We shall then describe the action of the derived Hecke algebra on the cohomology of an arithmetic group in Section 2.6, and then make it a bit more concrete in Section 2.10. Finally, Section 2.12 discusses some minor points to do with change of coefficient ring, and Section 2.13 discusses some other minor points about passage between \mathbb{Z}/ℓ^n coefficients and \mathbb{Z}_{ℓ} coefficients.

Appendix A expands on various points of homological algebra that are used in the current section.

2.1. As in Section 1.7, we fix a prime v of F , with residue field \mathbf{F}_v of characteristic p_v and size q_v , and set $G_v = \mathbf{G}(F_v)$. We denote by U_v an open compact subgroup of G_v . Eventually, we use only the case of U_v being either a maximal compact subgroup or an Iwahori subgroup, but there is no need to impose this. When we are working strictly in a local setting, we abbreviate these simply to G and U :

$$G = G_v, \quad U = U_v.$$

It will also be convenient to fix

$$V_v = \text{a pro-}p_v, \text{ normal, finite index subgroup of } U_v, \quad (20)$$

which we again abbreviate to V when it will cause no confusion.

Let S be a finite coefficient ring in which q_v is invertible. In what follows, by ‘ G -module’ we mean a module M under the group algebra SG with the property that every $m \in M$ has open stabilizer in G . The category of G -modules is an abelian category and it has enough projective objects (see Section A.2).

The usual Hecke algebra for the pair (G, U) can be defined as the endomorphism ring $\text{Hom}_{SG}(S[X], S[X])$, where $X = G/U$ and $S[X]$ denotes the free S -module on a set X . Motivated by this, we define:

DEFINITION 2.2. The derived Hecke algebra for (G, U) with coefficients in S is the graded algebra

$$\mathcal{H}(G, U)_S := \text{Ext}^*(S[G/U], S[G/U]), \tag{21}$$

where the Ext-group is taken inside the category of G -modules.

Let us record some variants on the notation:

- We write simply $\mathcal{H}(G, U)$ when the coefficients are understood to be S .
- We write $\mathcal{H}^j(G, U)$ or $\mathcal{H}^{(j)}(G, U)$ for the component in degree j , that is, the Ext^j summand on the right.
- In global situations where we have fixed a level structure $K_v \leq G_v$ for all v , or for almost all v , we often write simply $\mathcal{H}_{v,S}$ for the corresponding derived Hecke algebra $\mathcal{H}(G_v, K_v)$. Again we write simply \mathcal{H}_v if the coefficients are understood to be S .

If we choose a projective resolution \mathbf{P} of $S[G/U]$ as G -module, then $\mathcal{H}(G, U)$ is identified with the cohomology of the differential graded algebra $\text{Hom}_{SG}(\mathbf{P}, \mathbf{P})$. It will be convenient for later use to make an explicit choice of \mathbf{P} : Let \mathbf{Q} be a free resolution of the trivial module S in the category of $S[U/V]$ -modules. We may take \mathbf{P} to be the compact induction (from U to G) of \mathbf{Q} . Observe that all the groups \mathbf{P}_i of the resulting resolution are free S -modules.

2.3. Description in terms of invariant functions. We may also describe $\mathcal{H}(G, U)$ as the algebra of ‘ G -equivariant cohomology classes on $G/U \times G/U$ with finite support modulo G .’ We now spell out carefully what this means; an explicit isomorphism between this description and Definition 2.2 is constructed in Appendix A.

First some notation: for $x, y \in G/U$, we denote by G_{xy} the pointwise stabilizer of (x, y) inside G ; it is a profinite group. We denote by $H^*(G_{xy}, S)$ the continuous cohomology of G_{xy} with coefficients in S (discretely topologized).

In this model, an element of $\mathcal{H}(G, U)$ is an assignment h that takes as input $(x, y) \in G/U \times G/U$ and produces as output $h(x, y) \in H^*(G_{xy}, S)$, subject to the following conditions:

- h is G -invariant, that is to say, $[g]^*h(gx, gy) = h(x, y)$, where $[g]^* : H^*(G_{gx,gy}) \rightarrow H^*(G_{xy})$ is pullback by $\text{Ad}(g)$.
- h has finite support modulo G , that is, there is a finite subset $T \subset G/U \times G/U$ such that $h(x, y) = 0$ if (x, y) does not lie in the G -orbit of T .

The addition and S -module structure on $\mathcal{H}(G, U)$ is defined pointwise. The product is given by the rule

$$h_1 * h_2(x, z) = \sum_{y \in G/U} \underbrace{h_1(x, y)}_{H^*(G_{xy})} \cup \underbrace{h_2(y, z)}_{H^*(G_{yz})} \tag{22}$$

where we give the right-hand side the following meaning: The cup product on the right makes sense in $H^*(G_{xyz}, S)$, that is, first restrict h_1 and h_2 to $H^*(G_{xyz}, S)$, and take the cup product there. Now split G/U as a disjoint union $\coprod O_i$ of orbits under G_{xz} ; let O be one such orbit. We regard

$$\sum_{y \in O} h_1(x, y) \cup h_2(y, z) := \text{Cores}_{G_{xz}}^{G_{xy_0z}} (h_1(x, y_0) \cup h_2(y_0, z)) \tag{23}$$

where $y_0 \in O$ is any representative, and the ‘trace’ or corestriction is taken from G_{xy_0z} to G_{xz} ; note that the right-hand side of (23) is independent of choice of $y_0 \in O$. Adding up over orbits O gives the meaning of the right-hand side of (22).

REMARK. Suppose that Δ is a compact subgroup of G that stabilizes every point of G/U . In this case, we can restrict h to get a function $h_\Delta : G/U \times G/U \rightarrow H^*(\Delta)$. We also have $(hh')_\Delta = h_\Delta h'_\Delta$, where the right-hand multiplication is the more familiar

$$h_\Delta h'_\Delta(x, z) = \sum_{y \in G/U} h_\Delta(x, y) \cup h'_\Delta(y, z). \tag{24}$$

2.4. Double coset description. Finally, we can describe $\mathcal{H}(G, U)$ in terms of double cosets $U \backslash G/U$. For $x \in G/U$ let

$$U_x = U \cap \text{Ad}(g_x)U$$

where $g_x \in G$ represents x (that is to say, $x = g_x U$). Then U_x is the stabilizer of x in U .

Fix a set of representatives $[U \backslash G/U] \subset G/U$ for the left U -orbits on G/U . Then we have an isomorphism of S -modules

$$\bigoplus_{x \in [U \backslash G/U]} H^*(U_x, S) \xrightarrow{\sim} \mathcal{H}(G, U) \tag{25}$$

given thus: Fix a class $z \in [U \backslash G/U]$, and a representative $g_z \in G$ for z —thus $z = g_z U$. Let $\alpha \in H^*(U_z, S)$. Then the class of $\alpha \in H^*(U_z, S)$, considered as an element of the left-hand side of (25), is carried to the function $h_{z,\alpha}$ on $G/U \times G/U$ characterized by the following properties:

- (i) $h_{z,\alpha}(x, y) = 0$ unless (x, y) belongs to the G -orbit of (z, eU) .
- (ii) $h_{z,\alpha}$ sends (z, eU) to $\alpha \in H^*(U_z, S)$ —note that U_z is exactly the common stabilizer of z and eU .

This gives another description of $\mathcal{H}(G, U)$. It is harder to directly describe the multiplication rule in this presentation, and we use instead the isomorphism to the previous description. Later on we describe explicitly the action of $h_{z,\alpha}$ on the cohomology of an arithmetic manifold.

Now let us examine the ‘size’ of $\mathcal{H}(G, U)$; this discussion is really only motivational, and so we will be a little informal. Suppose, for example, that G is split and U is hyperspecial. In this case, the quotient $U \backslash G/U$ is parameterized by a dominant chamber $X_*(\mathbf{A})^+$ inside the cocharacter group $X_*(\mathbf{A})$ of a maximal split torus \mathbf{A} . Moreover, if $x \in G/U$ is a representative for a double coset parameterized by $\lambda \in X_*(\mathbf{A})$, then the group U_x is, modulo a pro- p -subgroup, the \mathbf{F}_v -points $M_\lambda(\mathbf{F}_v)$ of the centralizer M_λ of λ . Thus we obtain an isomorphism of S -modules:

$$\mathcal{H}(G, U) := \bigoplus_{\lambda \in X_*(\mathbf{A})^+} H^*(M_\lambda(\mathbf{F}_v), S).$$

For ‘generic’ λ —that is, away from the walls of $X_*(\mathbf{A})^+$ —the group M_λ is a split torus; the order of its \mathbf{F}_v -points is a power of $(q_v - 1)$. Thus if $(q_v - 1)$ is invertible on S , all the terms of $\mathcal{H}(G, U)$ corresponding to dominant λ vanish.

In this paper we will be primarily concerned with the case when $q_v - 1 = 0$ inside S . Although it is certainly interesting to study $\mathcal{H}(G, U)$ in general, the preceding discussion shows that this case (that is, $q_v = 1$ in S) is where $\mathcal{H}(G, U)$ is ‘largest.’

2.5. Derived invariants. If M is any complex of G -modules, we may form the derived invariants

$$\text{derived } U\text{-invariants on } M := \underline{\text{Hom}}_{S_G}(S[G/U], M) \in \mathbf{D}(\text{Mod}_S)$$

where $\underline{\text{Hom}}$ is now derived Hom in the derived category of G -modules, taking values in the derived category of S -modules.

Then the derived Hecke algebra automatically acts on the cohomology of the derived invariants:

$$\mathcal{H}(G, U) \curvearrowright H^*(\text{derived } U\text{-invariants on } M). \quad (26)$$

Indeed, the derived invariants are represented by the complex $\text{Hom}_{SG}(\mathbf{P}, M)$, where \mathbf{P} is as before any projective resolution of $S[G/U]$. The action of $\text{Hom}_{SG}(\mathbf{P}, \mathbf{P})$ on this complex furnishes the desired (right) action of $\mathcal{H}(G, U)$.

Let us describe the derived invariants in more familiar terms. Let V be as in (20), and consider the explicit projective resolution \mathbf{P} discussed in Section 2.2; we see that the derived U -invariants are computed by the complex $\text{Hom}_{SU}(\mathbf{Q}, M)$. This coincides with U/V -homomorphisms from \mathbf{Q} to the termwise invariants M^V ; since \mathbf{Q} is a projective resolution of S in the category of U/V -modules, we see that

$$\text{derived } U\text{-invariants on } M \simeq \underline{\text{Hom}}_{SU/V}(S, M^V) \in \mathbf{D}(\text{Mod}_S), \quad (27)$$

where the right-hand side is derived homomorphisms, in the derived category of U/V -modules. In other words, there is an identification

$$H^*(\text{derived } U\text{-invariants on } M) \simeq \mathbb{H}^*(U/V, M^V),$$

the group hypercohomology of the finite group U/V acting on the complex of termwise invariants M^V .

2.6. Arithmetic manifolds. In the remainder of this section, we describe how the derived Hecke algebras act on the cohomology of arithmetic manifolds.

We follow the notation of Section 1.7. In particular, we fix $K \subset \mathbf{G}(\mathbf{A}_f)$ an open compact subgroup, which we are supposing to have a product structure $K = \prod_w K_w$; let us split this as

$$K = K^{(v)} \times K_v$$

where $K^{(v)} = \prod_{w \neq v} K_w$ is the structure ‘away from v .’ Associated to this is an arithmetic manifold $Y(K)$, as in (17).

We construct an action of the derived Hecke algebra $\mathcal{H}_v = \mathcal{H}_v(G_v, K_v)$ on the cohomology of $Y(K)$. To do so, we exhibit $Y(K)$ as the derived K_v -invariants on a suitable G_v -module, and then apply (26).

For U_v any open compact subgroup of G_v , let us abridge:

$$C^*(U_v) = \text{cochain complex of } Y(K^{(v)} \times U_v) \text{ with } S \text{ coefficients.} \quad (28)$$

Now let

$$M = \varinjlim_{U_v} C^*(U_v),$$

where the limit is taken over open compact subgroups $U_v \leq G_v$. Visibly, M is a complex of smooth G_v -modules. Choosing $V_v \subset K_v$ as in (20), we have

$$M^{V_v} \simeq C^*(V_v),$$

since we may interchange invariants and the direct limit; and then for a finite cover $X \rightarrow Y$ with Galois group D we have an isomorphism $C^*(Y) \xrightarrow{\sim} C^*(X)^D$. However, the derived invariants of K_v/V_v on $C^*(V_v)$ ‘coincide with’ the cohomology of $Y(K)$: the natural map

$$C^*(Y(K)) = C^*(V_v)^{K_v/V_v} \rightarrow \underline{\mathrm{Hom}}_{S[K_v/V_v]}(S, C^*(V_v))$$

is a quasi-isomorphism, in the derived category of S -modules. This follows from the fact that the terms $C^*(V_v)$ have no higher cohomology as K_v/V_v -modules, because each $C^j(V_v)$ is the module of S -valued functions on a free K_v/V_v -set and is, in particular, induced from a representation of the trivial group.

We have exhibited a quasi-isomorphism

$$C^*(Y(K)) \simeq \text{derived } K_v\text{-invariants on } M$$

between $C^*(Y(K))$ and a complex that represents the derived invariants of K_v acting on M . Thus (26) gives a natural right action of $\mathcal{H}^\circ(G_v, K_v)$ on the cohomology of $Y(K)$.

REMARK. Although this is strictly a right action, we often write it on the left, which conforms more to the usual notation for Hecke operators; the reader should therefore remember that the multiplication needs to be appropriately switched at times, but this will cause almost no issue because the derived Hecke algebra will prove to be graded-commutative at all the places we use.

Of course, this description is totally incomprehensible; thus we now work on translating it to something more usable.

2.7. Digression: pullback from a congruence quotient. We first need a brief digression to construct certain cohomology classes on $Y(K)$. These are called ‘congruence classes’ in [10], because they capitulate in congruence covers of $Y(K)$.

There is a natural map

$$H^*(K_v, S) \longrightarrow H^*(Y(K), S), \quad (29)$$

where, on the left, $H^*(K_v, S)$ is the continuous cohomology of the profinite group K with coefficients in (discretely topologized) S . Indeed, any cohomology class for $H^*(K_v, S)$ is inflated from a quotient $K_v/K_{v,1}$. Let K_1 be the preimage of $K_{v,1}$ in K . The covering $Y(K_1) \rightarrow Y(K)$ has deck group $K_v/K_{v,1}$, and thus gives rise to a map, well defined up to homotopy,

$$Y(K) \longrightarrow \text{classifying space of } K_v/K_{v,1}. \quad (30)$$

We may then pull back cohomology classes along this map to get (29).

These ‘congruence’ cohomology classes have a very simple behavior under Hecke operators:

LEMMA 2.8. *Let h be in the image of the map (29). For any prime w of F that does not divide the level of K or the size of S , such that $\mathbf{G}(F_w)$ is split, and any usual Hecke operator T supported at w , we have*

$$Th = \deg(T)h.$$

We give a direct proof, but let us note that one can also deduce the result from the commutativity of the Hecke algebra at w (which is proved, under mild restrictions on w , in Section 3). It is also likely that the assumption that $\mathbf{G}(F_w)$ is split is unnecessary (since w does not divide the level of K , it is automatically quasisplit by our definitions, which should be enough for the argument below to go through).

Proof. It is easy to verify this if $w \neq v$, so we examine only the case $w = v$.

By the assumptions, we may suppose that $K_v = \mathcal{G}(\mathcal{O}_v)$, for a split reductive \mathcal{G} over \mathcal{O}_v . Suppose that T arises from the double coset $K_v a_v K_v$, where, without loss, a lies in a maximal split torus $\mathbf{A}(F_v)$ that is in good position relative to K_v —that is, it extends to a maximal split torus of \mathcal{G} .

We show that h has the same pullback under the two natural maps

$$\pi_1, \pi_2 : Y(K \cap \text{Ad}(a_v)K) \rightarrow Y(K),$$

namely, the natural map, and the map induced by multiplication by a_v ; this implies the Lemma.

There is an isomorphism $X_*(\mathbf{A}) \simeq A_v/(A_v \cap K_v)$; let \mathcal{M} be the centralizer in \mathcal{G} of the cocharacter in $X_*(\mathbf{A})$ that corresponds to the class of a . Let K_2 be the preimage, under $K_v \rightarrow \mathcal{G}(\mathcal{O}_v/\varpi_v^D)$, of $\mathcal{M}(\mathcal{O}_v/\varpi_v^D)$; here D is a large enough integer, and ϖ_v a uniformizer.

Then, on the one hand, the inclusion $K_2 \hookrightarrow K \cap \text{Ad}(a_v)K$ has index equal to a power of q_v . In particular, it induces an injection on $H^*(-, S)$, so it is enough to verify that $\pi_1^*h = \pi_2^*h$ after pullback under $Y(K_2) \rightarrow Y(K \cap \text{Ad}(a_v)K)$.

However, the pullback $H^*(\mathbf{G}(\mathbf{F}_v), S) \rightarrow H^*(K_v, S)$ is an isomorphism. The class h is therefore actually pulled back from $\mathbf{G}(\mathbf{F}_v)$. Our assertion then follows from the fact that the natural maps $K_2 \rightarrow \mathbf{G}(\mathbf{F}_v)$ —namely, the reduction map, and the conjugate of the reduction map by a_v —actually coincide. This proves that $\pi_1^*h = \pi_2^*h$, and concludes the proof of the Lemma. \square

This motivates the following definition:

DEFINITION 2.9. We say a class $h \in H^*(Y(K), S)$ is *Hecke-trivial* if, for all places v that do not divide the level of K and with residue characteristic invertible on S , and all Hecke operators T supported at v ,

$$(T - \deg(T))^n h = 0,$$

for a sufficiently large integer $n = n(T)$. We denote by $H^*(Y(K), S)_{\text{triv}}$ the submodule of Hecke-trivial classes.

2.10. Concrete expression for the action of \mathcal{H}_v on $H^*(Y(K), S)$. Let us now give a more down-to-earth description of the action of $\mathcal{H}_{v,S}$ on $H^*(Y(K), S)$, with notation as above. In particular, we show that the action of elements $h_{z,\alpha}$ can be described in a fashion that is very close to the usual definition of Hecke operators.

From $z = g_z K_v \in G_v/K_v$, and $\alpha \in H^*(K_v \cap \text{Ad}(g_z)K_v)$, we obtain a class $h_{z,\alpha} \in \mathcal{H}(G_v, K_v)$, by the recipe of Section 2.4. Then:

LEMMA 2.11. *Write*

$$K_z = K \cap \text{Ad}(g_z)K, \quad K'_z = K \cap \text{Ad}(g_z^{-1})K.$$

Also, let $\langle \alpha \rangle$ be the image of α under $H^*(K_v \cap \text{Ad}(g_z)K_v) \xrightarrow{(29)} H^*(Y(K_z), S)$. Then the action of $h_{z,\alpha}$ on $H^*(Y(K), S)$ coincides with the following composite

$$H^*(Y(K)) \xrightarrow{[g_z]^*} H^*(Y(K_z)) \xrightarrow{\cup \langle \alpha \rangle} H^*(Y(K_z)) \rightarrow H^*(Y(K)), \quad (31)$$

where all cohomology is taken with S coefficients, and the arrows are (respectively) pullback by the map $Y(K_z) \rightarrow Y(K)$ induced by the map $g \mapsto gg_z$, cup with $\langle \alpha \rangle$, and pushforward for the standard map $Y(K_z) \rightarrow Y(K)$.

Note that this is almost exactly the same as a usual Hecke operator; we have just inserted the operation of $\cup \langle \alpha \rangle$ on the way. The fact that $\langle \alpha \rangle$ is Hecke-trivial, in the sense of Definition 2.9, is the key point that makes this operation commute with usual Hecke operators.

Proof. Routine but extremely tedious; see [Appendix A](#). □

REMARK. Note also the following trivial case: taking $g_z = 1$, we see that the operation of ‘cup with $\alpha \in H^*(K_v, S)$ ’ always belongs to the derived Hecke algebra.

REMARK. As an example let us write out the argument that derived Hecke operators at different places always commute with one another. Fix places $v \neq w$, elements $g_v \in G_v, g_w \in G_w$, and classes $\alpha_v \in H^*(K_v \cap \text{Ad}(g_v)K_v)$ and $\alpha_w \in H^*(K_w \cap \text{Ad}(g_w)K_w)$. We claim that the composite of the two associated derived Hecke operators can be described in the following way, which makes graded commutativity clear: Push–pull along

$$Y(K) \leftarrow Y(K_{g_v} \cap K_{g_w}) \rightarrow Y(K),$$

but cup in the middle with the class of $\alpha_v \cup \alpha_w$. To verify this claim, examine the following diagram:

$$\begin{array}{ccccc}
 & & Y(K_{g_v} \cap K_{g_w}) & & (32) \\
 & & \swarrow & \searrow & \\
 & & \times g_w & & \\
 & Y(K_{g_v}) & & & Y(K_{g_w}) \\
 & \swarrow & & \swarrow & \searrow \\
 & \times g_v & & \times g_w & \\
 Y(K) & & Y(K) & & Y(K)
 \end{array}$$

When we write (for example) $\times g_v$ we mean that the map is induced by right multiplication by g_v . The composite of the derived Hecke operators is, by definition, obtained by going along the bottom two rows. However, the middle diagram is a fiber product square, and so the two ways of going from $Y(K_{g_v})$ to $Y(K_{g_w})$, via ‘push–pull’ or ‘pull–push,’ coincide. To prove the desired claim, then, it suffices to show that the two pullbacks of the class $\langle \alpha_v \rangle \in H^*(Y(K_{g_v}))$ to $H^*(Y(K_{g_v} \cap K_{g_w}))$ —via the natural map, and via the map $\times g_w$ —actually coincide.

Equivalently, the classes obtained from α_v , in the natural way on $Y(K_{g_v} \cap g_w K_{g_w}^{-1})$ and on $Y(K_{g_v} \cap g_w^{-1} K_{g_w})$ are in fact compatible, under the map $\times g_w$ from one space to the other. However, these cohomology classes are obtained from a certain covering of the spaces, obtained by adding extra level at v , and the compatibility follows from the fact that $\times g_w$ lifts to these coverings.

2.12. Change of coefficients. Let us examine what happens under a change of rings $S \rightarrow S'$. The description of Section 2.4 and the explicit action of Section 2.10 means that there is a map of Hecke algebra $\mathcal{H}_{v,S} \rightarrow \mathcal{H}_{v,S'}$ compatible with the actions on $H^*(Y(K), S) \rightarrow H^*(Y(K), S')$. However, this does not make quite clear that the change of rings map is an algebra homomorphism. For completeness let us explain this now, since we want to freely pass between \mathbf{Z}/ℓ^n coefficients for various ns .

The tensor product $\otimes_S S'$ is a right exact functor from SG -modules to $S'G$ modules and so it can be derived to a map of derived categories. Note that this carries projectives to projectives since $\text{Hom}_{S'G_v}(P \otimes_S S', -) = \text{Hom}_{SG_v}(P, -)$.

This derived tensor product (let us write it as \otimes) ‘carries $S[G_v/K_v]$ to $S'[G_v/K_v]$ ’: if we choose a projective replacement $\mathbf{P} \rightarrow S[G_v/K_v]$ the natural map

$$\mathbf{P} \otimes_S S' \longrightarrow S'[G_v/K_v]$$

is a quasi-isomorphism. Indeed it is possible to choose \mathbf{P} so that each term of \mathbf{P} is free as an S -module (see the explicit resolution after (21)). Then $\mathbf{P} \otimes_S S'$ has no cohomology in higher degree (since this complex computes the $\text{Tor}_S(S[G_v/K_v], S')$ and the former is free) and thus it is a resolution of $S'[G_v/K_v]$.

This yields at once a map

$$\mathcal{H}_{v,S} \rightarrow \mathcal{H}_{v,S'}$$

from the Hecke algebra with S coefficients, to the same with S' coefficients. Explicitly, the left-hand side is represented by the cohomology of the differential graded algebra $\text{Hom}_{SG}(\mathbf{P}, \mathbf{P})$, and this dga maps to $\text{Hom}_{S'G_v}(\mathbf{P} \otimes_S S', \mathbf{P} \otimes_S S')$, whose cohomology computes $\mathcal{H}_{v,S'}$. This is the desired algebra map ‘change of coefficients.’

Consider now the obvious map

$$\iota : \varinjlim_{U_v} C^*(U_v) \rightarrow \varinjlim_{U_v} C^*(U_v; S'),$$

where the notation is as in (28), and the right-hand side is defined the same way but with S' coefficients. This induces

$$\iota' : \text{Hom}_{SG} \left(\mathbf{P}, \varinjlim_{U_v} C^*(U_v) \right) \rightarrow \text{Hom}_{S'G} \left(\mathbf{P} \otimes_S S', \left(\varinjlim_{U_v} C^*(U_v; S') \right) \right)$$

wherein we compose with ι and extend by S -linearity. There are compatible actions of $\mathcal{H}_{v,S}$ and $\mathcal{H}_{v,S'}$ on the left and right sides. On the other hand, the map ι' induces on cohomology the natural map $H^*(Y(K), S) \rightarrow H^*(Y(K), S')$.

To summarize: the actions of $\mathcal{H}_{v,S}$ on $H^*(Y(K), S)$ and $\mathcal{H}_{v,S'}$ on $H^*(Y(K), S')$ are compatible, with respect to the natural maps $\mathcal{H}_{v,S} \rightarrow \mathcal{H}_{v,S'}$ and $H^*(Y(K), S) \rightarrow H^*(Y(K), S')$.

2.13. Passage from mod ℓ^n to ℓ -adic; the global derived Hecke algebra.

We now write out in grotesque detail certain minor details of the passage from mod ℓ^n to ℓ -adic coefficients, which will be used without comment in our later proofs. This section should probably be skipped by the reader and consulted only as needed.

In what follows, we describe the straightforward version of the global derived Hecke algebra using all good primes for K that do not divide ℓ ; here ‘good’ is defined after (19). *When we refer to the global derived Hecke algebra without any further remark, we are always referring to this version.* We remark, after the construction, how to make a definition using a restricted set of primes.

The action of the derived Hecke algebra gives an algebra of endomorphisms $\tilde{\mathbb{T}}_n \subset \text{End}(H^*(Y(K), \mathbf{Z}/\ell^n))$, namely the algebra of endomorphisms generated by all the derived Hecke algebras $\mathcal{H}_{v, \mathbf{Z}/\ell^n}$ where v varies over good primes of K that are not above ℓ .

Now we have

$$H^*(Y(K), \mathbf{Z}_\ell) = \varprojlim H^*(Y(K), \mathbf{Z}/\ell^n)$$

and we *define* the global derived algebra to be

$$\tilde{\mathbb{T}} \subset \text{End}(H^*(Y(K), \mathbf{Z}_\ell)) \tag{33}$$

to be those endomorphisms of the form $\varprojlim t_n$, for some compatible system $t_n \in \tilde{\mathbb{T}}_n$, that is, t_n ‘reduces to t_m ’ for $n > m$ in the sense that the following diagram should commute:

$$\begin{array}{ccc} H^*(Y(K), \mathbf{Z}/\ell^n) & \xrightarrow{t_n} & H^*(Y(K), \mathbf{Z}/\ell^n) \\ \downarrow & & \downarrow \\ H^*(Y(K), \mathbf{Z}/\ell^m) & \xrightarrow{t_m} & H^*(Y(K), \mathbf{Z}/\ell^m). \end{array} \tag{34}$$

Let $\tilde{\mathbb{T}}_n^*$ be the systems of elements $(t_n, t_{n-1}, \dots, t_1)$, where $t_r \in \tilde{\mathbb{T}}_r$ for $r \leq n$ are all compatible in the sense that the above diagram should commute for each $t_r, t_{r'}$. In particular, $\tilde{\mathbb{T}}_n^*$ acts on \mathbf{Z}/ℓ^r -valued cohomology for each $r \leq n$. The inverse limit $\varprojlim \tilde{\mathbb{T}}_n^*$ acts on $H^*(Y(K), \mathbf{Z}_\ell)$, and its image in $\text{End}H^*(Y(K), \mathbf{Z}_\ell)$ is precisely the global derived Hecke algebra $\tilde{\mathbb{T}}$.

Fix m . For $n \geq m$ consider the map

$$\tilde{\mathbb{T}}_n^* \rightarrow \tilde{\mathbb{T}}_m^*.$$

For increasing n and fixed m , the image of this map gives a decreasing sequence of subsets of the finite set $\tilde{\mathbb{T}}_m^*$. This sequence must stabilize. Call this stabilization

$\tilde{\mathbb{T}}_{\infty,m}$; it is a subring of $\tilde{\mathbb{T}}_m^*$ and thus acts by endomorphisms of $H^*(Y(K), \mathbf{Z}/\ell^m)$; also, there exists N_m such that $\tilde{\mathbb{T}}_{\infty,m}$ coincides with the image of $\tilde{\mathbb{T}}_{N_m}^*$ in $\tilde{\mathbb{T}}_m^*$.

The natural map

$$\varprojlim \tilde{\mathbb{T}}_n^* \rightarrow \tilde{\mathbb{T}}_{\infty,m} \tag{35}$$

is onto, since we are dealing with an inverse system of finite sets.

Let $\mathcal{H}_{v,\mathbf{Z}/\ell^n}$ be the local derived Hecke algebra at v with \mathbf{Z}/ℓ^n -coefficients. We show later (Section 3.4) that, if ℓ^n divides $q_v - 1$, then the natural map $\mathcal{H}_{v,\mathbf{Z}/\ell^n} \rightarrow \mathcal{H}_{v,\mathbf{Z}/\ell^m}$ is surjective. It follows that if $q_v - 1$ is divisible by ℓ^{N_m} , then the image of $\tilde{\mathbb{T}}_{N_m}^*$ acting on $H^*(Y(K), \mathbf{Z}/\ell^m)$ contains the image of $\mathcal{H}_{v,\mathbf{Z}/\ell^m}$. Therefore, the image of $\tilde{\mathbb{T}}_{\infty,m}$ acting on $H^*(Y(K), \mathbf{Z}/\ell^m)$ contains the image of $\mathcal{H}_{v,\mathbf{Z}/\ell^m}$.

In practice, we establish ‘bigness’ results of the following type:

(*) For all $m \leq n$, there exists sets of primes $Q_n = \{q_1, \dots, q_r\}$ such that ℓ^n divides $q_i - 1$ and the image of $\otimes_i \mathcal{H}_{q_i,\mathbf{Z}/\ell^m}$ acting on $H^*(Y(K), \mathbf{Z}/\ell^m)$ is ‘large:’ $H^*(Y(K), \mathbf{Z}/\ell^m)$ is generated over $\otimes_i \mathcal{H}_{q_i,\mathbf{Z}/\ell^m}$ by elements of some fixed degree D .

When we prove such results, it will not be for the full cohomology of $Y(K)$ but rather for its localization at some ideal of the Hecke algebra, but we suppress that for the current discussion.

Let us prove that, under this assumption (*), $H^*(Y(K), \mathbf{Z}_\ell)$ is generated over $\tilde{\mathbb{T}}$ by elements of degree D . The assumption implies (by the previous discussion, with $n = N_m$) that $H^*(Y(K), \mathbf{Z}/\ell^m)$ is generated over $\tilde{\mathbb{T}}_{\infty,m}$ by elements of degree D ; by (35), it is also generated over $\varprojlim \tilde{\mathbb{T}}_n^*$ by elements of degree D . That is to say,

$$\left(\varprojlim \tilde{\mathbb{T}}_n^*\right) \otimes H^D(Y(K), \mathbf{Z}/\ell^m)$$

surjects onto $H^*(Y(K), \mathbf{Z}/\ell^m)$ for every m . By a compactness argument the same assertion holds with \mathbf{Z}_ℓ -coefficients.

More generally, the same type of argument allows us to show that various types of ‘largeness’ can be passed from \mathbf{Z}/ℓ^m coefficients to \mathbf{Z}_ℓ .

2.14. Restricting places and the strict global derived Hecke algebra. We can restrict the primes and the powers of ℓ used in the above construction. It is convenient to index these restrictions by a function

$$V : \text{primes} \longrightarrow \{0, 1, 2, \dots, \} \cup \{\infty\}$$

where primes v with $V(v) = 0$ will be not used at all in the definition.

In the above construction, replace $\tilde{\mathbb{T}}_n$ by the algebra generated by $\mathcal{H}_{v,\mathbf{Z}/\ell^n}$ where $n \leq V(v)$. Proceeding as above, then, we obtain a restricted global Hecke

algebra $\tilde{\mathbb{T}}^{(V)}$ acting on cohomology with \mathbf{Z}_ℓ coefficients. Informally, each prime v can be used at most at torsion level $\mathbf{Z}/\ell^{V(v)}$.

For example, taking

$$V_0(v) = \begin{cases} 0, & v \text{ not good;} \\ \text{largest power of } \ell \text{ dividing } q_v - 1, & \text{else,} \end{cases} \tag{36}$$

the resulting algebra $\tilde{\mathbb{T}}^{(V_0)}$ has the advantage that it will be graded-commutative by the results of Section 3, at least if ℓ does not divide the order of the Weyl group.

It will be sometimes convenient to enlarge this by the usual Hecke algebra, that is, defining the ‘strict’ global derived Hecke algebra

$$\tilde{\mathbb{T}}' := \text{algebra generated by } \tilde{\mathbb{T}}^{(V_0)} \text{ and all undervived Hecke operators at good places prime to } \ell.$$

However, by default, when we write $\tilde{\mathbb{T}}$, we mean the ‘full’ version using $V = \infty$ for all good primes not above ℓ , and $V = 0$ at all other primes. Thus we have inclusions of algebras, each inside endomorphisms of cohomology:

$$\text{usual (undervived) Hecke algebra } \mathbb{T} \subset \tilde{\mathbb{T}}' \subset \tilde{\mathbb{T}}.$$

The advantage of $\tilde{\mathbb{T}}'$ is that it is clearly graded-commutative.

Thus, for example, if \mathfrak{m} is a maximal ideal of \mathbb{T} , the strict global derived Hecke algebra $\tilde{\mathbb{T}}'$ induces an algebra of endomorphisms of the \mathfrak{m} -completion $H^*(Y(K), \mathbf{Z}_\ell)_{\mathfrak{m}}$:

$$H^*(Y(K), \mathbf{Z}_\ell)_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}' \longrightarrow H^*(Y(K), \mathbf{Z}_\ell)_{\mathfrak{m}}.$$

While *a priori* we do not know that the full $\tilde{\mathbb{T}}$ preserves $H^*(Y(K), \mathbf{Z}_\ell)_{\mathfrak{m}}$, this is true under a mild additional assumption: For each good place w not equal to ℓ , let $\mathbb{T}^{(w)}$ be the prime-to- w usual Hecke algebra, and $\mathfrak{m}^{(w)}$ the induced maximal ideal. Suppose that the natural map

$$H^*(Y(K), \mathbf{Z}_\ell)_{\mathfrak{m}^{(w)}} \rightarrow H^*(Y(K), \mathbf{Z}_\ell)_{\mathfrak{m}} \tag{37}$$

is an isomorphism. This is true, for example, if there exists Galois representations associated to cohomology classes (by the argument of [21, Lemma 6.20]). In this case, the local derived Hecke algebra at w clearly preserves the left-hand side, and so also preserves the right-hand side. Since this is true for all good w not dividing ℓ , the full $\tilde{\mathbb{T}}$ also preserves the \mathfrak{m} -completion of cohomology.

3. Torus localization and Satake isomorphism

Our main goal here is to prove a version of the Satake isomorphism that applies to the derived Hecke algebra. Namely, take $m = \ell^r$ a prime power. Suppose $q \equiv 1$ modulo ℓ^r . We show (see (40) for the precise statement)

$$\begin{aligned} & \text{derived Hecke algebra for split } q\text{-adic group with } \mathbf{Z}/\ell^r\text{-coefficients} \\ & \cong (\text{derived Hecke algebra for maximal torus with } \mathbf{Z}/\ell^r \text{ coefficients})^W \end{aligned}$$

where the W superscript means Weyl-fixed, and we also require that ℓ does not divide the order of W .

For example, if $q \equiv 1$ modulo ℓ , the derived Hecke algebra of $\mathrm{PGL}_2(\mathbf{Q}_q)$ with coefficients in \mathbf{Z}/ℓ is isomorphic to

$$\mathbf{Z}/\ell[x_0, x_0^{-1}, y_1, z_2]^{\mathbf{Z}/2}$$

where x_0, y_1, z_2 have (respectively) degree 0, 1, 2, and the action of $\mathbf{Z}/2$ switches $x_0^{\pm 1}$ and negates y_1, z_2 .

A consequence of our results is that (under our assumptions on q, m) the derived Hecke algebra is graded-commutative. We do not know if this is valid without any assumption on q and the coefficient ring S . Recall, however, that $q \equiv 1$ in S is precisely the case where the Hecke algebra is largest, by the discussion of Section 2.4, and understanding this case will be enough for our global analysis.

3.1. It is a curious fact that, in characteristic dividing $q_v - 1$, the Iwahori–Hecke algebra of a split F_v -group is isomorphic to the group algebra of its affine Weyl group. A related interesting phenomenon is that, under the same assumptions, the Satake isomorphism

$$\text{Hecke algebra} \longrightarrow \text{Hecke algebra of torus}$$

is given simply by *restriction* (!)

These points can be explained by ‘torus localization,’ as we now explain. Using that method we derive our Satake isomorphism below. Of course this is a little bit cheap, but it turns out to be exactly what we need anyway.

I am very grateful to David Treumann for conversations about this material. In particular, I learned about localization in the context of local geometric Langlands from his paper *Smith theory and geometric Hecke algebras* [34].

3.2. In this section and the next, \mathbf{G} will be a *split group over the nonarchimedean local field F_v* . The coefficient ring for all our Hecke algebras will be taken to be $S = \mathbf{Z}/\ell^r$, for a prime ℓ and $r \geq 1$. We shall suppose that ℓ^r divides $(q_v - 1)$, where q_v is the cardinality of the residue field \mathbf{F}_v . We also assume that ℓ is relatively prime to the order of the Weyl group of \mathbf{G} .

We fix other notations as follows: Let \mathcal{G} be a split group over \mathcal{O}_v whose generic fiber is identified with \mathbf{G} . Let $K_v = \mathcal{G}(\mathcal{O}_v)$, a maximal compact subgroup of $G_v = \mathbf{G}(F_v)$. Let \mathbf{A} be a maximal torus in \mathbf{G} , and \mathbf{B} a Borel subgroup of \mathbf{G} containing \mathbf{A} ; we suppose them to extend to a torus \mathcal{A} and Borel \mathcal{B} inside \mathcal{G} . We write A_v, B_v for the F_v -points of \mathbf{A}, \mathbf{B} . We shall use the notation $\mathbf{A}(F_v)^\circ$ for the maximal compact subgroup of $\mathbf{A}(F_v)$, and similar notation whenever the maximal compact subgroup is unique.

Let W be the Weyl group for \mathbf{A} . We write $X_* = X_*(\mathbf{A})$ for the cocharacter group of \mathbf{A} . We identify X_* with $A_v/A_v \cap K_v \subset G_v/K_v$ by means of the map

$$\chi \in X_* = \text{Hom}(\mathbb{G}_m, \mathbf{A}) \longrightarrow \chi(\varpi_v), \tag{38}$$

with ϖ_v a uniformizer.

We write for short $T = \mathcal{A}(\mathbf{F}_v)$. The reduction map $A_v \cap K_v \rightarrow T$ splits uniquely, and so we obtain a ‘Teichmüller’ lift

$$T \hookrightarrow A_v \cap K_v. \tag{39}$$

This induces a cohomology isomorphism, with \mathbf{Z}/ℓ^r coefficients.

We have a Cartan decomposition

$$G_v = K_v \cdot A_v \cdot K_v.$$

The A_v component of this decomposition is unique up to the action of the Weyl group W .

THEOREM 3.3. *Let notations be as above; in particular, the coefficient ring is always $S = \mathbf{Z}/\ell^r$, where ℓ^r divides $q_v - 1$, and ℓ does not divide the order of the Weyl group.*

Then restriction (in the model of Section 2.3) defines an isomorphism

$$\begin{aligned} &\text{derived Hecke algebra for } (G_v, K_v) \\ &\xrightarrow{\sim} \text{derived Hecke algebra for } (A_v, A_v \cap K_v)^W. \end{aligned} \tag{40}$$

Let us explicate what we mean by ‘restriction.’ As per Section 2.3, an element h of the left-hand side is an association:

$$(x, y) \in (G_v/K_v)^2 \rightsquigarrow h(x, y) \in H^*(G_{xy}, S),$$

and its image h' on the right-hand side is obtained by restricting to $A_v/(A_v \cap K_v) \hookrightarrow G_v/K_v$ and pulling back cohomology classes under the inclusion $A_{xy} \hookrightarrow G_{xy}$. The element h' is clearly A_v -invariant, and it is also W -invariant:

$$[w]^*h'(wx, wy) = h(x, y)$$

because of the G -invariance of h .

Because $A_{xy} = A_v \cap K_v$ for each x, y , and the (Teichmüller) inclusion (39) $T \hookrightarrow A_v \cap K_v$ induces a cohomology isomorphism, we can regard h' as a function $X_* \times X_* \rightarrow H^*(T, S)$. We often regard h' as such without explicit comment. The multiplication in this model is usual convolution in the X_* variable, together with multiplication in $H^*(T, S)$. We may therefore identify the right-hand side of (40) with

$$(S[X_*] \otimes H^*(T, S))^W, \tag{41}$$

just as in (25), that is, restrict to $X_* \times \{0\}$ and identify functions on X_* with the group algebra in the obvious way.

3.4. A useful corollary to this result is the following (although even easier, as it does not use the algebra structure): The induced map

$$\text{derived Hecke algebra over } \mathbf{Z}/\ell^n \rightarrow \text{derived Hecke algebra over } \mathbf{Z}/\ell^m$$

is a *surjection* for $n > m$, under our assumption that ℓ^n divides $q_v - 1$. (We used this in the discussion of Section 2.13).

In fact, we are reduced to checking the same fact when C is a cyclic group of order divisible by ℓ^n , that is,

$$H^*(C, \mathbf{Z}/\ell^n) \rightarrow H^*(C, \mathbf{Z}/\ell^m)$$

is surjective. This follows from a straightforward computation.

3.5. Some useful Lemmas.

LEMMA 3.6. *Any nontrivial root α of \mathbf{A} on \mathbf{G} is nontrivial on the ℓ -Sylow of $A_v \cap K_v$. In particular, α induces a nontrivial character $\mathcal{A}(\mathbf{F}_v) \rightarrow \mathbf{F}_v^\times$.*

Proof. This is just a matter of checking the residue characteristic is forced to be big enough: if the claim is not true, the root α would be divisible by ℓ^r in $X^*(\mathbf{A})$; but roots are divisible at most by 2 because $\langle \alpha, \alpha^\vee \rangle = 2$, and $\ell > 2$ because it is prime to the order of the Weyl group. □

LEMMA 3.7. Use notation as above; in particular, \mathbf{F}_v is a finite field of cardinality $q_v \equiv 1$ modulo ℓ^r , and the order of the Weyl group is not divisible by ℓ .

Then the restriction map from the cohomology $H^*(\mathcal{G}(\mathbf{F}_v), \mathbf{Z}/\ell^r)$ to Weyl-fixed cohomology of the torus $H^*(\mathcal{A}(\mathbf{F}_v), \mathbf{Z}/\ell^r)^W$ is an isomorphism.

Proof. Write for short (and just for this proof) G, A, B for the \mathbf{F}_v -points of $\mathcal{G}, \mathcal{A}, \mathcal{B}$.

Consider the composite of restrictions

$$H^*(G, \mathbf{Z}/\ell^r) \rightarrow H^*(B, \mathbf{Z}/\ell^r) \xrightarrow{\sim} H^*(A, \mathbf{Z}/\ell^r). \tag{42}$$

The second map is an isomorphism and its inverse is specified by corestriction. Therefore we can transport the W -action on $H^*(A)$ to a W -action on $H^*(B, \mathbf{Z}/\ell^r)$; explicitly the action of w is

$$\text{Cores}_A^B \circ [w]_A \circ \text{Res}_A^B, \tag{43}$$

where $[w]_A$ is pullback of cohomology classes under $\text{Ad}(w^{-1}) : A \rightarrow A$.

We now show that $\text{Res}_B^G \circ \text{Cores}_B^G = \sum_{w \in W} w$, where the w -action on $H^*(B)$ is that just defined. Since $\text{Cores}_B^G \text{Res}_B^G = |W|$, which is invertible in \mathbf{Z}/ℓ^r , we see that Res_B^G is injective and Cores_B^G is surjective; so Res_B^G is an isomorphism onto the W -invariants on $H^*(B, \mathbf{Z}/\ell^r)$, which implies the Lemma.

By the usual formula [6, Proposition 9.5], using the Weyl group W as a system of representatives for double cosets, the composite equals

$$\sum_{w \in W} \text{Cores}_{wBw^{-1} \cap B}^B \cdot \text{Ad}(w^{-1})^* \cdot \text{Res}_{B \cap w^{-1}Bw}^B.$$

But $w^{-1}Bw \cap B$ contains A , and $[wBw^{-1} \cap B : A] = 1$ modulo ℓ^r . So we can rewrite the w -term as

$$\text{Cores}_{wBw^{-1} \cap B}^B \cdot \text{Ad}(w^{-1})^* \cdot \text{Cores}_A^{B \cap w^{-1}Bw} \text{Res}_A^B = \text{Cores}_A^B \cdot [w]_A \cdot \text{Res}_A^B$$

which is exactly the W -action on $H^*(B)$, by (43). □

LEMMA 3.8. Let G_1, G_2 be finite groups. Suppose that $G_1 \hookrightarrow G_1 \times G_2$ is the natural inclusion, and M is a module for $G_1 \times G_2$, with trivial G_2 action, and whose order $\#M$ is killed by the order $\#G_2$ of G_2 . Then the corestriction map $H^*(G_1, M) \rightarrow H^*(G_1 \times G_2, M)$ is zero.

Proof. Indeed, the composite $H^*(G_1 \times G_2) \xrightarrow{\text{Res}} H^*(G_1) \xrightarrow{\text{Cores}} H^*(G_1 \times G_2)$ is multiplication by the order of G_2 , and is therefore zero with M coefficients; but the first Res is surjective because $G_1 \rightarrow G_1 \times G_2$ is split. □

LEMMA 3.9. *Let $\Gamma \subset G_v$ be a finite ℓ -subgroup. Let \mathbf{S} be the double centralizer of Γ , considered as an algebraic F_v -subgroup of \mathbf{G} . Then:*

- (a) *\mathbf{S} has component group of prime-to- ℓ order,*
- (b) *the maximal compact subgroup $\mathbf{S}(F_v)^\circ$ of its F_v -points fixes every point of G_v/K_v that is fixed by Γ .*

Proof. Let $x \in G_v/K_v$ be fixed by Γ . Conjugating Γ by G_v we may suppose that $x = K_v$, the identity coset in G_v/K_v .

Now, the quotient of the orders of \mathcal{G} and \mathcal{A} over the finite field \mathbf{F}_v is congruent to $|W|$ modulo ℓ , because of our assumption ℓ divides $q - 1$. Therefore there is an ℓ -Sylow of K_v contained in $A_v \cap K_v$. Thus, further conjugating Γ by K_v we can further assume that $\Gamma \subset \mathbf{A}(F_v) \cap K_v$. The centralizer $\mathbf{Z}(\Gamma)$ of Γ is then a subgroup containing \mathbf{A} . The double centralizer \mathbf{S} is thus contained in \mathbf{A} and, of course, it contains Γ .

Because $\mathbf{S} \subset \mathbf{A}$, the maximal compact subgroup of $\mathbf{S}(F_v)$ is contained in the maximal compact $\mathbf{A}(F_v) \cap K_v$ of \mathbf{A} ; the latter fixes x . This proves (b).

To verify the assertion about the component group of \mathbf{S} , we first verify that $\mathbf{Z}(\Gamma)$ is connected. Note that \mathbf{S} is contained in $\mathbf{Z}(\Gamma)$ by the analysis above, so it is in fact the center of $\mathbf{Z}(\Gamma)$. Then we are reduced to the following assertion: for any reductive group \mathbf{Z} , the component group of the center of \mathbf{Z} is only divisible by primes dividing the order of the Weyl group. Replacing \mathbf{Z} by its quotient by the connected center, we can check the same assertion for \mathbf{Z} semisimple; so it is enough to check for \mathbf{Z} simply connected semisimple. There it is obvious case by case.

To see that $\mathbf{Z} = \mathbf{Z}(\Gamma)$ is connected, we can reason as follows: \mathbf{A} is a maximal split torus within \mathbf{Z} , so any element of \mathbf{Z}/\mathbf{Z}^0 has a representative in \mathbf{Z} that belongs to the normalizer of \mathbf{A} . Here \mathbf{Z}^0 denotes the connected component. So it is enough to show that any \overline{F}_v -point n in the normalizer of \mathbf{A} that belongs to \mathbf{Z} actually belongs to \mathbf{Z}^0 . Let $w \in \text{Aut}(\mathbf{A})$ be the element of the Weyl group of \mathbf{A} corresponding to such an n . Fix $\gamma \in \Gamma$. Since n centralizes Γ , we see that w fixes γ . Write N for the ℓ -part of $q - 1$. So $\gamma \in \mathbf{A}[N] \simeq X_* \otimes \mu_N$; fixing a primitive N th root, we can identify $\mathbf{A}[N]$ with X_*/N . Since the order of w is relatively prime to ℓ , we see—by taking invariants in $X_* \rightarrow X_* \rightarrow X_*/N$ —that γ actually lies in the image of some w -fixed character $\mathbb{G}_m \rightarrow \mathbf{A}$.

Applying this reasoning for each $\gamma \in \Gamma$, we see that w actually centralizes a subtorus of \mathbf{A} containing Γ . But the centralizer of that torus is a connected group, thus contained in \mathbf{Z}^0 . We conclude that \mathbf{Z} is connected, as we claimed. \square

LEMMA 3.10. *Suppose $z \in G_v/K_v$ does not belong to the image of X_* (where the map $X_* \hookrightarrow G_v/K_v$ was defined in (38)).*

Let Γ be an ℓ -Sylow of $A_v \cap K_v$. Let Γ_z be the stabilizer of z in Γ .

Then the corestriction map $H^(\Gamma_z) \rightarrow H^*(\Gamma)$ is zero with \mathbf{Z}/ℓ^r coefficients.*

Proof. Note that the centralizer and so also the double centralizer of Γ is simply \mathbf{A} . (Any root of \mathbf{A} is nontrivial on the ℓ -Sylow of $A_v \cap K_v$, by Lemma 3.6, so the connected centralizer is \mathbf{A} ; the centralizer cannot be larger than \mathbf{A} because any element of the Weyl group acts nontrivially on $\Gamma \simeq X_*(\mathbf{A})/\ell^r$.)

Let \mathbf{S} be the double centralizer of Γ_z . Since $\Gamma_z \subset \Gamma$ we also have $\mathbf{S} \subset \mathbf{A}$.

Let \mathbf{S}^0 be the identity component of \mathbf{S} ; it is a split torus. Because ((a) of Section 3.9) the component group of \mathbf{S} is prime-to- ℓ , we see that Γ_z lies inside \mathbf{S}^0 , and thus inside the maximal compact subgroup of $\mathbf{S}^0(F_v)$. Let Γ_z^* be the ℓ -Sylow of $\mathbf{S}^0(F_v)$. Thus $\Gamma_z \subset \Gamma_z^*$.

Choose a complement $\mathbf{S}' \subset \mathbf{A}$ to \mathbf{S}^0 , that is, a subtorus with the property that $\mathbf{S}^0 \times \mathbf{S}' \rightarrow \mathbf{A}$ is an isomorphism. Now $\Gamma \subset \mathbf{A}(F_v)^\circ$ is an ℓ -Sylow by computation of orders, so therefore

$$\Gamma = \Gamma_z^* \times \Gamma'$$

where Γ' is the ℓ -Sylow of $\mathbf{S}'(F_v)^\circ$.

If \mathbf{S}' were trivial, then $\mathbf{S}^0 = \mathbf{A}$; in that case, by (b) of Section 3.9, z lies in the fixed set of $\mathbf{A}(F_v)^\circ$, which is none other than

$$X_* \subset G_v/K_v,$$

which contradicts our assumption. (To see this, let \mathbf{N} be the unipotent radical of the Borel $\mathbf{B} \supset \mathbf{A}$. If $\mathbf{A}(F_v)^\circ x K_v = x K_v$, we have $x^{-1} \mathbf{A}(F_v)^\circ x \subset K_v$. By using the Iwasawa decomposition, it is enough to check that if this inclusion holds for some $x = n \in \mathbf{N}(F_v)$, then in fact $n \in K_v$. In that case we have $n^{-1} a n \in K_v$ for all $a \in \mathbf{A}(F_v)^\circ$, and in particular, $n^{-1}(\text{Ad}(a)n) \in K_v$ for all such a . Choose a generic positive element $\lambda \in X_*(\mathbf{A})$, giving an enumeration of the positive roots $\alpha_1, \dots, \alpha_s$ so that $\langle \alpha_i, \lambda \rangle$ is increasing. For each such root we have a root subspace $u_i : \mathbf{G}_a \rightarrow \mathbf{N}$, and the product map $u_s(x_s)u_{s-1}(x_{s-1}) \dots u_1(x_1)$, from $\mathbf{G}_a^s \rightarrow \mathbf{N}$, extends to an isomorphism of schemes over \mathcal{O}_v . In this ordering, the commutator $[u_i, u_j]$ involves only u_k with $k > \max(i, j)$. Let x_1 be the α_1 coordinate of n . We have $(\alpha_1(a) - 1)x_1 \in \mathcal{O}_v$ for all $a \in \mathbf{A}(F_v)^\circ$, which implies $x_1 \in \mathcal{O}_v$, cf. second paragraph of the proof of Lemma 3.10. Adjust n on the right by $u_1(-x_1)$ to arrange that x_1 is trivial. Now proceed the same way for the $\alpha_2, \alpha_3, \dots$ coordinate.)

Therefore, \mathbf{S}' is nontrivial. We see at once that the order of Γ' is divisible by the ℓ -part of $q - 1$.

Thus, by Lemma 3.8, the corestriction from Γ_z^* to Γ is zero with \mathbf{Z}/ℓ^r coefficients. The corestriction from Γ_z to Γ factors through this one, so it is zero too. \square

3.11. Proof of Theorem 3.3. Throughout the proof, as in the statement of the theorem, we take coefficients in $S = \mathbf{Z}/\ell^r$. Accordingly, we drop explicit mention of the coefficients from the notation.

Recall the explicit description of the Satake map, using the identification (41) of the toral derived Hecke algebra:

Given an assignment $(x, y) \in G_v/K_v \mapsto h(x, y) \in H^*(G_{xy})$, we associate to it the function $X_* \times X_* \rightarrow H^*(T)$, given by

$$h' : (x, y) \in (A_v/A_v \cap K_v)^2 \mapsto \text{Res}_T^{G_{xy}} h(x, y) \in H^*(T).$$

We must show that the rule $h \mapsto h'$ gives an isomorphism

$$\begin{aligned} &\text{derived Hecke algebra for } (G_v, K_v) \\ &\simeq \text{derived Hecke algebra for } (A_v, A_v \cap K_v)^W. \end{aligned} \tag{44}$$

We first verify that $h \mapsto h'$ is bijective. Each element of the derived Hecke algebra for (G_v, K_v) is uniquely of the form $\sum h_{a,\alpha}$ where $a \in X_*^+$ and $\alpha \in H^*(K_v \cap \text{Ad}(a)K_v)$, with notation as in Section 2.4. The intersection of $K_v a K_v$ with X_* is precisely given by the W -orbit of a by uniqueness of the Cartan decomposition. So the map $h \mapsto h'$ sends $h_{a,\alpha}$ to the function $h'_{a,\alpha}$ on $X_* \times X_*$ characterized by W -invariance and:

- (i) $h_{z,\alpha}(x, e) = 0$ unless $x \in Wa$;
- (ii) $h_{z,\alpha}$ sends (a, e) to the image of $\alpha \in H^*(K_v \cap \text{Ad}(a)K_v) \rightarrow H^*(T)$.

It is enough, then, to show that each element of derived Hecke algebra for $(A_v, A_v \cap K_v)^W$ is uniquely a sum of such elements $h'_{a,\alpha}$. This comes down to the fact that the map

$$H^*(K_v \cap \text{Ad}(a)K_v) \longrightarrow H^*(T)^{W_a} \tag{45}$$

is an isomorphism, where W_a is the stabilizer of a in the Weyl group. But, if we write M for the Levi subgroup of \mathcal{G} that centralizes a , then $K_v \cap \text{Ad}(a)K_v$ is, modulo a pro- p -group, the k -points $M(k)$, and W_a is identified with the Weyl group of M . So (45) follows from Lemma 3.7.

To show that $h \mapsto h'$ preserves multiplication, we compute $(h_1 h_2)'(x, z)$; it equals the restriction, from G_{xz} to T , of

$$\sum_{O \subset G_v/K_v} \sum_{y \in O} h_1(x, y) \cup h_2(y, z),$$

the sum being grouped, as before, over orbits O of G_{xz} on such y . Recall that the inner sum is understood by computing the cup product $h_1(x, y) \cup h_2(y, z)$ for a single $y \in O$, and then corestricting from G_{xyz} to G_{xz} . Therefore,

$$(h_1 h_2)'(x, z) = \sum_O \underbrace{\text{Res}_T^{G_{xz}} \text{Cores}_{G_{xyz}}^{G_{xz}} h_1(x, y) \cup h_2(y, z)}_{:=H(O)}$$

and as usual we can express $H(O)$ a sum over T -orbits on G_{xz}/G_{xyz} , that is to say, as a sum of T orbits $O' \subset O$:

$$H(O) = \sum_{O'} \text{Cores}_{T_{y'}}^T \text{Res}_{T_{y'}}^{G_{xyz}}(\dots) \tag{46}$$

where we have chosen a representative $y' \in O'$ for each T -orbit O' upon O ; and the injection $T_{y'} \rightarrow G_{xyz}$ that defines the restriction map is induced by an element of G_{xz} conjugating y' to y .

We saw in Lemma 3.10 that the corestriction map vanishes unless y' actually belongs to X_* . (Indeed, writing Γ for the unique ℓ -Sylow of the abelian group T , then $\Gamma_{y'}$ is an ℓ -Sylow of $T_{y'}$, and the corestriction map induced by $\Gamma_{y'} \rightarrow T_{y'}$ is surjective on cohomology.) In the case when $y' \in X_*$, we have $T_{y'} = T$ in which case $O' = \{y'\}$. We conclude that

$$H(O) = \sum_{y' \in O \cap X_*} \text{Res}_T^{G_{xyz}}(h_1(x, y) \cup h_2(y, z))$$

and finally adding up all O we get

$$\begin{aligned} (h_1 h_2)'(x, z) &= \sum_{y \in X_*} \text{Res}_T^{G_{xyz}}(h_1(x, y) \cup h_2(y, z)) \\ &= \sum_{y \in X_*} \text{Res}_T^{G_{xy}} h_1(x, y) \cup \text{Res}_T^{G_{yz}} h_2(y, z) = h'_1 h'_2(x, z). \end{aligned}$$

This concludes the proof of the theorem.

4. Iwahori–Hecke algebra

In this section, we collect a few important facts about Iwahori–Hecke algebras. In particular, we discuss the structure of the Iwahori–Hecke algebra at a Taylor–Wiles prime (Section 4.2), the relation between modules over the (usual, that is, nonderived) Iwahori–Hecke algebra and modules over the (usual) spherical Hecke algebras (Section 4.4) and finally briefly discuss a localization result for the derived Iwahori–Hecke algebra (Section 4.6).

These results are presumably well known to experts but they help us polish our presentation of the Taylor–Wiles method—indeed similar ideas appear in the paper of Khare and Thorne [21].

4.1. We continue with the notation of the prior section (Section 3.2). In particular, G_v is the F_v -points of a reductive split group.

In this section, we also make use of the affine Weyl group \tilde{W} attached to \mathbf{G} ; by definition this is the semidirect product $X_* \rtimes W$ where X_* is the cocharacter group of the maximal torus \mathbf{A} , and W is the Weyl group of \mathbf{A} .

Let S be the ring \mathbf{Z}/ℓ^r , for a prime ℓ ; this will be the coefficient ring for all our Hecke algebras and derived Hecke algebras. We suppose that $q_v \equiv 1$ modulo ℓ^r and that ℓ does not divide the order of the Weyl group.

Let I_v be an Iwahori subgroup of G_v contained inside K_v and in good position with reference to \mathbf{A} . By this we mean that I_v stabilizes a chamber of the building that lies inside the apartment defined by \mathbf{A} . An explicit choice of such an I_v can be obtained from an integral model \mathcal{B} of a Borel subgroup containing \mathbf{A} :

$$I_v = \text{preimage of } \mathcal{B}(\mathbf{F}_v) \text{ inside } \mathcal{G}(\mathcal{O}_v),$$

and the other such I_v s are W -conjugate to this one.

It will be helpful to keep in mind that the index $[K_v : I_v] \equiv |W|$ modulo ℓ^r , in particular, this index is invertible in S . Take the Haar measure on G_v which assigns I_v mass 1.

4.2. The structure of the Iwahori algebra. Let H_I be the Hecke algebra for I_v . We understand this to be defined as

$$H_I := \text{Hom}_{S G_v}(S[G_v/I_v], S[G_v/I_v]).$$

This is identified with the set of S -valued and finitely supported functions f on $I_v \backslash G_v / I_v$. Namely, identifying such functions with measures (multiplying by the Haar measure on G_v , thought of as valued in S), each such function f acts by right convolution on $S[G_v/I_v]$, and therefore defines an element of H_I .

Therefore, in the text, we often produce elements of H_I by describing the associated bi-invariant function.

Warning: the resulting identification

$$H_I \simeq (\text{functions on } I_v \backslash G_v / I_v) \tag{47}$$

is an anti-isomorphism of algebras if we equip the right-hand side with the convolution product. When we multiply elements of H_I , we always understand

the multiplication to be that of H_I , not the multiplication induced from the right-hand side of (47).

We similarly define

$$\begin{aligned} H_K &= \text{Hom}_{S_{G_v}}(S[G_v/K_v], S[G_v/K_v]), \\ H_{IK} &= \text{Hom}_{S_{G_v}}(S[G_v/K_v], S[G_v/I_v]), \\ H_{KI} &= \text{Hom}_{S_{G_v}}(S[G_v/I_v], S[G_v/K_v]). \end{aligned}$$

If V is any G -representation, the algebra $\text{Hom}_{S_{G_v}}(S[G_v/K_v], S[G_v/K_v])$ acts on the right on $V^{K_v} = \text{Hom}_{G_v}(S[G_v/K_v], V)$. Similarly, elements H_{IK} induces endomorphisms $V^{I_v} \rightarrow V^{K_v}$. Indeed a useful mnemonic for the subscript ‘ IK ’ is that, acting as explained above, H_{IK} goes from I -invariants to K -invariants, and so on. Also H_{IK}, H_{KI} are bimodules for H_K and H_I .

As before, each of these can be identified with a space of functions. Thus, for example,

$$H_{IK} \simeq \text{functions on } K_v \backslash G_v / I_v, \quad (48)$$

and similarly for H_{IK}, H_K . As before, these identifications arise from the right convolution action of the functions on $S[G_v/K_v]$ or $S[G_v/I_v]$.

Note that, somewhat contrary to what the notation might suggest, an element of H_{IK} considered as a function is *left* K_v -invariant and *right* I_v -invariant. Again, we use these identifications such as (48) without comment to produce elements inside the various Hecke algebras. The same warning applies here: the identifications do not preserve multiplication; the order must be switched, just as (47) is an antiautomorphism. To avoid confusion here, our convention is that products are always to be understood in the sense of the H_I, H_K , and so on, and *not* via convolution of functions.

It is useful later to define

$$e_K = \frac{1_{K_v}}{\text{measure}(K_v)}. \quad (49)$$

Considered as an element of H_I it is idempotent. When considered as an element of H_{IK} , it carries the identity coset of $S[G_v/K_v]$ to $\sum_{k \in K/I} kI_v \in S[G_v/I_v]$, and when considered as an element of H_{KI} , carries the identity coset of $S[G_v/I_v]$ to $\frac{1}{[K_v:I_v]} eK_v \in S[G_v/K_v]$. In particular, $e_K \in H_{IK}$ induces the corestriction map $V^{I_v} \rightarrow V^{K_v}$, and $e_K \in H_{KI}$ induces the map $V^{K_v} \rightarrow V^{I_v}$ which is the natural inclusion divided by the index $[K_v : I_v]$.

Because q is congruent to 1 modulo ℓ^r , the structure of H_I is very simple. It is isomorphic simply to the group algebra of the affine Weyl group:

$$H_I \simeq S[\tilde{W}]. \quad (50)$$

An explicit anti-isomorphism sends the characteristic function of $I_v w I_v$ to the element w , for $w \in \tilde{W}$; in particular, e_K is sent to $\frac{1}{|\tilde{W}|} \sum_{w \in \tilde{W}} w$, the sum over the usual Weyl group.

This follows from the standard presentation of the Iwahori–Hecke algebra (for a reference with complex coefficients, see [12, Theorem 4.2]); the key point is that the relation $(T_s - q)(T_s + 1)$ simplifies to $T_s^2 = 1$ when $q = 1$ in the coefficients. Actually it is also possible to verify H_I is isomorphic to $S[\tilde{W}]$ by using torus localization, although we omit the details.

4.3. Central element and discriminant. Every element of $S[X_*]^W$ is central in $S[\tilde{W}]$. Therefore, (50) yields a natural map (indeed an isomorphism) from $Z := S[X_*]^W$ to the center of H_I .

Then H_I, H_K have structures of Z -algebra and H_{IK}, H_{KI} have structure of Z -module, all of which are compatible in the obvious way.

For example, the ring homomorphism $Z \rightarrow H_K$ is given by $z \mapsto e_K z e_K = e_K z$. In fact this is a ring *isomorphism*, as follows easily from the explicit presentation. Then for example, H_{IK} has two structures of Z -module, one via $Z \rightarrow H_I$ and one via $Z \rightarrow H_K$, and the ‘compatibility’ is that these two structures coincide.

4.4. Iwahori–Hecke algebra. The next statement asserts that the bimodules H_{KI}, H_{IK} give equivalences of categories, at least over the open subset of $\text{Spec}(Z)$ where the ‘ W -covering’ $\text{Spec } S[X_*] \rightarrow \text{Spec } S[X_*]^W$ is étale. (I am grateful to Peter Schneider for correcting an error. In an earlier version of this paper, this Lemma was formulated over a larger open subset of $\text{Spec } Z$, but one step in the proof was not correct in this greater generality.)

LEMMA 4.5. *Let \mathfrak{m} be a maximal ideal of Z over which the map $Z \rightarrow S[X_*]$ is étale, and write Z' for the localization of Z at \mathfrak{m} .*

Write $H'_K = H_K \otimes_Z Z'$ and define similarly H'_I, H'_{IK}, H'_{KI} . Then the bimodules H'_{IK} and H'_{KI} induce inverse equivalences of categories between H'_K modules and H'_I modules.

This is probably well known in characteristic zero at least.

Proof. (Sketch). Let us show, for example, that the natural map induced by multiplication

$$H_{IK} \otimes_{H_K} H_{KI} \rightarrow H_I \tag{51}$$

yields an isomorphism after localization at \mathfrak{m} . In what follows, we denote such localization with a prime: $S[X_*]' = S[X_*] \otimes_Z Z'$.

Note that H_I is free of rank w as a right module over $S[X_*]$ (clear from (50)) and, being finite étale, $S[X_*]'$ is free as a right module over Z' of rank w .

So H_I' is free of rank w^2 over Z' . Similarly, H'_{IK} and H'_{KI} are free of rank w and H'_K is locally free of rank 1 as a Z' -module. The obstruction to (51) being an isomorphism then given by the vanishing of a suitable determinant; it is enough, therefore, to show that (51) is onto after reducing modulo the maximal ideal \mathfrak{m}' of Z' .

We can extend the natural homomorphism $Z' \rightarrow Z'/\mathfrak{m}'$ to a homomorphism $\chi : S[X_*]' \rightarrow k$, with k an algebraically closed field containing the finite field Z'/\mathfrak{m}' . Note that k has characteristic ℓ , and that χ is not fixed by any element of W .

Now a homomorphism from $X_* \rightarrow k^\times$ is the same as an unramified k -valued character χ of the maximal torus A_v ; so we may identify χ to a character of A_v . We form the corresponding induced representation $V = V_\chi$. Its elements consist of locally constant k -valued functions on G_v that transform according to χ on a Borel subgroup containing A_v . Now V^{I_v} is a k -vector space of rank w , and V^{K_v} is a k -vector space of rank 1, and Z acts on these spaces via the character χ (as follows, for example, from (53) below).

We now show that the natural maps

$$H_{ab} \otimes_Z k \rightarrow \text{Hom}(V^a, V^b)$$

are isomorphisms for a and b belonging to $\{I, K\}$; that implies the desired claim, that is to say, that (51) is onto after reducing modulo \mathfrak{m} .

Because the two sides have the same rank it is enough to check surjectivity. In fact, it is enough to show surjectivity in the case of H_{II} and to show that all the other maps are nonzero (because then, for example, the image for H_{IK} would be a nonzero subspace of $\text{Hom}(V^{I_v}, V^{K_v})$ which is stable under $\text{Hom}(V^{I_v}, V^{I_v})$). The other maps are clearly nonzero: the element e_K induces a nonzero map in each of the cases IK, KI, KK . So we are reduced to seeing that

$$H_I \twoheadrightarrow \text{Hom}(V^{I_v}, V^{I_v}). \tag{52}$$

But there is a standard basis for V^{I_v} indexed by the Weyl group: v_w ($w \in W$), whose restriction to K is the characteristic function of the Bruhat cell indexed by w . The group algebra of W , inside H_I , acts by permuting the elements v_w . Also the element $\lambda \in X_*$, considered again inside H_I , acts by

$$\lambda \cdot v_w = \langle w\chi, \lambda \rangle v_w. \tag{53}$$

In other words, as a representation of \tilde{W} , this is the representation induced from the generic character χ , and so clearly irreducible. The surjectivity of (52) follows. \square

4.6. Localization for the derived Iwahori–Hecke algebra. It will later on be helpful to make use of localization for the *derived* Iwahori–Hecke algebra.

We define the derived Iwahori–Hecke algebra as per the recipe of Section 2, that is,

$$\mathcal{H}_I := \text{Ext}_{S G_v}^*(S[G_v/I_v], S[G_v/I_v]).$$

As before this is isomorphic to the algebra of functions h that associate to $(x, y) \in G_v/I_v \times G_v/I_v$ a class $h(x, y) \in H^*(G_{xy}, S)$, with the product as described in Section 2.3. In a similar way, we get derived versions $\mathcal{H}_{IK}, \mathcal{H}_{KI}$ of the bimodules H_{KI}, H_{IK} defined earlier.

Now, we can consider ‘restriction to \tilde{W} ’, that is,

$$h_1 \in \mathcal{H}_{II} \longrightarrow h'_1 \in \text{functions } \tilde{W} \times \tilde{W} \rightarrow H^*(T, S) \tag{54}$$

where T is as in Section 3.2 and we identify $w \in \tilde{W}$ with $wI \in G_v/I_v$; and

$$h_2 \in \mathcal{H}_{IK} \longrightarrow h'_2 \in \text{functions } \tilde{W} \times X_* \rightarrow H^*(T, S) \tag{55}$$

where here we identify $x \in X_*$ with the associated coset xK_v ; and we used the fact that T stabilizes pointwise both $\tilde{W} \cdot I_v$ and $X_* \cdot K_v$ to restrict cohomology classes to T . Finally we have a similar map for \mathcal{H}_{KI} .

Note that the right-hand side of (54) has an algebra structure, at least restricting to functions supported on finitely many \tilde{W} -orbits, by means of the formula $H_1 H_2(x, y) = \sum_{z \in \tilde{W}} H_1(x, z) \cup H_2(z, y)$ (where all of x, y, z all belong to \tilde{W}). This algebra acts on the right-hand side of (55), by means of the same formula, but H_2 now belongs to the right-hand side of (55), and therefore y is now taken to belong to X_* .

LEMMA 4.7. *Under our current notation and assumptions, (see Section 3.2), the map (54) is an algebra morphism. Similarly, the map (55) is compatible with the map (54) and the product $\mathcal{H}_{II} \times \mathcal{H}_{IK} \rightarrow \mathcal{H}_{IK}$; similarly for \mathcal{H}_{KI} .*

Proof. We want to show that (where $h_1, h_2 \in \mathcal{H}_{II}$)

$$\text{Res}(h_1 h_2) = h'_1 h'_2,$$

where Res means to restrict all G_v/I_v arguments to \tilde{W} and restrict cohomology classes to T ; we also want similar statements for the \mathcal{H}_{II} -action on \mathcal{H}_{KI} and \mathcal{H}_{IK} .

By precisely the same argument as in Section 3.11, we are reduced to the following claim:

Claim: Let y' belong to either G_v/K_v or G_v/I_v . Let Γ be an ℓ -Sylow of $A_v \cap K_v$. Let $\Gamma_{y'}$ be the stabilizer of y' in Γ . Then the corestriction $H^*(\Gamma_{y'}, S) \rightarrow H^*(\Gamma, S)$ vanishes, unless $y' \in X_* \subset G_v/K_v$ or $y' \in \tilde{W} \subset G_v/I_v$.

We repeat the reasoning of Lemma 3.10: let \mathbf{S} be the algebraic double centralizer of $\Gamma_{y'}$. As before, $\Gamma_{y'} \subset \Gamma$ gives $\mathbf{S} \subset \mathbf{A}$. Let \mathbf{S}^0 be the identity component of \mathbf{S} . By Lemma 3.9, the component group of \mathbf{S} is prime-to- ℓ . Therefore, $\Gamma_{y'}$ lies inside \mathbf{S}^0 . Let $\Gamma_{y'}^* \subset \mathbf{S}^0(F_v)^\circ$ be an ℓ -Sylow of the maximal compact subgroup. Thus $\Gamma_{y'} \subset \Gamma_{y'}^*$.

Choose a complement $\mathbf{S}' \subset \mathbf{A}$ to \mathbf{S}^0 , that is, a subtorus with the property that $\mathbf{S}^0 \times \mathbf{S}' \rightarrow \mathbf{A}$ is an isomorphism. Then $\Gamma_{y'}^* \subset \mathbf{S}^0(F_v)^\circ$ is an ℓ -Sylow, and $\Gamma \subset \mathbf{A}(F_v)^\circ$ is an ℓ -Sylow. Therefore

$$\Gamma = \Gamma_{y'}^* \times \Gamma'$$

where Γ' is the ℓ -Sylow of $\mathbf{S}'(F_v)^\circ$.

If \mathbf{S}' were trivial, then $\mathbf{S}^0 = \mathbf{A}$. In that case y' lies in the fixed set of $\mathbf{A}(F_v)^\circ$. In the case where $y' \in G_v/K_v$ this was proved in Section 3.9. However, the proof of this assertion also applies word for word to establish the same assertion in the case $y' \in G_v/I_v$.

The fixed set of $\mathbf{A}(F_v)^\circ$ on G_v/K_v is X_* , as before, and the fixed set of $\mathbf{A}(F_v)^\circ$ on G_v/I_v is precisely $\tilde{W}I_v \subset G_v/I_v$. (Here is a proof of the latter claim: if gI_v is fixed, then $gK_v \in X_*K_v$, and modifying g by an element of $\mathbf{A}(F_v)$, we can suppose $g \in K_v$. We are reduced to computing the $\mathbf{A}(F_v)^\circ$ -fixed points on K_v/I_v , which amount to the torus fixed points on a flag variety over \mathbf{F}_v —using Lemma 3.6 to avoid problems with small residue field, these fixed points are precisely the wI_v with $w \in W$, as desired.)

Otherwise, \mathbf{S}' is not trivial, the corestriction $\Gamma_{y'}^* \rightarrow \Gamma$ vanishes as before, and so the corestriction $\Gamma_{y'} \rightarrow \Gamma$ vanishes too. □

5. The trivial representation

In this section we give our first piece of global evidence that the derived Hecke algebra can account for the ‘degree spreading’ of Hecke eigenclasses.

5.1. Let D be a division algebra of dimension d^2 over an imaginary quadratic field F . Let \mathbf{G} be the algebraic group of elements of norm 1 inside D . Let $Y(K)$ be the arithmetic manifold (17) associated to \mathbf{G} and a level structure K . We shall suppose K to be contained in the stabilizer of some maximal order \mathcal{O}_D . Observe that $\dim Y(K) = d^2 - 1$.

In this section, we study the derived Hecke action on ‘Hecke-trivial part’ (see Definition 2.9) of the cohomology of $Y(K)$.

Recall that the global derived Hecke algebra $\tilde{\mathbb{T}}$ consists of all endomorphisms of $H^*(\mathbf{Z}_\ell)$ that are limits (Section 2.13), under $H^*(\mathbf{Z}_\ell) \simeq \varprojlim H^*(\mathbf{Z}/\ell^n)$, of endomorphisms that lie in the algebra generated by all $\mathcal{H}_{v, \mathbf{Z}/\ell^n}$.

THEOREM 5.2. *For all but finitely many primes ℓ , the action of $\tilde{\mathbb{T}}$ on $H^*(Y(K), \mathbf{Z}_\ell)$ preserves the trivial summand $H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}}$, and:*

- (i) *The trivial part of the cohomology $H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}}$ is cyclic over $\tilde{\mathbb{T}}$, generated by the trivial class;*
- (ii) *The image $\tilde{\mathbb{T}}_{\text{triv}}$ of $\tilde{\mathbb{T}}$ in $\text{End } H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}}$ is graded-commutative. Also $\tilde{\mathbb{T}}_{\text{triv}} \otimes \mathbf{Q}_\ell$ coincides with \mathbf{Q}_ℓ -algebra generated by $H^*(Y(K), \mathbf{Q})_{\text{triv}}$ acting on itself by means of the cup product.*

Note the significance of the second part of the statement: inside the \mathbf{Q}_ℓ -derived Hecke algebra there is a natural ‘preferred’ rational structure. Our general conjecture (Conjecture 8.8) says that this should be true in great generality and the preferred rational structure is related to motivic cohomology. Certainly the situation that we discuss here is quite easy compared to the general case, but nonetheless it has several points of interest.

We also note that the theorem is almost certainly *false* (in the form stated above) if F is not totally imaginary, for reasons related to (a) of Section 1.6.

We deduce the Theorem from the following:

LEMMA 5.3. *Notation as above, so that \mathbf{G} is the algebraic group arising from a division algebra over the imaginary quadratic field F . For all sufficiently large ℓ , the following statement holds:*

For each integer n there are infinitely many places v of the field F , with $q_v \equiv 1 \pmod{\ell^n}$ and where the division algebra is locally split, such that the pullback map of (29)

$$H^*(\mathbf{G}(\mathbf{F}_v), \mathbf{Z}/\ell) \rightarrow H^*(Y(K), \mathbf{Z}/\ell)_{\text{triv}}$$

is surjective.

Note that the map above really does take values in the Hecke-trivial cohomology, by Lemma 2.8.

Proof. (Summary) The proof of Lemma (5.3) occupies Sections 5.5–5.11. After some initial setup, we give in Section 5.8 certain conditions (a), (b), (c), (d)

which imply the Lemma; and then after Section 5.8 we check these conditions can actually be satisfied. □

First of all, let us explain why the Lemma implies the theorem:

5.4. Lemma (5.3) implies the theorem.

Proof. First of all, $\tilde{\mathbb{T}}$ preserves $H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}}$. This follows by the argument described around (37). Explicitly, fixing any prime w , we may find a prime-to- w usual Hecke operator T for which the trivial part of \mathbf{Z}_ℓ -cohomology coincides with the generalized zero eigenspace for $T - \text{deg}(T)$. (This can be checked over \mathbf{C} , since we are supposing the prime ℓ to be large enough. See for example, discussion after (59)).

By avoiding a finite set of ℓ , we may clearly suppose that $H^*(Y(K), \mathbf{Z})$ has no ℓ -torsion, and that $\ell > d$. Similarly, we suppose that

$$H^*(Y(K), \mathbf{Z})_{\text{triv}} \otimes_{\mathbf{Z}} \mathbf{Z}/\ell \rightarrow H^*(Y(K), \mathbf{Z}/\ell)_{\text{triv}}$$

is an isomorphism: see Section 5.6 for an explanation.

By Lemma 3.7, the map $H^*(\mathbf{G}(\mathbf{F}_v), \mathbf{Z}/\ell^n) \rightarrow H^*(\mathbf{G}(\mathbf{F}_v), \mathbf{Z}/\ell)$ is surjective if $q_v \equiv 1$ modulo ℓ^n . It follows from this that the surjectivity assertion of Lemma 5.3 continues to hold with coefficients modulo ℓ^n .

Now we can consider an element of $H^*(\mathbf{G}(\mathbf{F}_v), \mathbf{Z}/\ell^n)$ as an element of the derived Hecke algebra for \mathbf{G} at v (see the Remark of Section 2.10). So the assertion implies that the cup product action of each $h \in H^*(Y(K), \mathbf{Z}/\ell^n)_{\text{triv}}$ is contained in the action of the derived Hecke algebra; by passage to the limit, the cup product action of $H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}}$ on itself is contained in the action of the derived Hecke algebra.

Let B_ℓ be the image of cup product $H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}} \rightarrow \text{End}(H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}})$. Let $\tilde{\mathbb{T}}_{\text{triv}}$ be the image of $\tilde{\mathbb{T}}$ inside $\text{End}(H^*(Y(K), \mathbf{Z}_\ell)_{\text{triv}})$. It remains to show that these are equal. This comes down to the fact that $\tilde{\mathbb{T}}_{\text{triv}}$ is contained in the commutant of B_ℓ and so cannot be larger than it. In more detail:

Let $\tilde{\mathbb{T}}_{\text{triv}}^{(V_0)}$ be the subring of $\tilde{\mathbb{T}}_{\text{triv}}$ defined by only using local derived Hecke algebras $\mathcal{H}_{q, \mathbf{Z}/\ell^n}$ with ℓ^n dividing $q - 1$, that is, the restricted variant of the global derived Hecke algebra defined after (36).

We have inclusions

$$B_\ell \subset \tilde{\mathbb{T}}_{\text{triv}}^{(V_0)} \subset \tilde{\mathbb{T}}_{\text{triv}}, \tag{56}$$

where the first inclusion follows from the argument just given. To conclude the proof, we show that these are both equalities. Note, first of all, that each element of $\tilde{\mathbb{T}}_{\text{triv}}^{(V_0)}$ commutes, in the graded sense, with each element of $\tilde{\mathbb{T}}_{\text{triv}}$. Indeed, if

$q \equiv 1$ modulo ℓ^n , the global action of $\mathcal{H}_{q, \mathbf{Z}/\ell^n}$ is readily seen to commute with $\mathcal{H}_{v, \mathbf{Z}/\ell^n}$ for $v \neq q$, and it commutes in the case $v = q$ because $\mathcal{H}_{q, \mathbf{Z}/\ell^n}$ is known to be graded-commutative.

Choose now $h \in \tilde{\mathbb{T}}_{\text{triv}}$. There is $b \in B_\ell$ such that $h.1 = b.1$ (here 1 is the trivial class in H^0). Then $(h - b).1 = 0$. The same is true then for both the even and odd components of $(h - b)$. But, as we just saw, both components commute in the graded sense, with $\tilde{\mathbb{T}}_{\text{triv}}^{(V_0)}$, so in fact both of these components kill all of H_{triv}^* . Thus $h - b = 0$, so $h \in B_\ell$, as required. \square

5.5. Recollections. Let N be an integer. (We shortly fix it to be ‘large enough.’)

Let U_N be the standard unitary group, the stabilizer of $\sum_{i=1}^N |z_i|^2$. There are natural maps

$$\text{bi-invariant differential forms on } U_N \xrightarrow{\sim} H^*(U_N, \mathbf{C}) \longrightarrow H^*(\text{GL}_N(\mathcal{O}_F), \mathbf{C}) \tag{57}$$

obtained by the natural identification of $\text{GL}_N(\mathbf{C})$ -invariant differential forms on $\text{GL}_N(\mathbf{C})/U_N$ with bi-invariant differential forms on U_N (and then Hodge theory). The notation is a little confusing: $H^*(U_N)$ above refers to the singular cohomology of U_N as a topological space, whereas $H^*(\text{GL}_N(\mathcal{O}_F))$ refers to the group cohomology of $\text{GL}_N(\mathcal{O}_F)$.

Moreover, the algebra of invariant differential forms on U_N is a free-exterior algebra with primitive generators $\Omega_1, \Omega_3, \dots, \Omega_{2N-1}$ in degree $1, 3, 5, \dots, 2N - 1$; ‘primitive’ is taken with respect to the coproduct on cohomology, induced by $U_N \times U_N \rightarrow U_N$. An explicit representative for Ω_j can be taken as

$$X_1, \dots, X_j \in \text{Lie}(U_N) \mapsto \text{antisymmetrization of trace}(X_1 \cdots X_j). \tag{58}$$

The same symbols Ω_i will also be used to denote the corresponding invariant differential forms on $\text{GL}_N(\mathbf{C})/U_N$. For later use, note that these can be restricted to cohomology classes for SU_N and also to invariant differential forms on $\text{SL}_N(\mathbf{C})/\text{SU}_N$; these restrictions kill Ω_1 .

There are natural inclusions $U_N \hookrightarrow U_{N+1}$ and $\text{GL}_N(\mathcal{O}_F) \hookrightarrow \text{GL}_{N+1}(\mathcal{O}_F)$. For fixed j and large enough N , these induce isomorphisms in $H^j(-, \mathbf{C})$. Moreover, these isomorphisms are compatible with increasing N . By passage to the inverse limit we get

$$H^*(U_\infty, \mathbf{C}) \longrightarrow H^*(\text{GL}_\infty(\mathcal{O}_F), \mathbf{C}).$$

Here (for example) GL_∞ means in fact $\varinjlim \text{GL}_N$.

Both sides here carry compatible coproducts; for the right-hand side we can take the coproduct induced by ‘intertwining’ map (see for example, [33, Ch. 2])

$\mathrm{GL}_\infty \times \mathrm{GL}_\infty \rightarrow \mathrm{GL}_\infty$. (To see that the coproducts are compatible amounts to say that the group multiplication and the intertwining map both induce the same homomorphism on the stable cohomology of U_N . This follows from the Eckmann–Hilton argument, or more directly as follows: Embed two commuting copies $U_N^{(a)} \times U_N^{(b)} \rightarrow U_{2N}$ by the intertwining map. The group multiplication $U_{2N} \times U_{2N} \rightarrow U_{2N}$ when restricted to $U_N^{(a)} \times U_N^{(b)}$ on the source gives the intertwining map $U_N^{(a)} \times U_N^{(b)} \rightarrow U_{2N}$. This fact, together with stability of homology, shows that the two coproducts are compatible, as claimed.)

The corresponding Pontryagin product on homology will be denoted by $*$.

In what follows, we fix N to be divisible by d^2 and chosen so large that

- the inclusion $\mathrm{GL}_N \hookrightarrow \mathrm{GL}_\infty$ induces an isomorphism of integral group homology in degrees $\leq d^2$, both with entries in \mathcal{O}_F and with entries in any residue field. (This is possible because, by a theorem of van der Kallen [20, page 289], the range of homological stability can be taken uniformly in these cases; indeed, van der Kallen’s bounds for stability involve only the Krull dimension in the case of commutative rings.)

We denote the stabilization map

$$H_i(\mathrm{GL}_N(\mathcal{O}_F)) \rightarrow H_i(\mathrm{GL}_\infty(\mathcal{O}_F))$$

by $a \mapsto a^{(\infty)}$ and its inverse by $b \mapsto b^{(N)}$. We use this notation for any choice of coefficients, not merely \mathbf{Z} .

- The map (57) induces an surjection in degrees $< d^2$. (That this is possible is a consequence of Borel’s result [3, (7.5)] asserting that cohomology classes in sufficiently low degree are representable by invariant differential forms on the symmetric space; Borel’s result is for a semisimple group, but we readily deduce the claimed result by applying it to SL_N .) We have written ‘surjection’ instead of isomorphism just because of the issue of working with GL rather than SL : the differential form corresponding to Ω_1 dies under (57)).

In a similar way to (57), we have an isomorphisms

$$\begin{aligned} H^*(\mathrm{SU}_d, \mathbf{C}) &\rightarrow \mathrm{SL}_d(\mathbf{C}) \text{ invariant diff. forms on } \mathrm{SL}_d(\mathbf{C})/\mathrm{SU}_d \\ &\xrightarrow{\sim} H^*(Y(K), \mathbf{C})_{\mathrm{triv}}. \end{aligned} \quad (59)$$

For the surjectivity of the final map: if a differential form ω on $Y(K)$ satisfies $T\omega = \deg(T)\omega$ for even one Hecke operator T , then by an easy ‘maximum modulus’ argument it must be invariant, that is, represented by a $\mathbf{G}(F \otimes \mathbf{R})$ invariant form on the corresponding symmetric space. (This uses compactness

of $Y(K)$; in the general case the answer is substantially more complicated.) In particular, the cohomology $H^*(Y(K), \mathbf{C})_{\text{triv}}$ is a free-exterior algebra, generated in degrees $3, 5, \dots, 2d - 1$.

Finally let us recall (Borel) that the K -theory of \mathcal{O}_F is, modulo torsion, one-dimensional in each odd degree, and that (Quillen) for any finite field \mathbf{F} of size q , the even K -groups vanish and the odd $K_{2s+1}(\mathbf{F}) \simeq \mathbf{Z}/(q^s - 1)$.

5.6. The constraints on ℓ . We impose the following constraints on ℓ :

- (i) The cohomology of $Y(K)$ with coefficients in ℓ is torsion-free.
- (ii) ℓ does not divide $\text{gcd}_v((q_v - 1) \cdots (q_v^{2d-1} - 1))$, where the gcd is taken over all $q_v \geq q_0$ for large enough q_0 . (This gcd stabilizes for q_0 large enough.)
- (iii) ℓ is relatively prime to the numerator and denominator of the rational number $M \in \mathbf{Q}^*$ defined in (62).
- (iv) the cohomology $H^*(Y(K), \mathbf{Z}/\ell)_{\text{triv}}$ is a free-exterior algebra on generators in degree $3, \dots, 2d - 1$.
- (v) $\ell > d^2$
- (vi) The cohomology of $\text{GL}_\infty(\mathcal{O}_F)$ is free of ℓ -torsion in degrees less than d^2 .

All these assertions are automatically true for ℓ big enough. This is obvious for (i), (ii), (iii), (v) and follows from the standard stability results for (vi). We examine (iv): We saw after (59) that $H^*(Y(K), \mathbf{Q})_{\text{triv}}$ is a free-exterior algebra; fix generators e_3, e_5, \dots that belong to $H^*(Y(K), \mathbf{Z})$. The products of the e_i are linearly independent over \mathbf{Q} , so their reductions are also linearly independent over \mathbf{Z}/ℓ for large enough ℓ . It remains to show that they span $H^*(Y(K), \mathbf{Z}/\ell)_{\text{triv}}$. But that is obvious by counting dimensions: if we fix a Hecke operator T , then for sufficiently large ℓ the generalized zero eigenspace of $T - \text{deg}(T)$ on $H^*(Y(K), \mathbf{Z}/\ell)$ has the same dimension as the generalized zero eigenspace of $T - \text{deg}(T)$ on $H^*(Y(K), \mathbf{C})$.

This concludes the proof that all of (i)–(vi) above are automatically valid for large enough ℓ . It would be interesting to see what happens for ‘bad’ ℓ .

5.7. Let N be a large integer, as chosen in Section 5.5. Fix an embedding

$$\iota : \mathbf{G} \hookrightarrow \text{SL}_N \tag{60}$$

for some large N , by taking a sum of many copies of the representation that arises from the division algebra acting on itself. Then (in suitable coordinates)

we may suppose that the open compact subgroup K is carried into the standard maximal compact $\prod_v \mathrm{SL}_N(\mathcal{O}_v)$ of $\mathrm{SL}_N(\mathbf{A}_f)$, and the arithmetic group $\mathbf{G}(F) \cap K$ is consequently carried into $\mathrm{SL}_N(\mathcal{O}_F)$.

The map ι gives rise to a map of symmetric spaces, that is, a map

$$\mathbf{G}(\mathbf{C})/\mathrm{SU}_d \rightarrow \mathrm{SL}_N(\mathbf{C})/\mathrm{SU}_N,$$

where we have chosen a maximal compact for $\mathbf{G}(\mathbf{C})$, which is isomorphic to SU_d , and then we have chosen a hermitian form on \mathbf{C}^N whose stabilizer SU_N contains $\iota(\mathrm{SU}_d)$. Thus we get an embedding of locally symmetric spaces, also denoted by ι :

$$\iota : Y(K) \rightarrow \underbrace{\mathrm{SL}_N(\mathcal{O}_F) \backslash \mathrm{SL}_N(\mathbf{C})/\mathrm{SU}_N}_{\simeq B(\mathrm{SL}_N(\mathcal{O}_F))}. \tag{61}$$

We can further compose ι with the inclusion of $\mathrm{SL}_N(\mathcal{O}_F)$ to $\mathrm{GL}_N(\mathcal{O}_F)$ to give a map

$$Y(K) \rightarrow B(\mathrm{GL}_N(\mathcal{O}_F)).$$

5.8. In this section, we formulate four claims (a)–(d) that will imply Lemma 5.3. We verify the claims in the remainder of the section.

Let $3 \leq i \leq 2d - 1$ be odd and let $a_i \in K_i(\mathcal{O}_F)$ be chosen so that it generates $K_i(\mathcal{O}_F)$ modulo torsion. Let $[a_i]$ be the image of a_i inside $H_i(\mathrm{GL}_\infty(\mathcal{O}_F), \mathbf{Z}_\ell)$; as per our notation above, $[a_i]^{(N)}$ is its preimage under the isomorphism $H_i(\mathrm{GL}_N(\mathcal{O}_F), \mathbf{Z}_\ell) \rightarrow H_i(\mathrm{GL}_\infty(\mathcal{O}_F), \mathbf{Z}_\ell)$.

We show that (for any n) there are infinitely many places v , splitting the division algebra and with $q_v \equiv 1$ modulo ℓ^n , and classes $\xi_i \in H^i(\mathrm{GL}_N(\mathbf{F}_v), \mathbf{Z}/\ell)$ with the property that:

- (a) The image of a_i in $K_i(\mathbf{F}_v)/\ell$ is nonzero. Call its image $b_i \in K_i(\mathbf{F}_v)/\ell$; therefore, b_i is a generator of $K_i(\mathbf{F}_v)/\ell$.
- (b) The pairing $\langle \xi_i, [b_i]^{(N)} \rangle \neq 0$ where $[b_i]$ is defined similarly to $[a_i]$: it is the associated homology class under $K_i(\mathbf{F}_v) \rightarrow H_i(\mathrm{GL}_\infty(\mathbf{F}_v), \mathbf{Z}/\ell)$, and correspondingly we have $[b_i]^{(N)} \in H_i(\mathrm{GL}_N(\mathbf{F}_v), \mathbf{Z}/\ell)$.
- (c) $\langle \xi_3 \cup \dots \cup \xi_{2d-1}, ([b_3] * \dots * [b_{2d-1}])^{(N)} \rangle \neq 0$, where $*$ is the Pontryagin product on the homology of $\mathrm{GL}_\infty(\mathbf{F}_v)$.
- (d) Let $[Y(K)] \in H_{d^2-1}(Y(K), \mathbf{Q})$ be the fundamental class of $Y(K)$. Then there exists $M \in \mathbf{Q}^*$ such that the image $\iota_*[Y(K)] \in H_*(\mathrm{GL}_N(\mathcal{O}_F), \mathbf{Q})$ satisfies:

$$\iota_*[Y(K)] = M \cdot ([a_3] * [a_5] * \dots * [a_{2d-1}])^{(N)}, \tag{62}$$

where on the right we have the Pontryagin product for $GL_\infty(\mathcal{O}_F)$, and ι is as in (61).

Let us first see why (a)–(d) implies Lemma 5.3. Recall that we chose ℓ to not divide the numerator or denominator of M , and also that the cohomology of $GL_\infty(\mathcal{O}_F)$ and so also $GL_N(\mathcal{O}_F)$ is ℓ -torsion free in degrees $< d^2$; therefore, condition (d) implies an equality in \mathbf{Z}_ℓ -homology:

$$\iota_*[Y(K)] = (\text{unit}) \cdot ([a_3] \cdots * [a_{2d-1}])^{(N)}. \tag{63}$$

Let π be the projection from $GL_N(\mathcal{O}_F)$ to $GL_N(\mathbf{F}_v)$. Write $\mathcal{E}_i = \pi^* \xi_i \in H^*(GL_N(\mathcal{O}_F), \mathbf{Z}/\ell)$ and $\eta_i = \iota^* \mathcal{E}_i \in H^*(Y(K), \mathbf{Z}/\ell)$. We have then

$$\langle \mathcal{E}_i, [a_i]^{(N)} \rangle = \langle \pi^* \xi_i, [a_i]^{(N)} \rangle = \langle \xi_i, \pi_* [a_i]^{(N)} \rangle = \langle \xi_i, [b_i]^{(N)} \rangle \neq 0$$

since $\pi_* [a_i] = [b_i]$. Also, in a similar way,

$$\begin{aligned} &\langle \mathcal{E}_3 \cup \cdots \cup \mathcal{E}_{2d-1}, ([a_3] * \cdots * [a_{2d-1}])^{(N)} \rangle \\ &= \text{unit multiple of } \langle \xi_3 \cup \cdots \cup \xi_{2d-1}, ([b_3] * \cdots * [b_{2d-1}])^{(N)} \rangle \neq 0 \end{aligned} \tag{64}$$

because the Pontryagin products and the stabilization maps are compatible with π_* . From this and (63) we get

$$\langle \eta_3 \cup \cdots \cup \eta_{2d-1}, [Y(K)] \rangle = \langle \mathcal{E}_3 \cup \cdots \cup \mathcal{E}_{2d-1}, \iota_* [Y(K)] \rangle \neq 0.$$

But the Hecke-trivial cohomology of $Y(K)$ modulo ℓ is a free-exterior algebra on generators in degrees $3, \dots, (2d - 1)$. Fix such generators—call them v_3, v_5, \dots . Each η_i is also an element of this Hecke-trivial cohomology by Lemma 2.8. It follows that

$$\eta_i = \text{unit} \cdot v_i + (\text{product of } v_j\text{s with } j < i)$$

because otherwise the cup product $\eta_3 \cup \cdots \cup \eta_{2d-1}$ would be trivial. We conclude that, in fact, the map

$$H^*(\mathbf{G}(\mathbf{F}_v), \mathbf{Z}/\ell) \rightarrow H^*(Y(K), \mathbf{Z}/\ell)_{\text{triv}} \tag{65}$$

is onto as required. Therefore, to prove Lemma 5.3 it is sufficient to prove (a)–(d) above.

5.9. Verification of (d) from Section 5.8. Since we are supposing (57) to be a surjection in degrees up to d^2 , it is enough to verify that there is a nonzero $M \in \mathbf{C}$ such that

$$\langle \iota_*[Y(K)], \omega \rangle = M \langle ([a_3] * \cdots * [a_{2d-1}])^{(N)}, \omega \rangle \quad (66)$$

whenever ω is an invariant differential form on $\mathrm{GL}_N(\mathbf{C})$. We compute in the case (see (58) for the definition):

$$\omega = \Omega_J = \Omega_{j_1} \wedge \cdots \wedge \Omega_{j_l}$$

where $J = \{j_1, \dots, j_l\}$, and show that both sides are nonzero if and only if $J = \{3, 5, \dots, 2d-1\}$. That is enough to prove (66).

The right-hand side of (66) equals

$$\langle [a_3] * \cdots * [a_{2d-1}], \omega^{(\infty)} \rangle = \langle [a_3] \otimes \cdots \otimes [a_{2d-1}], \mathrm{coproduct}(\omega^{(\infty)}) \rangle, \quad (67)$$

where we allow ourselves to write $\omega^{(\infty)} \in H^*(\mathrm{GL}_{\infty}(\mathcal{O}_F), \mathbf{C})$ for the stabilization of the cohomology class corresponding to ω . Now $\mathrm{coprod}(\omega^{(\infty)})$ is the product of various terms of the shape

$$(\Omega_{j_1} \otimes 1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \Omega_{j_1} \otimes 1 \otimes \cdots \otimes 1 + \cdots)^{(\infty)}$$

and from this we see (just for degree reasons) that the term on the right of (67) vanishes if there is even one j_i larger than $2d-1$ or one j_i equal to 1. So $J \subset \{3, \dots, 2d-1\}$. Again, for degree reasons, equality must hold. That shows the right-hand side is zero unless $J = \{3, 5, \dots, 2d-1\}$. When $J = \{3, 5, \dots, 2d-1\}$, the right-hand side becomes

$$\langle a_3, \Omega_3 \rangle \cdot \langle a_5, \Omega_5 \rangle \cdots \langle a_{2d-1}, \Omega_{2d-1} \rangle$$

and each factor $\langle a_j, \Omega_j \rangle$ is nonzero: this is the nontriviality of the Borel regulator.

Now let us examine the left-hand side of (66), which equals $\langle [Y(K)], \iota^* \Omega_{j_1} \wedge \cdots \wedge \iota^* \Omega_{j_d} \rangle$. It is easy to see that $\iota^* \Omega_1$ vanishes. We also claim that $\iota^* \Omega_j$ must vanish for $j > 2d-1$. Indeed we claim that $\iota^* \Omega_j$, which defines an invariant form on $Y(K)$ and thus corresponds by (59) to an invariant differential form in the cohomology of SU_d , is *primitive* as such. For that consider this

diagram:

$$\begin{array}{ccc}
 \text{invariant forms on } \mathrm{SL}_N(\mathbf{C})/\mathrm{SU}_N & \longrightarrow & \text{invariant forms on } \mathbf{G}(\mathbf{C})/\mathrm{SU}_d & (68) \\
 \downarrow & & \downarrow & \\
 \mathrm{SU}_N\text{-invariant forms on } i\mathfrak{su}_N & \longrightarrow & \mathrm{SU}_d\text{-invariant forms on } i\mathfrak{su}_d & \\
 \downarrow & & \downarrow & \\
 H^*(\mathrm{SU}_N) & \longrightarrow & H^*(\mathrm{SU}_d) &
 \end{array}$$

where the various maps of groups arise from the map ι of (60), and the top vertical maps arise by restriction to the tangent space of the identity coset. In other words, the element of $H^*(\mathrm{SU}_d)$ corresponding to $\iota^*\Omega_j$ is just the pullback of the element of $H^*(U_N)$ corresponding to Ω_j under the group homomorphism

$$\varphi : \mathrm{SU}_d \rightarrow \mathrm{U}_N$$

induced by ι . In particular, $\varphi^*\Omega_j$ is also primitive.

This shows that the left-hand side of (66) vanishes unless $J = \{3, 5, \dots, 2d - 1\}$. We must still check that it is actually nonvanishing in this case. For this, we must show that $\iota^*(\Omega_3 \wedge \dots \wedge \Omega_{2d-1})$ is nonvanishing, equivalently that

$$\varphi^*\Omega_3 \wedge \dots \wedge \varphi^*\Omega_{2d-1}$$

is a nonvanishing element of the cohomology of SU_d . Since each $\varphi^*\Omega_j$ is primitive it is enough to see that they are all nonzero. The N -dimensional representation of SU_d defined by φ is isomorphic to the sum of many copies of the standard representation of SU_d . Now one can just compute explicitly with (58).

5.10. Verification of (b) and (c) from Section 5.8. In words, what we have to do is produce elements $\theta_i \in H^i(\mathrm{GL}_\infty(\mathbf{F}_v), \mathbf{Z}/\ell)$ for each odd $3 \leq i \leq 2d - 1$; these θ_i should detect (pair nontrivially with) a generator of $K_i(\mathbf{F}_v)$, and the cup product $\theta_3 \cup \dots \cup \theta_{2d-1}$ should detect the Pontryagin product of the homology classes associated to those generators. Then we may take $\xi_i = \theta_i^{(N)} \in H^i(\mathrm{GL}_N(\mathbf{F}_v), \mathbf{Z}/\ell)$.

Quillen shows a natural choice for θ_i : an equivariant Chern class derived from the standard representation of GL_N . In other words, write $G = \mathrm{GL}_N(\mathbf{F}_v)$ and write Γ for the Galois group of \mathbf{F}_v . The standard representation of GL_N can be considered a G -equivariant vector bundle on $\mathrm{Spec}(\mathbf{F}_v)$, and thus we get a Chern class

$$c_{2i} \in H_{G,\mathrm{et}}^{2i}(\mathrm{Spec}\mathbf{F}_v, \mathbf{Z}/\ell(i)) \rightarrow H_{\mathrm{et}}^1(\mathrm{Spec}\mathbf{F}_v, \mathbf{Z}/\ell(i)) \otimes H^{2i-1}(G, \mathbf{Z}/\ell)$$

where on the left we have equivariant étale cohomology, and on the right we have usual étale cohomology; the arrow is explicated in [32, II.1.2, Lemma 1]. We may identify the étale cohomology of \mathbf{F}_v with the (continuous) group cohomology of Γ , and here Γ is acting trivially on μ_ℓ , and so $H^1(\text{Spec } \mathbf{F}_v, \mathbf{Z}/\ell(i)) = H^1(\Gamma, \mathbf{Z}/\ell(i))$ is identified with simply $\mu_\ell^{\otimes i}$. Fix a generator α for μ_ℓ . Thus the image of c_{2i} is of the form $\alpha^i \otimes \theta_{2i-1}$, for some $\theta_{2i-1} \in H^{2i-1}(G, \mathbf{Z}/\ell)$. Similarly we can map c_{2i} into $H^0(\Gamma, \mathbf{Z}/\ell(i)) \otimes H^{2i}(G, \mathbf{Z}/\ell)$; in that way we get a class $\theta'_{2i} \in H^{2i}(G, \mathbf{Z}/\ell)$ so that the image of c_{2i} is $\alpha^i \otimes \theta'_{2i}$. The class c_{2i} gives a morphism (the ‘Soulé Chern class’)

$$s : H_{2i-1}(G, \mathbf{Z}/\ell) \rightarrow H^1(\Gamma, \mathbf{Z}/\ell(i)) \tag{69}$$

which sends $\lambda \in H_{2i-1}$ to $\langle \theta_{2i-1}, \lambda \rangle \alpha^{\otimes i}$.

These constructions are ‘stable’ under increasing N —see [32, page 257]—and so we can consider $\theta_{2i-1}, \theta'_{2i}$ as classes in $H^*(\text{GL}_\infty(\mathbf{F}_v), \mathbf{Z}/\ell)$, and the Soulé map as a map $H_{2i-1}(\text{GL}_\infty(\mathbf{F}_v), \mathbf{Z}/\ell) \rightarrow H^1(\Gamma, \mathbf{Z}/\ell(i))$.

Now Soulé Chern class is known to be surjective when precomposed with $K_{2i-1} \rightarrow H_{2i-1}$ ([32, Proposition 5, page 284]) (so long as $i < \ell$ —true by assumption on ℓ). This immediately verifies property (b), that is to say if we fix a generator b_{2i-1} for $K_{2i-1}(\mathbf{F}_v)$ with associated homology class $[b_{2i-1}]$ we have

$$0 \neq s([b_{2i-1}]) = \langle \theta_{2i-1}, [b_{2i-1}] \rangle \alpha^{\otimes i}$$

so $\langle \theta_{2i-1}, [b_{2i-1}] \rangle \neq 0$ as desired.

To compute, for example, $\langle \theta_3 \cup \theta_5 \cup \theta_7, [b_3] * [b_5] * [b_7] \rangle$ we rewrite it as

$$= \langle \text{coprod}(\theta_3) \text{coprod}(\theta_5) \text{coprod}(\theta_7), b_3 \otimes b_5 \otimes b_7 \rangle. \tag{70}$$

Let us note that each $[b_i]$ is primitive in homology—that is, $\langle [b_i], \alpha \cup \beta \rangle = 0$ if α, β are cohomology classes both in positive degree. This is because $[b_i]$ comes from the image of the Hurewicz map $\pi_i \rightarrow H_i$, so we can just pull back to the sphere S^i and compute.

Quillen has shown [26, Proposition 2] (see also [26, Remark 2, p.569]) that the coproduct of (for example) θ_7 equals

$$\text{coprod}(\theta_7) = \underbrace{\theta'_0}_{1} \otimes \theta_7 + \theta'_2 \otimes \theta_5 + \theta'_4 \otimes \theta_3 + \theta'_6 \otimes \theta_1 + \text{symmetric terms.}$$

Thus, when we take the product $\text{coprod}(\theta_3) \text{coprod}(\theta_5) \text{coprod}(\theta_7)$, we get a sum of several terms; because of the *primitivity* of $[b_i]$ just noted, the only terms that contribute to (70) will be those coming from

$$(1 \otimes 1 \otimes \theta_3 + 1 \otimes \theta_3 \otimes 1 + \theta_3 \otimes 1 \otimes 1)$$

$$\begin{aligned} &(1 \otimes 1 \otimes \theta_5 + 1 \otimes \theta_5 \otimes 1 + \theta_5 \otimes 1 \otimes 1) \\ &(1 \otimes 1 \otimes \theta_7 + 1 \otimes \theta_7 \otimes 1 + \theta_7 \otimes 1 \otimes 1) \end{aligned}$$

and the only term of these with degree 3, 5, 7 in the first, second and third factors is $\theta_3 \otimes \theta_5 \otimes \theta_7$. Therefore,

$$\begin{aligned} \langle \theta_3 \cup \theta_5 \cup \theta_7, [b_3] * [b_5] * [b_7] \rangle &= \langle \theta_3 \otimes \theta_5 \otimes \theta_7, [b_3] \otimes [b_5] \otimes [b_7] \rangle \\ &= \prod \langle \theta_3, [b_3] \rangle \langle \theta_5, [b_5] \rangle \langle \theta_7, [b_7] \rangle \neq 0 \end{aligned}$$

which gives (c) in the case $d = 4$; the general case is the same.

5.11. Verification of (a) from Section 5.8. Write $\mathcal{O}' = \mathcal{O}_F[1/\ell]$. The Soulé maps from (69) fit in a commutative diagram

$$\begin{CD} K_{2i-1}(\mathcal{O}') \otimes \mathbf{Z}_\ell @>>> K_{2i-1}(\mathbf{F}_v) \otimes \mathbf{Z}_\ell \\ @VVV @VVV \\ H^1(\Gamma, \mathbf{Z}_\ell(i)) @>>> H^1(\text{Gal}(\overline{\mathbf{F}}_v/\mathbf{F}_v), \mathbf{Z}_\ell(i)) \end{CD} \tag{71}$$

where Γ is now the Galois group of the maximal unramified extension of \mathcal{O}' ; note that $\mathcal{O}_F \rightarrow \mathcal{O}'$ induces an isomorphism on K_{2i-1} for $i > 1$ (see [36, Theorem 4.6]). The right-hand vertical arrow is an isomorphism for $i < \ell$ (Soulé, *loc. cit.*), and the left-hand vertical arrow is a surjection (see [19]).

The map $H^1(\Gamma, \mathbf{Z}_\ell(i))/\ell \rightarrow H^1(\Gamma, \mathbf{F}_\ell(i))$ is an injection. Choose an element $a_{2i-1} \in K_{2i-1}(\mathcal{O}')$ whose image in $H^1(\Gamma, \mathbf{F}_\ell(i))$ is nonzero (note that $H^1(\Gamma, \mathbf{Z}_\ell(i))/\ell$ is nonzero, by computing Euler characteristic). A nontrivial class in $H^1(\Gamma, \mathbf{F}_\ell(i))$ is represented by a nontrivial extension

$$\mathbf{F}_\ell(i) \rightarrow M \rightarrow \mathbf{F}_\ell, \tag{72}$$

in other words, by a homomorphism $\Gamma \rightarrow \text{GL}_2(\mathbf{Z}/\ell)$ of the form

$$\begin{pmatrix} \omega^i & \rho \\ 0 & 1 \end{pmatrix}.$$

Note that the image of this homomorphism must have size divisible by ℓ ; for otherwise the extension (72) splits. By Chebotarev density, we may find infinitely many Frobenius elements Frob_v which map to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which implies that the restriction of the extension class to $H^1(\mathbf{F}_v, \mathbf{F}_\ell(i))$ is nontrivial.

In other words, the image of a_{2i-1} in $H^1(\text{Gal}(\overline{\mathbf{F}}_v/\mathbf{F}_v), \mathbf{F}_\ell(i))$, and so also in K_{2i-1}/ℓ , is nontrivial for finitely many v . This proves our assertion.

6. Setup for patching

We continue in a global setting, but now turn to the study of tempered cohomology. This study will occupy most of the remainder of the paper (Sections 6–8).

In the current section (Section 6), we set up the various assumptions needed, and in the next section (Section 7) we use the relationship between the Taylor–Wiles method and the derived Hecke algebra (outlined in Section 1.4) to prove our target theorem, Theorem 7.6: it says the global derived Hecke algebra actually is big enough to be able to account for the degree spread of cohomology.

In Section 8, we explain how to index elements of the global derived Hecke algebra by a Selmer group, and use this to formulate our main Conjecture 8.8.

A few apologies are in order:

- We switch notation slightly, working with mod p^n coefficients rather than mod ℓ^n , to better make contact with the standard presentations of Galois representations and the Taylor–Wiles method.
- We have made no attempt to optimize the method for small primes, And, in particular, make rather strong assumptions; in particular, we assume both that we are in the ‘minimal case’ of formally smooth local deformation rings, and that the Hecke ring at base level is isomorphic to \mathbf{Z}_p (no congruences).

6.1. Assumptions.

- (1) Our general notations are as in Section 1.7, but we now specialize to the case that the number field F is \mathbf{Q} , and that \mathbf{G} is a *simply connected*, semisimple \mathbf{Q} -group. We assume that \mathbf{G} is split, and we fix a Borel subgroup \mathbf{B} and a maximal torus \mathbf{A} contained inside \mathbf{B} .
- (2) We fix a level structure $K_0 \subset \mathbf{G}(\mathbf{A}_f)$, the ‘base level.’ We write

$$Y(1) = Y(K_0)$$

for the arithmetic manifold (17) of level K_0 . We sometimes refer to this as the ‘level 1’ arithmetic manifold even though it is not literally so.

- (3) Let Π be a tempered cohomological automorphic cuspidal representation for \mathbf{G} , factorizing as $\Pi = \Pi_\infty \otimes \Pi_f$ over archimedean and finite places. We suppose that $\Pi^{K_0} \neq 0$, so that Π actually contributes to the cohomology of $Y(1)$.
- (4) We write \mathbb{T}_{K_0} for the Hecke algebra at level K_0 . It will be convenient to follow the definition of [21] and define this in a derived sense: Consider the chain complex of $Y(K_0)$, with \mathbf{Z}_p coefficients, as an object in the derived category of \mathbf{Z}_p -modules; each (prime to the level, and to p) Hecke operator gives an endomorphism of this object. Define \mathbb{T}_{K_0} to be the \mathbf{Z}_p -algebra generated by such endomorphisms. This has the advantage that \mathbb{T}_{K_0} acts on cohomology with coefficients in any \mathbf{Z}_p -module.
- We may similarly form the Hecke algebra \mathbb{T}_K at a deeper level $K \subset K_0$; unless specified, it will be generated only by Hecke operators at good primes for K .
- (5) We shall suppose that the coefficient field of Π is \mathbf{Q} , for simplicity—by this, we mean that the eigenvalues of Hecke correspondences at all good places for K_0 lie in \mathbf{Q} , or equivalently the underlying representation Π_v has a \mathbf{Q} -rational structure for all such v . Under this assumption, Π gives rise to a ring homomorphism

$$\mathbb{T}_{K_0} \rightarrow \mathbf{Z}_p. \quad (73)$$

- (6) Let T be the set of ramified places for Π , together with any places at which K_0 is not hyperspecial.
- (7) Write

$$k = \mathbf{Z}/p\mathbf{Z}$$

and fix an algebraic closure \bar{k} for k where $p > 5$ is a prime such that:

- (a) $H^*(Y(K_0), \mathbf{Z})$ is p -torsion free.
- (b) p does not divide the order of the Weyl group of \mathbf{G} , and also $p \notin T$, that is, p is not a bad place for Π or K .
- (c) ‘No congruences between Π and other forms at level K_0 :’

Consider the composite homomorphism

$$\chi : \mathbb{T}_{K_0} \rightarrow \mathbf{Z}_p \rightarrow k, \quad (74)$$

where the first map is the action on Π , and the second map the obvious one. Let $\mathfrak{m} = \ker(\chi)$ be its kernel. We shall require that the induced map of completions is an *isomorphism*:

$$\mathbb{T}_{K_0, \mathfrak{m}} \xrightarrow{\sim} \mathbf{Z}_p \quad (75)$$

and moreover we shall assume the vanishing of homology after completion at the maximal ideal \mathfrak{m} :

$$H_j(Y(1), \mathbf{Z}_p)_{\mathfrak{m}} = 0, j \notin [\mathfrak{q}, \mathfrak{q} + \delta]. \quad (76)$$

Observe that in favorable situations (75) implies (76), and both should be true for all large enough p — see the Remark below for a further discussion. Informally, (75) and (76) enforce that there are no congruences, modulo p , between Π and other cohomological forms at level K_0 .

Note that the definitions of χ , \mathfrak{m} make sense at any level. Thus we use the notation χ , \mathfrak{m} sometimes for the corresponding notions for other level structures $Y(K)$, where $K \subset K_0$.

(8) We put

$$S = T \cup \{p\},$$

the collection of all primes that we have to worry about.

REMARK. We expect that 7(c) should be automatically valid for p sufficiently large; in practice, for the purposes of this paper, it is not an onerous assumption (the minimal level conditions, enforced in (e) of Section 6.2, is more restrictive).

We give a proof that 7(c) is valid for all large enough p , for \mathbf{G} an inner form of SL_n such that $Y(K)$ is compact. It is likely this can be generalized to other settings, with more work. In what follows, denote by \mathbb{T}_{K_0} the Hecke algebra defined as above, but with \mathbf{Z} coefficients; this is easily seen to be finitely generated over \mathbf{Z} . Firstly, the algebra $\mathbb{T}_{K_0} \otimes \mathbf{C}$ is semisimple, because it acts faithfully on $H^*(Y(K_0), \mathbf{C})$ and this action is semisimple (there is an invariant metric on harmonic forms). Thus, for all large enough primes p , \mathbb{T}_{K_0} is étale over \mathbf{Z} and thus (75) must be valid. If the homology in (76) is nonvanishing, then there exists an eigenclass for \mathbb{T}_{K_0} on H_j whose associated character factors through $\mathbb{T}_{K_0, \mathfrak{m}}$; by (75), this character must coincide with the action of \mathbb{T}_{K_0} on Π . In other words, the Hecke eigensystem associated to Π occurs in degree j . This eigensystem corresponds to an automorphic representation Π' such that $\Pi'_v = \Pi_v$ for almost all v . By the strong multiplicity one theorem for \mathbf{GL}_n , this implies that Π'_∞ is tempered cuspidal, and then it has nonvanishing (\mathfrak{g}, K) cohomology only in degrees $[\mathfrak{q}, \mathfrak{q} + \delta]$.

That is the basic setup; now for Galois representations.

6.2. Assumptions about Galois representations; the deformation ring $R_{\bar{\rho}}$.

We make assumptions very close to [14, Conjecture 6.1]. We briefly summarize them and refer the reader to [14] for full details:

Let $K \subset K_0$ be a deeper level structure, \mathbb{T}_K be as above, and let $\mathbb{T}_K \rightarrow k$ be a character. We require that there exist a Galois representation

$$\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow G^\vee(\bar{k})$$

satisfying the usual unramified compatibility, see (a) below.

Moreover, for the specific character $\mathbb{T}_K \rightarrow k$ as in (74), that is, the map associated to the fixed automorphic representation Π , reduced modulo p , we require more precise statements: Let \mathfrak{m} be the kernel of $\mathbb{T}_K \rightarrow k$, and $\mathbb{T}_{K,\mathfrak{m}}$ the completion of \mathbb{T}_K at \mathfrak{m} . We require there to exist a Galois representation

$$\tilde{\rho} : \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \longrightarrow G^\vee(\mathbb{T}_{K,\mathfrak{m}})$$

with the following properties:

- (a) (Unramified compatibility): Fix a representation τ of G^\vee . For all primes q not dividing the level of K , the representation ρ is unramified at q , and the action of $\text{trace}(\tau \circ \tilde{\rho})(\text{Frob}_q) \in \mathbb{T}_{K,\mathfrak{m}}$ coincides with the image of the (Satake)-associated Hecke operator $T_{q,\tau}$.
- (b) Let $\bar{\rho}$ be the reduction modulo p of $\tilde{\rho}$, so that

$$\bar{\rho} : \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow G^\vee(k).$$

Then $\bar{\rho}$ has big image: when restricted to the Galois group of $\mathbf{Q}(\zeta_{p^\infty})$, the image of $\bar{\rho}$ contains the image of the k -points of the simply connected cover of G^\vee .

- (c) (Vague version: see [14, Conjecture 6.1] for precise formulation): There is a reasonable notion of ‘crystalline at p ’ representation into G^\vee , and the representation $\tilde{\rho}$ is ‘crystalline at p .’
- (d) (Vague version: see [14, Conjecture 6.1] and references therein for precise version): The representation $\tilde{\rho}$ satisfies the expected local constraints (‘local–global compatibility’) when restricted to \mathbf{Q}_q ; here q is a Taylor–Wiles prime (Section 6.3) that divides the level of K . We *also* assume a natural version of local–global compatibility at Iwahori level $Y_0(q)$, formulated before Lemma 6.6.
- (e) ‘All local deformation rings are all formally smooth.’ we suppose that

$$H^0(\mathbf{Q}_q, \text{Ad}\bar{\rho}) = H^2(\mathbf{Q}_q, \text{Ad}\bar{\rho}) = 0 \quad \text{for all } q \in S = T \cup \{p\}.$$

This means that the local deformation ring of $\bar{\rho}$ at primes $q \in T$ is isomorphic to \mathbf{Z}_p , and the local deformation ring of $\bar{\rho}$ at $q = p$ is formally smooth. This keeps our notation as light as possible. (We also use the formal smoothness to squeeze the most out of the Taylor–Wiles method, but probably one can get something without it.)

(Note that the assumption that there is a \mathbb{T} -valued Galois representation, rather than a weaker notion such as a determinant, is not reasonable unless one has a condition like ‘residual irreducibility.’ In our case, however, we are assuming that the residual representation $\bar{\rho}$ has very large image anyway—see (b) above.)

In particular, the natural map $\mathbb{T}_{\mathcal{K}_0, \mathfrak{m}} \rightarrow \mathbf{Z}_p$ of (75) gives rise to a Galois representation valued in $G^\vee(\mathbf{Z}_p)$, which we shall just call ρ :

$$\rho : \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \longrightarrow G^\vee(\mathbf{Z}_p), \tag{77}$$

which of course lifts the residual representation:

$$\bar{\rho} : \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \longrightarrow G^\vee(k).$$

Let $R_{\bar{\rho}}$ be the universal crystalline deformation ring of $\bar{\rho}$, allowing ramification only at the set S . Good references for deformation rings are [13] or [15].

Let us now set up notations for Taylor–Wiles primes.

6.3. Taylor–Wiles primes and auxiliary level structures. A *Taylor–Wiles prime* of level n is a prime q (we also occasionally use the letter ℓ), not dividing the level of K_0 , such that:

- p^n divides $q - 1$, and
- $\bar{\rho}(\text{Frob}_q)$ is conjugate to a strongly regular element of $T^\vee(k)$.

Here an element $t \in T^\vee(k)$ is *strongly regular* if its centralizer inside G^\vee is equal to T^\vee .

We are really interested in systems of such primes, and it is useful to keep track of the strongly regular element as part of the data. Fix once and for all a sufficiently large integer s . We work with collections of such primes of cardinality s , which we call Taylor–Wiles data:

- A *Taylor–Wiles datum* of level n is a set of primes $Q_n = (q_1, \dots, q_s)$ together with strongly regular elements $(\text{Frob}_{q_1}^T, \dots, \text{Frob}_{q_s}^T) \in T^\vee(k)$ such that
 - p^n divides $q_i - 1$, and
 - $\bar{\rho}(\text{Frob}_{q_i})$ is conjugate to $\text{Frob}_{q_i}^T$.

We usually use q to denote a typical element of a Taylor–Wiles set of primes, but occasionally we also use the letter ℓ . Note that the set of possible choices for each $\text{Frob}_{q_i}^T$ has size $|W|$, the order of the Weyl group.

- Let r be the rank of the maximal torus \mathbf{A} . Set

$$R = rs, \quad T_s^\vee = (T^\vee)^s, \quad W_s = W^s \tag{78}$$

thus T_s^\vee is a torus and R is the rank of T_s^\vee , and W_s acts on T_s^\vee .

- Level structures: If $q \notin S$ is prime, we denote by

$$Y_0(q) \rightarrow Y(1) \tag{79}$$

the covering obtained by adding Iwahori level structure at q , that is, we replace K_0 by the preimage of a Borel subgroup under $K_0 \rightarrow \mathbf{G}(\mathbf{F}_q)$. Similarly, we get $Y_1(q)$ by taking the preimage of a unipotent radical of a Borel subgroup under the same mapping. The covering $Y_1(q) \rightarrow Y_0(q)$ is ‘Galois,’ with Galois group $\mathbf{A}(\mathbf{F}_q) \simeq (\mathbf{F}_q^\times)^r$.

Suppose that p^n divides $q - 1$. In that case, define $Y_1(q, n)$ to be the unique subcovering of $Y_1(q) \rightarrow Y_0(q)$ such that the covering $Y_1(q, n) \rightarrow Y_0(q)$ has covering group $(\mathbf{Z}/p^n)^r$. In summary:

$$Y_1(q) \rightarrow \underbrace{Y_1(q, n) \rightarrow Y_0(q)}_{\mathbf{A}(\mathbf{F}_q)/p^n \simeq (\mathbf{Z}/p^n)^r} \tag{80}$$

$\mathbf{A}(\mathbf{F}_q) \simeq (\mathbf{F}_q^\times)^r$

- For a Taylor–Wiles datum Q_n of level n , we let $Y_1^*(Q_n)$ be the fiber product, over $Y(1)$, of all the coverings $Y_1(q_i, n) \rightarrow Y(1)$. Similarly we define $Y_0(Q_n)$. Therefore, $Y_1^*(Q_n) \rightarrow Y_0(Q_n)$ is Galois; we write

$$T_n = (\text{Galois group of } Y_1^*(Q_n) \rightarrow Y(1)) = \prod_{q \in Q_n} \mathbf{A}(\mathbf{F}_q)/p^n, \tag{81}$$

thus we have (noncanonically)

$$T_n \simeq (\mathbf{Z}/p^n)^R. \tag{82}$$

As a shorthand we write

$$H_*(Q_n, S) := H_*(Y_1^*(Q_n), S)$$

where S is a ring of coefficients.

6.4. Rings of diamond operators. We now set up the rings that are generated by ‘diamond operators,’ that is, the deck transformation groups of our various coverings $Y_1^*(Q_n) \rightarrow Y_0(Q_n)$. Continuing with the notation of the prior section, put

$$S_n = \mathbf{Z}/p^n[T_n], \quad S'_n = \mathbf{Z}_p[T_n], \quad (83)$$

so these act on the \mathbf{Z}/p^n and \mathbf{Z}_p -valued chain complex of $Y_1^*(Q_n)$. Fixing, as in (82), an isomorphism of T_n with $(\mathbf{Z}/p^n)^R$, we can identify this ring as follows.

$$S_n \simeq \mathbf{Z}/p^n[x_1, \dots, x_R]/((1+x_i)^{p^n} - 1), \quad (84)$$

where $x_i = [e_i] - 1$, e_i being a generator of the i th factor \mathbf{Z}/p^n under the isomorphism (82). Recall $R = rs$ as in (78).

Finally we form a ‘limit ring’

$$\mathbf{S} := \mathbf{Z}_p[[x_1, \dots, x_R]]. \quad (85)$$

The presentation (84) gives rise to obvious maps $\mathbf{S} \rightarrow S_n$ and (compatible) augmentations of \mathbf{S} and S_n to \mathbf{Z}_p and \mathbf{Z}/p^n respectively, carrying all the x_i variables to zero; we denote by \mathfrak{l} and I_n the corresponding kernels, so that $\mathbf{S}/\mathfrak{l} \simeq \mathbf{Z}_p$ and $S_n/I_n \simeq \mathbf{Z}/p^n$. We need the following easy Lemma:

LEMMA. *Let notations be as above. The natural map*

$$\mathrm{Ext}_{S_n}^*(\mathbf{Z}/p^n, \mathbf{Z}/p^n) \rightarrow \mathrm{Ext}_{\mathbf{S}/p^n}^*(\mathbf{Z}/p^n, \mathbf{Z}/p^n) \quad (86)$$

(change of rings) is surjective. Also, the natural map

$$\mathrm{Ext}_{\mathbf{S}/p^n}^*(\mathbf{Z}/p^n, \mathbf{Z}/p^n) \rightarrow \mathrm{Ext}_{\mathbf{S}}^*(\mathbf{Z}_p, \mathbf{Z}/p^n) \quad (87)$$

(change of ring, and functoriality of Ext in the first argument) is an isomorphism.

Recall that the change of ring map $\mathrm{Ext}_B \rightarrow \mathrm{Ext}_A$ induced by a ring map $A \rightarrow B$ can be realized by thinking of Ext_B in terms of extensions of B -modules, and then just regarding it as an extension of A -modules.

Proof. For (86), it is sufficient to check surjectivity on Ext^1 because the right-hand side is generated by $\mathrm{Ext}^0, \mathrm{Ext}^1$: it can be computed to be an exterior algebra using a Koszul resolution, see Lemma B.1. Now ‘morally speaking’ this is because it is the pullback on group $H^1(-, \mathbf{Z}/p^n)$ induced by $\mathbf{Z}_p^R \rightarrow (\mathbf{Z}/p^n)^R$, but this is not a real proof (at least without discussing the relation between cohomology of profinite groups, and Exts over the corresponding ‘completed’ group algebra).

So we just compute both sides by using the compatible (under $\mathbf{S}/p^n \rightarrow S_n$) sequences $I_n \rightarrow S_n \rightarrow \mathbf{Z}/p^n$ and $l/p^n \rightarrow \mathbf{S}/p^n \rightarrow \mathbf{Z}/p^n$; taking homomorphisms into \mathbf{Z}/p^n , we get

$$\text{Ext}_{S_n}^1(\mathbf{Z}/p^n, \mathbf{Z}/p^n) \simeq \text{Hom}_{S_n}(I_n, \mathbf{Z}/p^n) (\simeq \text{Hom}(I_n/I_n^2, \mathbf{Z}/p^n)) \tag{88}$$

and similar. (Note that the image of $\text{Hom}_{S_n}(S_n, \mathbf{Z}/p^n)$ in $\text{Hom}_{S_n}(I_n, \mathbf{Z}/p^n)$ is zero.)

So we need to check that the map

$$l/p^n \rightarrow I_n \tag{89}$$

induces an isomorphism when we take $\text{Hom}_{\mathbf{S}}(-, \mathbf{Z}/p^n)$.

The homomorphisms $\text{Hom}_{\mathbf{S}}(I_n, \mathbf{Z}/p^n)$ are precisely given by homomorphisms $\varphi : T_n \rightarrow \mathbf{Z}/p^n$, namely, we send $x_i = [e_i] - 1 \in I_n$ to $\varphi(e_i)$, using the notation after (84). Since $l/(p^n, l^2)$ is a free \mathbf{Z}/p^n module on x_1, \dots, x_R , it follows at once that (89) induces an isomorphism as desired.

Now we discuss (87), which is not hard but we spell it out, mainly to be clear because I find change of rings confusing. Let $\pi : P \rightarrow \mathbf{Z}_p$ be the Koszul resolution of \mathbf{Z}_p as an \mathbf{S} -module, and $\overline{P} = P/p^n$ its reduction mod p^n , so that $\overline{\pi} : \overline{P} \rightarrow \mathbf{Z}/p^n$ is the Koszul resolution of \mathbf{Z}/p^n as an \mathbf{S}/p^n -module. There is an identification $\text{Ext}_{\mathbf{S}/p^n}^*(\mathbf{Z}/p^n, \mathbf{Z}/p^n)$ with the cohomology of $\text{Hom}(\overline{P}, \mathbf{Z}/p^n)$; it sends a closed element $C \in \text{Hom}(\overline{P}, \mathbf{Z}/p^n[m])$ to the class $\alpha \in \text{Ext}_{\mathbf{S}/p^n}^m(\mathbf{Z}/p^n, \mathbf{Z}/p^n)$ represented by the diagram $\mathbf{Z}/p^n \xleftarrow{\overline{\pi}} \overline{P} \xrightarrow{C} \mathbf{Z}/p^n[m]$ in the derived category of \mathbf{S}/p^n -modules (one can invert quasi-isomorphisms in the derived category).

Consider now the diagram

$$\begin{array}{ccc} \mathbf{Z}/p^n & \xleftarrow{\overline{\pi}} & \overline{P} & \xrightarrow{C} & \mathbf{Z}/p^n[m] \\ & \uparrow A & & \uparrow B & \\ \mathbf{Z}_p & \xleftarrow{\pi} & P & & \end{array} \tag{90}$$

where \sim means quasi-isomorphism; A, B are the natural projections.

Now, the image $\alpha' \in \text{Ext}_{\mathbf{S}}(\mathbf{Z}_p, \mathbf{Z}/p^n)$ of α , under the map (87), is represented by the map $C \circ \overline{\pi}^{-1} \circ A$ inside the derived category of \mathbf{S} -modules, equivalently, by the composition $C \circ B \circ \pi^{-1}$. In other words, the image α' of α is represented by the class $C' \in \text{Hom}(P, \mathbf{Z}/p^n[m])$ obtained by pulling back $C \in \text{Hom}(\overline{P}, \mathbf{Z}/p^n[m])$ via $P \rightarrow \overline{P}$. That pullback induces an isomorphism of complexes

$$\text{Hom}_{\mathbf{S}/p^n}(\overline{P}, \mathbf{Z}/p^n) \simeq \text{Hom}_{\mathbf{S}}(P, \mathbf{Z}/p^n),$$

and thus, passing to cohomology, the desired isomorphism of Ext-groups. □

6.5. Adding level q structure: The relationship between homology at level $Y(1)$ and level $Y_0(q)$. Fix a prime q that does not divide the level of K_0 , satisfying $q \equiv 1$ modulo p^n . This section and the next two Sections 6.5, 6.8, 6.11 all gather some ‘standard’ properties of passing between level 1 and level q , working with \mathbf{Z}/p^n -cohomology. We remind the reader that ‘level 1’ does not literally mean level 1, but just the base level K_0 at which we work.

In the current section, we discuss the relationship between the homology of $Y(1)$ and $Y_0(q)$; recall that $Y_0(q)$ was obtained by adding Iwahori level at q (see (79)).

We can use the discussion of Section 4.4: For $S = \mathbf{Z}/p^n$, let H_I, H_K and so on be the (usual, that is, no ‘derived’!) Hecke algebras for (G_q, K_q) , with S -coefficients and let H_{IK}, H_{KI} be the bimodules previously defined in Section 4.2. There are natural maps

$$\begin{aligned}
 H^*(Y(1), S) \otimes_{H_K} H_{KI} &\longrightarrow H^*(Y_0(q), S), \\
 H^*(Y_0(q), S) \otimes_{H_I} H_{IK} &\longrightarrow H^*(Y(1), S).
 \end{aligned}
 \tag{91}$$

These maps are defined by the ‘usual double coset formulas.’ More formally, we may identify H_K with $\text{Hom}_{SG_q}(S[G_q/K_q], S[G_q/K_q])$, H_{KI} with $\text{Hom}_{SG_q}(S[G_q/K_q], S[G_q/I_q])$ and so forth; one may then proceed as in the discussion of Section 2.6 to define the maps of (91).

Before we go further, let us formulate a natural notion of ‘local–global compatibility’ at level $Y_0(q)$: the center of the Iwahori–Hecke algebra H_I at level q , which is identified (Section 4.3) with the Hecke algebra H_K . We shall suppose:

(Local–global compatibility at level $Y_0(q)$): H_K , identified with the center of H_I as just explained, acts on $H_*(Y_0(q), k)_{\mathfrak{m}}$ by means of the same (generalized) eigencharacter $H_K \rightarrow k$ by which H_K acts on Π .

It is feasible that this assumption could be avoided entirely but since it is very likely to be proven along with the other, more essential, local–global compatibility at Taylor–Wiles primes it seems harmless to assume it.

LEMMA 6.6. *Let notation be as above, and assume the local–global compatibility just mentioned. Suppose that the prime q is such that $\bar{\rho}(\text{Frob}_q)$ is conjugate to a strongly regular element of $T^\vee(k)$. Then the maps (91) are isomorphisms when we localize at a maximal ideal of H_K induced by \mathfrak{m} . In particular, if q is part of a Taylor–Wiles datum of level n , then (91) induces*

$$H^*(Y_0(q), \mathbf{Z}/p^n)_{\mathfrak{m}} \xleftarrow{\sim} H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}} \otimes_{H_K} H_{KI}.
 \tag{92}$$

Proof. The assertion that (92) is an isomorphism follows from the assertion that (91) is an isomorphism by ‘localization at \mathfrak{m} ’ (taking a little care because the Hecke algebra for $Y_0(q)$ and $Y(1)$ are not quite identically defined, the former omitting the prime above q . For a more detailed treatment of this point, please see [21, Lemma 6.20]).

So it is enough to prove the first assertion. Everything will be with \mathbf{Z}/p^n coefficients. First note that the two natural ways of making a map

$$H^*(Y(1)) \otimes_{H_K} H_{KI} \otimes_{H_I} H_{IK} \longrightarrow H^*(Y(1)).$$

(that is, first contracting the first two coordinates, or first contracting the second two coordinates) both coincide; similarly the other way around. We make the rest of the argument in a more abstract setting.

Suppose R_1, R_2 are two rings, and we are given an (R_1, R_2) -bimodule M_{12} and an (R_2, R_1) -bimodule M_{21} , giving associated functors

$$F(-) = - \otimes_{R_1} M_{12}, \quad G(-) = - \otimes_{R_2} M_{21}$$

from (right) R_1 -modules to (right) R_2 -modules and vice versa, respectively.

We assume that F, G define an equivalence of categories, that is, there are natural equivalences from FG to the identity functor and from GF to the identity functor. (In our setting above $R_1 = H'_K, R_2 = H'_I$ —the primes denote localization at the ideal of H_K induced by \mathfrak{m} —and the bimodules are H'_{KI}, H'_{IK} ; the assumptions are satisfied by Lemma 4.5.)

Next let X be an R_1 -module, let Y be an R_2 -module. (In our setting these are given by the localized homology of $Y(1)$ and $Y_0(q)$, respectively.) Suppose also we are given maps $\alpha : F(X) \rightarrow Y, \beta : G(Y) \rightarrow X$ in such a way that the induced maps

$$G(F(X)) \xrightarrow{G(\alpha)} G(Y) \xrightarrow{\beta} X \tag{93}$$

$$F(G(Y)) \xrightarrow{F(\beta)} F(X) \xrightarrow{\alpha} Y \tag{94}$$

arise from the specified maps $GF \rightarrow \text{id}$ and $FG \rightarrow \text{id}$.

Then the maps $F(X) \rightarrow Y$ and $G(Y) \rightarrow X$ must be surjections (by inspection of (93) and (94)); then in the diagram $G(F(X)) \rightarrow G(Y) \rightarrow X$ we have a composite of surjections giving an isomorphism, so both are isomorphisms; in particular, $G(Y) \rightarrow X$ and $F(X) \rightarrow Y$ must be isomorphisms.

Applied to our original context, this concludes the proof of the first assertion of the Lemma. □

Note also that inside H_I we have a copy of the monoid algebra $k[X_*^+]$, namely, the action of ‘ U_q -operators’ $I_q \chi I_q$ for $\chi \in X_*^+$ (see (38) for identification of

X_* with a coset space; X_*^+ is the positive cone corresponding to the Borel subgroup \mathbf{B}). Each element $t \in T^\vee(k)$ defines a character $\chi_t : k[X_*^+] \rightarrow k$ of this monoid algebra: t corresponds (by ‘Local Langlands’, easy in this split case) to an unramified character $\mathbf{A}(\mathbf{Q}_q) \rightarrow k^\times$, that is, to a homomorphism $X_* \rightarrow k^\times$ and then we just take the linear extension

$$\chi_t : k[X_*^+] \rightarrow k. \tag{95}$$

Using the previous Lemma, it is easy to compute the action of this monoid algebra:

COROLLARY 6.7. *Suppose that q is part of a Taylor–Wiles datum. Then the generalized eigenvalues of $k[X_*^+]$ acting on $H^*(Y_0(q), k)_{\mathfrak{m}}$ are all of the form χ_{Frob^T} , where $\text{Frob}^T \in T^\vee(k)$ is conjugate to the Frobenius at q , and the notation is defined in (95).*

The corollary asserts just the usual relationship between ‘ T_q eigenvalues at level 1 and U_q eigenvalues at level q .’ We omit the proof; it follows from the prior Lemma and a straightforward computation.

REMARK. Let $H^*(Y_0(q), \mathbf{Z}/p^n)_{\mathfrak{m}, \chi_{\text{Frob}^T}}$ be the summand of $H^*(Y_0(q), \mathbf{Z}/p^n)_{\mathfrak{m}}$ corresponding to the χ_{Frob^T} -eigenspace of $k[X_*^+]$. For later use, we note that there is an isomorphism:

$$H^*(Y_0(q), \mathbf{Z}/p^n)_{\mathfrak{m}, \chi_{\text{Frob}^T}} \cong H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}} \tag{96}$$

where the map from left to right is the pushforward π_* , and an inverse in the other direction is given by the pullback π^* together with projection to the χ_{Frob^T} eigenspace.

To see these are inverses we compute in H_I : with reference to the actions of (91), the forward (pushforward) map corresponds to $e_K \in H_{IK}$, and the reverse map corresponds to $|W|e_K q \in H_{KI}$ where $q \in S[X_*^+] \hookrightarrow H_I$ is chosen to realize to the projection on the χ_{Frob^T} eigenspace, and $e_K q$ means the product of $e_K \in H_{KI}$ with $q \in H_I$. (See remarks after (49)).

When we compose them we get $|W|e_K q \in H_I$ or $|W|e_K q e_K \in H_K$; to see, for example, that the former acts as the identity endomorphism on $H^*(Y_0(q), \mathbf{Z}/p^n)_{\mathfrak{m}, \chi_{\text{Frob}^T}}$, observe that it can be written as $\sum_{w \in W} e_w q$ with $e_w = IwI$, and then we just use the fact that q annihilates all the $k[X_*^+]$ -eigenspaces except the one indexed by Frob_T , whereas the e_w permutes the various eigenspaces.

6.8. Adding level q structure continued: The homology at level $Y_1(q, n)$. This section together with the previous and subsequent ones—Sections 6.5, 6.8, 6.11—all gather some ‘standard’ properties of passing between level 1 and level q . So continue with q as in the prior Section, that is, part of a Taylor–Wiles datum of level n .

We now consider the homology of $Y_1(q, n)$ —as defined in (80)—and the action of its Hecke algebra at level $Y_1(q, n)$. Now this homology is basically glued from the homology of $Y_0(q)$: if we let \mathcal{F} be the pushforward of the constant sheaf k from $Y_1(q, n)$ to $Y_0(q)$, then \mathcal{F} is a successive extension of copies of the constant sheaf k , in a Hecke equivariant way.

We state some consequences of this more formally:

LEMMA 6.9. *Assume that q is part of a Taylor–Wiles datum of level n . Assume local–global compatibility for $Y_0(q)$, in the sense described after (91); let other assumptions be as in Section 6.1. Let \mathfrak{q}, δ be as in (18). Then the homology $H_j(Y_1(q, n), \mathbf{Z}_p)_{\mathfrak{m}}$ vanishes for $j \notin [\mathfrak{q}, \mathfrak{q} + \delta]$.*

Proof. Clearly we can replace the role of \mathbf{Z}_p by $\mathbf{F}_p = k$, and then by the remark before the proof, it is enough to prove the same for $Y_0(q)$. By (92), it suffices to prove the same vanishing statement for $Y(1)$ with $\mathbf{F}_p = k$ coefficients. But this is part of our assumption (7(a) from Section 6.1). □

Now we want to say that this relationship between the homology of $Y_1(q, n)$ and $Y_0(q)$ is equivariant for Hecke operators at q . The full Hecke algebra is somewhat complicated and we just deal with its ‘positive, commutative subalgebra.’ Set $\widetilde{\mathcal{X}}_*$ to be the quotient of $\mathbf{A}(\mathbf{Q}_q)$ by the subgroup $p^n \mathbf{A}(\mathbf{Z}_q)$. Denote by Δ_q the quotient of \mathbf{Q}_q^\times by the subgroup of \mathbf{Z}_q^\times of index p^n . Therefore Δ_q and $\widetilde{\mathcal{X}}_*$ depend on n but we suppress that from the notation for simplicity. We may identify

$$\widetilde{\mathcal{X}}_* = X_*(\mathbf{A}) \otimes \Delta_q,$$

and we think of $\widetilde{\mathcal{X}}_*$ as a thickened version of the character lattice $X_*(\mathbf{A})$. From the valuation $\Delta_q \rightarrow \mathbf{Z}$ we get

$$\widetilde{\mathcal{X}}_* \rightarrow X_* \tag{97}$$

and we can define the ‘positive cone’ $\widetilde{\mathcal{X}}_*^+ \subset \widetilde{\mathcal{X}}_*$ as the preimage of X_*^+ .

Let I be an Iwahori subgroup of $\mathbf{G}(\mathbf{Q}_q)$, and $I' \triangleleft I$ the subgroup corresponding to the covering $Y_1(q, n)$, that is, $I/I' \simeq (\mathbf{Z}/p^n)'$. Then there is a natural map from $\mathbf{Z}_p[\widetilde{\mathcal{X}}_*^+]$ to the Iwahori–Hecke algebra at level I' sending

$\chi \in \widetilde{\mathcal{X}}_*$ to the coset $I'\chi I'$. In particular, we get an action of $\mathbf{Z}_p[\widetilde{\mathcal{X}}_*^+]$ on the homology of $Y_1(q, n)$.

LEMMA 6.10. *Any generalized eigenvalue of $k[\widetilde{\mathcal{X}}_*^+]$ acting on $H_*(Y_1(q, n), k)_{\mathfrak{m}}$ is also a generalized eigenvalue of $k[\widetilde{\mathcal{X}}_*^+]$ acting on $H_*(Y_0(q), k)_{\mathfrak{m}}$ via the map $k[\widetilde{\mathcal{X}}_*^+] \rightarrow k[X_*^+]$ induced by (97).*

Thus, by Corollary 6.7, we get a splitting

$$H_*(Y_1(q, n), k)_{\mathfrak{m}} = \bigoplus_{\substack{\text{Frob}^T \in T^\vee(k) \\ \text{Frob}^T \sim \bar{\rho}(\text{Frob}_q)}} (H_*(Y_1(q, n), k))_{\mathfrak{m}, \chi_{\text{Frob}^T}} \tag{98}$$

into the sum of generalized eigenspaces associated to the characters $\chi_{\text{Frob}^T} : k[\widetilde{\mathcal{X}}_*^+] \rightarrow k$. (In the subscript, \sim means ‘is conjugate to.’) Again, this is nothing but a fancy way of talking about the decomposition into ‘ U_q -eigenspaces.’ The only point to note is that the decomposition is canonically indexed by elements of $T^\vee(k)$ conjugate to the Frobenius.

Proof. Write for short $G = \mathbf{G}(\mathbf{Q}_q)$; let I be an Iwahori subgroup of G , and $I' \triangleleft I$ the subgroup corresponding to the covering $Y_1(q, n)$, that is, $I/I' \simeq (\mathbf{Z}/p^n)^r$. We prove the same statement in cohomology and without the \mathfrak{m} ; the desired statement follows by dualizing and localizing.

By the discussion of Section 2.6 we can identify

$$H^*(Y_1(q, n), k) \simeq \text{Ext}_{kG}^*(k[G/I'], M)$$

$$H^*(Y_0(q), k) \simeq \text{Ext}_{kG}^*(k[G/I], M)$$

where M is the direct limit of cochain complexes of a family of coverings, obtained by adding more and more level structure at q .

Claim: We may filter $k[G/I']$ by G -submodules $F^0 \subset F^1 \subset F^2 \subset \dots$ such that:

- (i) each successive quotient F^{i+1}/F^i is isomorphic as G -module to a sum of copies of $k[G/I]$,
- (ii) For every $\chi \in \widetilde{\mathcal{X}}_*^+$, the action of $I'\chi I'$ preserves the filtration, and the action on the quotients coincides with the action of $I\chi I$.

Assuming the existence of this filtration, the result follows easily: we get long exact sequences of the form

$$\text{Ext}_{kG}^j(F^{i-1}, M) \rightarrow \text{Ext}_{kG}^j(F^i, M) \rightarrow \text{Ext}^j(k[G/I], M)^{\oplus e} \rightarrow$$

and these are equivariant for the action of $I'\chi I'$, which acts by via $I\chi I$ on the right-hand summand. The result follows immediately by a descending induction.

To construct the desired filtration regard $k[G/I']$ as the compact induction from I to G of the I -representation $k[I/I']$, that is,

$$k[G] \otimes_{k[I]} k[I/I']$$

In particular, $\Delta = I/I'$ acts by G -endomorphisms on $k[G/I']$, which is the action ‘by right multiplication.’ This action of $\delta \in \Delta$ coincides with the action of $I'\delta I'$ and, in particular, commutes with the action of $k[\widetilde{\mathcal{X}}_*^+]$.

Now we filter $k[G/I']$ by the kernels $k[G/I']\langle \mathfrak{m}^j \rangle$ of successive powers \mathfrak{m}^j of the maximal ideal \mathfrak{m} in $k[\Delta]$. This filtration is stable for $k[\widetilde{\mathcal{X}}_*^+]$ because the actions of $\Delta, k[\widetilde{\mathcal{X}}_*^+]$ commute. Also the j th term of the resulting filtration is thus

$$F^j = k[G] \otimes_{k[I]} k[\Delta]\langle \mathfrak{m}^j \rangle$$

and the j th graded is just $k[G] \otimes_{k[I]} \frac{\langle \mathfrak{m}^j \rangle}{\langle \mathfrak{m}^{j+1} \rangle}$, that is, a direct sum of copies of $k[G/I]$, as claimed.

It remains to check assertion (ii) in the *Claim*. For any $y \in \mathfrak{m}^{j-1}$, multiplication by y gives a map

$$F^j / F^{j-1} \rightarrow F^1 = F^1 / F^0$$

and a suitable sum of such maps is an isomorphism (as we see by checking the corresponding assertion for $k[\Delta]$). Since these multiplication maps commute with the action of $k[\widetilde{\mathcal{X}}_*^+]$ we are reduced to computing the action of $I'\chi I'$ on F^1 ; now F^1 is identified with $k[G/I]$ in a natural way and the assertion is clear. \square

6.11. Adding level q structure continued: Galois representations for level $Y_1(q, n)$. This section and the prior two Sections 6.5, 6.8, 6.11 all gather some ‘standard’ properties of passing between level 1 and level q . We now consider more closely the action of the Iwahori–Hecke algebras at level $Y_1(q, n)$ and formulate local–global compatibility. First let us look at the Galois side.

LEMMA 6.12. *Let q be a Taylor–Wiles prime of level n and assume that the unramified representation $\bar{\rho}|_{G_{\mathbb{Q}_q}}$ has image inside T^\vee . Then any deformation of $\bar{\rho}|_{G_{\mathbb{Q}_q}}$ can be conjugated to one taking values in T^\vee . In particular, any such deformation of $\bar{\rho}|_{G_{\mathbb{Q}_q}}$ actually factors through $G_{\mathbb{Q}_q}^{\text{ab}} \simeq \text{profinite completion of } \mathbb{Q}_q^\times$.*

Proof. We may present the tame quotient of $G_{\mathbb{Q}_q}$ as $\langle F, t : FtF^{-1} = t^q \rangle$, where t is a generator of tame inertia.

Suppose that A is an Artin local ring with maximal ideal \mathfrak{m} , and $\mathfrak{m}^d = 0$. We are given F, t in $G^\vee(A)$ that satisfy $FtF^{-1} = t^q$, where t reduces to the identity in $G^\vee(k)$ and F reduces to (after conjugating) a strongly regular element of $T^\vee(k)$. Conjugating, we may suppose that F belongs to the maximal torus $T^\vee(A)$. We prove by induction on d that this forces $t \in T^\vee(A)$ too. By the inductive hypothesis (obvious for $d = 1$) the image of t in $G^\vee(A/\mathfrak{m}^{d-1})$ belongs to $T^\vee(A/\mathfrak{m}^{d-1})$. Write thus $t = t_0\delta_t$ where $t_0 \in T^\vee(A)$ and $\delta_t \in G^\vee(A)$ lies in the kernel of reduction modulo \mathfrak{m}^{d-1} .

Now in fact t_0 lies in the kernel of reduction modulo \mathfrak{m} , and so t_0 and δ_t actually commute; indeed, t_0 commutes with anything in the kernel of reduction modulo \mathfrak{m}^{d-1} . Also $\delta_t^{q-1} = e$. (To check these statements, just compute in the formal group of G^\vee at the identity.) Now, $FtF^{-1} = t^q$ so that

$$F(t_0\delta_t)F^{-1}(t_0\delta_t)^{-1} = (t_0\delta_t)^{q-1} \in T^\vee(A).$$

But the left-hand side equals $t_0(\text{Ad}(F)\delta_t \cdot \delta_t^{-1})t_0^{-1}$, and so

$$\text{Ad}(F)\delta_t \cdot \delta_t^{-1} \in T^\vee(A)$$

and since F is strongly regular this means that $\delta_t \in T^\vee(A)$ as desired. □

We now want to connect the Galois deformation ring relevant to $Y_1(q, n)$ with the Iwahori–Hecke algebra.

Let q be a Taylor–Wiles prime of level n . Suppose fixed an element $\text{Frob}^T \in T^\vee(k)$ conjugate to the Frobenius at q . Consider a deformation $\sigma : G_{\mathbf{Q}} \rightarrow G^\vee(R)$ of $\bar{\rho}$, where one allows now ramification at q , and R is an Artin local ring with residue field k . Then we can uniquely conjugate σ so its restriction to $G_{\mathbf{Q}_q}$ factors as

$$G_{\mathbf{Q}_q}^{\text{ab}} \simeq \widehat{\mathbf{Q}_q^\times} \longrightarrow T^\vee(R),$$

and the image of a uniformizer in \mathbf{Q}_q^\times reduces to Frob^T . This map factors through (the profinite completion of) $\mathbf{Q}_q^\times / (1 + q\mathbf{Z}_q)$. Restricting to \mathbf{F}_q^\times we get

$$\mathbf{F}_q^\times \rightarrow T^\vee(R)$$

and pairing with characters of T^\vee we get $\mathbf{F}_q^\times \times X^*(T^\vee) \rightarrow R^*$; by the duality of T^\vee and \mathbf{A} , this is the same thing as

$$\mathbf{A}(\mathbf{F}_q) \rightarrow R^*. \tag{99}$$

We emphasize that the map (99) depended on the choice of a toral element Frob^T conjugate to Frobenius; changing this element changes the map through the action of the Weyl group.

Let $R_{S \cup \{q\}}^{\text{univ}}$ be Mazur’s universal deformation ring for $\bar{\rho}$, allowing ramification at q . By our assumptions (Section 6.2) on the existence of Galois representations, there is a map

$$R_{S \cup \{q\}}^{\text{univ}} \rightarrow \mathbb{T}_{K_1(q,n), \mathfrak{m}},$$

where $K_1(q, n)$ is the level structure for $Y_1(q, n)$. By means of this map, $R_{S \cup \{q\}}^{\text{univ}}$ acts on $H^*(Y_1(q, n), \mathbf{Z}_p)_{\mathfrak{m}}$. And, in particular, on the summand $H^*(Y_1(q, n), \mathbf{Z}_p)_{\mathfrak{m}, \text{Frob}_q^r}$ under (98). Thus, by (99), we get an action of $\mathbf{A}(\mathbf{F}_q)$ this cohomology group.

Now the assumption of local–global compatibility alluded to in Section 6.2 is a strengthened version of the following:

Local–global compatibility: The action $\mathbf{A}(\mathbf{F}_q) \curvearrowright H^*(Y_1(q, n), \mathbf{Z}_p)_{\mathfrak{m}, \text{Frob}_q^r}$ just defined coincides with the ‘geometric’ action, that is, wherein $\mathbf{A}(\mathbf{F}_q)$ acts by deck transformations on $Y_1(q, n)$ (see (80)).

By ‘strengthened version,’ we mean that we require a similar assertion at a derived category level, not just at the level of cohomology, and we also require the assertion for several auxiliary primes q rather than a single one. For details, see [14, Section 13.5].

We say a deformation of $\bar{\rho}|G_{\mathbf{Q}_q}$ is of ‘inertial level $\leq n$ ’ if, when considered as a representation of \mathbf{Q}_q^\times by Lemma 6.12, and restricted to \mathbf{F}_q^\times , it factors through the quotient \mathbf{F}_q^\times/p^n . We denote by $R_{S \cup \{q\}}^{\text{univ}, \leq n}$ the quotient of $R_{S \cup \{q\}}^{\text{univ}}$ that classifies deformations of $\bar{\rho}$ such that $\bar{\rho}|G_{\mathbf{Q}_q}$ has inertial level $\leq n$. Explicitly, $R_{S \cup \{q\}}^{\text{univ}, \leq n}$ is the quotient of

$$R_{S \cup \{q\}}^{\text{univ}} / \langle t - 1 : t \text{ is in the image of } p^n \mathbf{A}(\mathbf{F}_q) \text{ under (99)} \rangle. \tag{100}$$

Then local–global compatibility implies that the action of $R_{S \cup \{q\}}^{\text{univ}}$ on the homology of $Y_1(q, n)$ factors through $R_{S \cup \{q\}}^{\text{univ}, \leq n}$, according to our previous discussion.

7. Patching and the derived Hecke algebra

We continue with the notation and assumptions of the previous section Section 6; see in particular Sections 6.1 and 6.2. That section was primarily setup, and now we get down to proving that the global derived Hecke algebra is ‘big enough,’ in the sense discussed around (2). The main result is Theorem 7.6.

We use the patching of the Taylor–Wiles method; more specifically, we use the version of that method that was discovered [9] by Calegari and Geraghty,

which applies to situations where the same Hecke eigensystem occurs in multiple degrees. We also use heavily the presentation of the Calegari–Geraghty method given by Khare and Thorne [21].

7.1. Deformation rings and chain complexes at level Q_n . Fix now a Taylor–Wiles datum Q_n of level n . (We abusively use Q_n both to denote the Taylor–Wiles datum and simply the set of primes associated to that datum.)

Recall the definition of $Y_1^*(Q_n)$ from Section 6.3: it is the fiber product of coverings $Y_1(q, n) \rightarrow Y(1)$ over $q \in Q_n$. We now collect together various results about the homology of $Y_1^*(Q_n)$, which are essentially the same results as those already discussed for $Y_1(q, n)$, but using all the primes in Q_n instead of just $\{q\}$.

LEMMA 7.2. *The homology $H_j(Y_1^*(Q_n), \mathbf{Z}_p)_\mathfrak{m}$ vanishes for $j \notin [q, q + \delta]$.*

Proof. As in Lemma 6.9. □

Just as in (98), this homology group is split (by ‘ U -operators’) into summands indexed by collections $\text{Frob}_q^T \in T^\vee(k)$ ($q \in Q_n$), where each Frob_q^T is conjugate to the Frobenius at q . In particular, since the Taylor–Wiles datum is equipped (Section 6.3) with a specific choice of such a Frob_q^T for each $q \in Q_n$, we can consider the summand

$$H_*(Y_1^*(Q_n), \mathbf{Z}_p)_{\mathfrak{m}, \text{Frob}_{Q_n}^T} \subset H_*(Y_1^*(Q_n), \mathbf{Z}_p)_\mathfrak{m} \tag{101}$$

indexed by these prescribed lifts.

Recall from (81) that $Y_1^*(Q_n) \rightarrow Y_0(Q_n)$ is Galois, with Galois group T_n . Let us introduce notation for the deformation rings of interest to us: let

$$\begin{aligned} R_{Q_n} &= \text{universal deformation ring at level } S \coprod Q_n, \\ R_n &= \text{quotient of } R_{Q_n} \text{ classifying deformations of inertial level} \\ &\leq n \text{ at primes in } Q_n. \end{aligned} \tag{102}$$

For example, in the case when $Q_n = \{q\}$ this was the ring $R_{S \cup \{q\}}^{\text{univ}, \leq n}$ discussed around (100).

By the discussion of (99) and after, we get a morphism $\mathbf{A}(\mathbf{F}_{q_i})/p^n \rightarrow R_n^\times$, and therefore we get (see (81)):

$$T_n \rightarrow R_n^\times.$$

What we know (local–global compatibility, assumed in Section 6.11) is that the natural action of T_n (deck transformations) on homology of $Y_1^*(Q_n)$ is compatible with its action via $T_n \rightarrow R_n^\times$. To say differently, we get a map

$$S_n := \mathbf{Z}/p^n[T_n] \rightarrow R_n/p^n, \tag{103}$$

and the natural action of S_n on the \mathbf{Z}/p^n homology of $Y_1^*(Q_n)$ is compatible with that via the map to R_n/p^n .

Now consider the complex of singular chains

$$\tilde{C}_n = \text{Chains}(Y_1^*(Q_n); \mathbf{Z}/p^n)$$

with \mathbf{Z}/p^n coefficients. We think of it as a complex of S_n modules, because of the action of T_n by deck transformations on $Y_1^*(Q_n)$. It is quasi-isomorphic to a bounded complex of finite free S_n -modules and we have canonical identifications:

$$H_*\tilde{C}_n \simeq H_*(Y_1^*(Q_n), \mathbf{Z}/p^n) \tag{104}$$

$$H^*\text{Hom}_{S_n}(\tilde{C}_n, \mathbf{Z}/p^n) \simeq H^*(Y_0(Q_n), \mathbf{Z}/p^n) \tag{105}$$

$$H_*(\tilde{C}_n \otimes_{S_n} \mathbf{Z}/p^n) \simeq H_*(Y_0(Q_n), \mathbf{Z}/p^n). \tag{106}$$

Note that \tilde{C}_n is a free S_n -module, with basis given by the characteristic functions of an arbitrarily chosen set of representatives for T_n -orbits on singular simplices. Therefore, the homology of $\text{Hom}_{S_n}(\tilde{C}_n, \mathbf{Z}/p^n)$ computes the homomorphisms from \tilde{C}_n to \mathbf{Z}/p^n in the derived category of S_n -modules. By composition of homomorphisms in this derived category, we get a map

$$\underbrace{H^*\left(\text{Hom}_{S_n}(\tilde{C}_n, \mathbf{Z}/p^n)\right)}_{\simeq H^*(Y_0(Q_n), \mathbf{Z}/p^n)} \times \underbrace{\text{Ext}_{S_n}^*(\mathbf{Z}/p^n, \mathbf{Z}/p^n)}_{\simeq H^*(T_n, \mathbf{Z}/p^n)} \rightarrow \underbrace{H^*\left(\text{Hom}_{S_n}(\tilde{C}_n, \mathbf{Z}/p^n)\right)}_{\simeq H^*(Y_0(Q_n), \mathbf{Z}/p^n)}. \tag{107}$$

With respect to the identifications noted underneath the respective terms, this is precisely the ‘natural’ action of $H^*(T_n, \mathbf{Z}/p^n)$ on $H^*(Y_0(Q_n), \mathbf{Z}/p^n)$. This natural action arises thus: the covering $Y_1^*(Q_n) \rightarrow Y_0(Q_n)$ has covering group T_n , that is, can be regarded as a map

$$Y_0(Q_n) \rightarrow \text{BT}_n \tag{108}$$

from $Y_0(Q_n)$ to the classifying space of T_n ; this allows one to pull back cohomology classes from T_n and take cup product. The coincidence of (107) and this ‘natural action’ is a general fact; for lack of a reference we sketch a proof in Section B.4.

It is possible to ‘cut down’ \tilde{C}_n in a fashion that corresponds to the summand (101), as is explained in [21] (see Lemma 2.12 thereof, and surrounding discussion). This can be done compatibly for $Y_0(Q_n)$ and $Y_1^*(Q_n)$ and thus one

gets a perfect complex C_n of S_n -modules, equipped with identifications that are analogous to (104) and (105):

$$H_*(C_n; \mathbf{Z}/p^n) \simeq H_*(Y_1^*(Q_n), \mathbf{Z}/p^n)_{\mathfrak{m}, \text{Frob}^T}, \tag{109}$$

$$H^*(\text{Hom}_{S_n}(C_n, \mathbf{Z}/p^n)) \simeq H^*(Y_0(Q_n), \mathbf{Z}/p^n)_{\mathfrak{m}, \text{Frob}^T} \tag{110}$$

and again the action of $\text{Ext}_{S_n}^*(\mathbf{Z}/p^n, \mathbf{Z}/p^n)$ on the latter group corresponds to the natural action by pulling back cohomology classes via (108). (As a sanity check on this, note that the action of $H^*(T_n, \mathbf{Z}/p^n)$ on $H^*(Y_0(Q_n), \mathbf{Z}/p^n)$ indeed does preserve the splitting into summands of the type (101); one can see this directly by seeing that $H^*(T_n, \mathbf{Z}/p^n)$, considered inside the derived Iwahori–Hecke algebra, commutes with the ‘positive subalgebra’ used to define the splitting (101).)

7.3. Extracting the limit. Now we ‘pass to the limit’ as per Taylor–Wiles and Calegari–Geraghty. The idea is roughly speaking to extract, by a compactness argument, a subsequence of n along which the C_n, S_n, R_n are compatible, and then get limits $\mathbf{C}, \mathbf{S}, \mathbf{R}$ by an inverse limit. Usually in modularity lifting one is only concerned with the limit of the process; but in our case we also want to remember some facts about how this relates to the C_n, S_n, R_n . A discussion of this process which emphasizes exactly what we need is given in [14, Section 13], see in particular Theorem 13.1 therein.

We choose a sequence of Taylor–Wiles data Q_n with $n \rightarrow \infty$. After replacing the Q_n by a suitable subsequence and then reindexing—that is to say, replacing Q_i by Q_{n_i} for some $n_i > i$, and then regarding Q_{n_i} as a set of Taylor–Wiles primes of level i —we can arrange that we can ‘pass to the limit.’ After having done this, we obtain at last the following data:

- (a) A sequence of Taylor–Wiles data Q_n of level n .

Recall to this we have associated coverings $Y_1^*(Q_n) \rightarrow Y_0(Q_n) \rightarrow Y(1)$, as in Section 6.3, and the Galois group of the former map is called T_n ; also (103) we set $S_n = \mathbf{Z}/p^n[T_n]$, the group algebra of T_n .

- (b) With $\mathbf{S} = \mathbf{Z}_p[[x_1, \dots, x_R]]$ as in Section 6.4, a complex \mathbf{C} of finite free \mathbf{S} -modules, equipped with a quasi-isomorphism

$$\mathbf{C} \otimes_{\mathbf{S}} S_n \simeq C_n, \tag{111}$$

where C_n is as described in Section 7.1: a version of the chain complex of $Y_1^*(Q_n)$ with \mathbf{Z}/p^n coefficients, but localized at \mathfrak{m} and Frob^T .

(c) A quotient \bar{R}_n of R_n , defined as follows:

Recall from (102) the definition of R_n ; a quotient of the crystalline deformation ring at level $S \coprod Q_n$. We set

$$\bar{R}_n = R_n / (p^n, \mathfrak{m}^{K(n)}) \tag{112}$$

for a certain explicit function $K(n)$, chosen so that, for example, action of R_n on $H_*(C_n)$ automatically factors through \bar{R}_n . We can and will assume $K(n) \geq 2n$. (The main function of $K(n)$ is to make \bar{R}_n Artinian, while still retaining enough information about all of R_n for our purposes.)

(d) A ‘limit deformation ring’ $R \simeq \mathbf{Z}_p[[x_1, \dots, x_{R-\delta}]]$ equipped with maps $S \rightarrow R$ and maps $R \rightarrow \bar{R}_n$, $R \rightarrow R_{\bar{\rho}}$ which are compatible, in the sense that this diagram commutes:

$$\begin{array}{ccccc} S & \longrightarrow & R & \longrightarrow & R_{\bar{\rho}} \\ \downarrow & & \downarrow & & \downarrow \\ S_n & \longrightarrow & \bar{R}_n & \longrightarrow & R_{\bar{\rho}} / (p^n, \mathfrak{m}^{K(n)}) \end{array} \tag{113}$$

(Recall here that $R_{\bar{\rho}}$ is the deformation ring of $\bar{\rho}$, with crystalline conditions imposed, without adding any level, cf. Section 6.2.)

Moreover, the composite $S \rightarrow R \rightarrow R_{\bar{\rho}}$ factors through the augmentation $S \rightarrow \mathbf{Z}_p$; and also the left-hand square induces an isomorphism

$$R \otimes_S S_n \simeq \bar{R}_n. \tag{114}$$

(e) An action of R on $H_*(\mathbf{C})$, compatible with the S action, and with the maps $H_*(\mathbf{C}) \rightarrow H_*(C_n)$, where R acts on $H_*(C_n)$ via $R \rightarrow \bar{R}_n$.

(f) An identification of

$$H^*(\text{Hom}_S(\mathbf{C}, \mathbf{Z}_p)) \xrightarrow{\sim} H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}, \tag{115}$$

compatible under (111) with the identification $H^*(\text{Hom}_{S_n}(C_n, \mathbf{Z}/p^n)) \simeq H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$ that is the composition of (110) with the pushforward. (This is described in dual form in [14, Theorem 13.1(d)] but one gets similarly this result, and the statement about compatibility is just a matter of looking at the definition of the map (115).

(g) (These last results use heavily the formal smoothness, assumption (e) from Section 6.2): \mathbf{C} has homology only in degree q , and its homology there $H_q(\mathbf{C})$ is free as R -module. Moreover, one has an ‘ $R = T$ result’

$$R \otimes_S \mathbf{Z}_p \simeq R_{\bar{\rho}} \simeq \text{image of } \mathbb{T}_{K_0} \text{ in } \text{End } H_q(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}. \tag{116}$$

7.4. The structure of $\mathbf{S} \rightarrow \mathbf{R}$. The limit process has given a map of rings $\mathbf{S} \rightarrow \mathbf{R}$, where \mathbf{S} and \mathbf{R} are formal power series rings that represent, roughly speaking, ‘limits’ of the rings S_n, R_n as $n \rightarrow \infty$.

As in (77), the representation Π gives a lift ρ to \mathbf{Z}_p of the residual representation $\bar{\rho}$; this corresponds to an augmentation $\mathbf{R}_{\bar{\rho}} \rightarrow \mathbf{Z}_p$. Thus we also get an augmentation

$$f : \mathbf{R} \rightarrow \mathbf{Z}_p,$$

and the pullback of this to \mathbf{S} is the natural augmentation of $\mathbf{S} \rightarrow \mathbf{Z}_p$ ((d) of Section 7.3). In particular, the kernel of f on \mathbf{S} is precisely the ideal \mathfrak{I} .

Our assumptions imply that the map $\mathbf{S} \rightarrow \mathbf{R}$ is surjective. Indeed, because \mathbf{S} is complete for the l -adic topology it is enough to verify that $\mathbf{S}/\mathfrak{I} \rightarrow \mathbf{R}/\mathfrak{I}\mathbf{R}$ is surjective. But \mathbf{R}/\mathfrak{I} is a Hecke ring by (116) and so isomorphic to \mathbf{Z}_p by (75). Note, in particular, that this also means that $\mathfrak{I}\mathbf{R}$ is precisely the kernel of f .

The following easy lemma is now useful for explicit computations.

LEMMA 7.5. *We can choose generators x_i, y_j for \mathbf{S}, \mathbf{R} , that is,*

$$\mathbf{S} = \mathbf{Z}_p[[x_1, \dots, x_R]], \quad \mathbf{R} = \mathbf{Z}_p[[y_1, \dots, y_{R-\delta}]]$$

such that the x_i, y_j s lie in the kernel of the compatible augmentations

$$\mathbf{S} \rightarrow \mathbf{R} \rightarrow \mathbf{Z}_p,$$

and the map $\mathbf{S} \rightarrow \mathbf{R}$ is given by $x_i \mapsto y_i$ for $i \leq R - \delta$ and $x_i \mapsto 0$ for $i > R - \delta$.

Proof. Write $f : \mathbf{R} \rightarrow \mathbf{Z}_p$ for the augmentation. Abstractly, $\mathbf{R} \simeq \mathbf{Z}_p[[u_1, \dots, u_{R-\delta}]]$ where all the u_i lie in the maximal ideal. Set $y_i = u_i - f(u_i) \in \ker(\mathbf{R} \rightarrow \mathbf{Z}_p)$. Then still $\mathbf{R} \simeq \mathbf{Z}_p[[y_1, \dots, y_{R-\delta}]]$. We have noted above that $\mathfrak{J} := \mathfrak{I}\mathbf{R}$ is precisely the kernel of the augmentation $\mathbf{R} \rightarrow \mathbf{Z}_p$; thus, the y_i freely span as \mathbf{Z}_p -module the quotient $\mathfrak{J}/\mathfrak{J}^2$.

Lift the y_i to $x_1, \dots, x_r \in \ker(\mathbf{S} \rightarrow \mathbf{Z}_p)$. Necessarily the x_i span a saturated \mathbf{Z}_p -submodule of rank s inside $\mathfrak{I}/\mathfrak{I}^2 \simeq \mathbf{Z}_p^s$; they are \mathbf{Z}_p -independent because any linear relation $\sum a_i x_i \in \mathfrak{I}^2$ (with $a_i \in \mathbf{Z}_p$) would give rise to a corresponding linear relation in \mathbf{R} , a contradiction. (Here we say that a submodule Q of a free \mathbf{Z}_p -module Q' is saturated if the quotient Q'/Q is torsion-free.) Similarly, they are saturated because given x' and (a_1, \dots, a_r) with $\gcd(a_1, \dots, a_r) = 1$ and $px' = \sum a_i x_i + \mathfrak{I}^2$, we would get a corresponding relation in \mathbf{R} , again a contradiction.

Now extend the x_i to a full \mathbf{Z}_p -basis x_{r+1}, \dots, x_s for $\mathfrak{I}/\mathfrak{I}^2$. Each x_j for $j > r$ is sent under $\mathbf{S} \rightarrow \mathbf{R}$ an element of $\mathfrak{J} \subset \mathbf{R}$, which means that it can be written as a formal polynomial $P_j(y_1, \dots, y_r)$ in y_1, \dots, y_r , with no constant term; so replacing x_j by $x_j - P_j(x_1, \dots, x_r)$ we may suppose that $x_j \mapsto 0$ in \mathbf{R} . \square

Now we come to the main theorem of the section.

THEOREM 7.6. *Let assumptions be as in Sections 6.1 and 6.2. The cohomology $H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$ is generated, as a module over the strict global derived Hecke algebra (see Sections 2.13 and 2.14 for definition with \mathbf{Z}_p coefficients), by its minimal degree component $H^q(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$.*

Proof. We use the setup of the Taylor–Wiles limit process (Section 7.3), beginning with the fact that the natural map

$$H^q(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}_p)) \otimes \mathrm{Ext}_{\mathbf{S}}^i(\mathbf{Z}_p, \mathbf{Z}_p) \twoheadrightarrow H^{q+i}(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}_p)) \tag{117}$$

is a surjection for all j : by (g) of Section 7.3 and Lemma 7.5, we can choose coordinates so that $\mathbf{S} \simeq \mathbf{Z}_p[[x_1, \dots, x_R]]$, and the complex \mathbf{C} is quasi-isomorphic to a sum of copies of $\mathbf{S}/(x_R, \dots, x_{R-\delta+1})$ concentrated in a single degree. So the surjectivity of (117) follows from the ‘Koszul algebra’ computations in Appendix B.

Examine now the diagram, where all the maps are the obvious ones;

$$\begin{array}{ccccc}
 H^q(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}_p)) & \times & \mathrm{Ext}_{\mathbf{S}}^i(\mathbf{Z}_p, \mathbf{Z}_p) & \longrightarrow & H^{q+i}(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}_p)) \\
 \downarrow = & & \downarrow & & \downarrow V \\
 H^q(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}_p)) & \times & \mathrm{Ext}_{\mathbf{S}}^i(\mathbf{Z}_p, \mathbf{Z}/p^n) & \longrightarrow & H^{q+i}(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}/p^n)) \\
 \downarrow V & & \uparrow U & & \downarrow = \\
 H^q(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}/p^n)) & \times & (\mathrm{Ext}_{\mathbf{S}}^i(\mathbf{Z}/p^n, \mathbf{Z}/p^n)) & \longrightarrow & H^{q+i}(\mathrm{Hom}_{\mathbf{S}}(\mathbf{C}, \mathbf{Z}/p^n)) \\
 \uparrow f, \sim & & \uparrow U' & & \uparrow \sim \\
 H^q(\mathrm{Hom}_{S_n}(\mathbf{C} \otimes_{\mathbf{S}} S_n, \mathbf{Z}/p^n)) & \times & (\mathrm{Ext}_{S_n}^i(\mathbf{Z}/p^n, \mathbf{Z}/p^n)) & \xrightarrow{\Omega} & H^{q+i}(\mathrm{Hom}_{S_n}(\mathbf{C} \otimes_{\mathbf{S}} S_n, \mathbf{Z}/p^n)).
 \end{array} \tag{118}$$

Here, the middle square ‘commutes’ in the sense that the image of (x, Uy) is the same as the image of (Vx, y) , that is, U, V are adjoint for the pairing. The top and bottom squares commute. All this is obvious, except for perhaps the bottom square which involves change of rings, so let us talk through it: The map $\mathbf{S} \rightarrow S_n$ induces a forgetful map T from the derived category of S_n -modules to the derived category of \mathbf{S} -modules. Take

$$\alpha \in H^q(\mathrm{Hom}_{S_n}(\mathbf{C} \otimes_{\mathbf{S}} S_n, \mathbf{Z}/p^n)), \quad \beta \in \mathrm{Ext}_{S_n}^i(\mathbf{Z}/p^n, \mathbf{Z}/p^n).$$

We can regard α as a map $\mathbf{C} \otimes_{\mathbf{S}} S_n \rightarrow \mathbf{Z}/p^n[q]$ and β as a map $\mathbf{Z}/p^n[q] \rightarrow \mathbf{Z}/p^n[q+i]$, both in the derived category of S_n -modules. Applying the functor T , we see that $T\beta \cdot T\alpha = T(\beta\alpha)$. On the other hand, $T\alpha$ is simply the morphism

$\mathbb{C} \otimes_{\mathbb{S}} S_n \rightarrow \mathbb{Z}/p^n[q]$ considered as a map of \mathbb{S} -modules. If we precompose with $\gamma : \mathbb{C} \rightarrow \mathbb{C} \otimes_{\mathbb{S}} S_n$, considered as a map of \mathbb{S} -modules, we get $f(\alpha)$. Similarly, $T(\beta\alpha)\gamma = f(\beta\alpha) \cdot \gamma$. So $T\beta.f(\alpha) = f(\beta\alpha)$: that is the commutativity of the bottom square.

It follows from the Lemma of Section 6.4 that the composite $U \circ U'$ is surjective. Also the map V is surjective (because the cohomology of $\text{Hom}_{\mathbb{S}}(\mathbb{C}, \mathbb{Z}_p)$ is torsion-free, by (115) and assumption 7(a) of Section 6.1). Tracing through the above diagram, this is enough to show that the image of Ω generates the codomain of Ω .

Now, recall from (111) the quasi-isomorphism $\mathbb{C} \otimes_{\mathbb{S}} S_n \simeq C_n$; we have therefore shown that $H^*(\text{Hom}_{S_n}(C_n, \mathbb{Z}/p^n))$ is generated by $H^q(\text{Hom}_{S_n}(C_n, \mathbb{Z}/p^n))$ as a module over $\text{Ext}_{S_n}^*(\mathbb{Z}/p^n, \mathbb{Z}/p^n)$. As in the discussion after (107), this is equivalent to saying that $H^*(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}, \text{Frob}^T}$ is generated by its degree q component as a $H^*(T_n, \mathbb{Z}/p^n)$ -module, that is,

$$H^q(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}, \text{Frob}^T} \otimes H^*(T_n, \mathbb{Z}/p^n) \twoheadrightarrow H^*(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}, \text{Frob}^T}. \tag{119}$$

In what follows, let us write $\mathcal{H}_I, \mathcal{H}_K$ for the tensor product of (derived) Iwahori–Hecke algebras $\mathcal{H}_{I,q}$ and derived Hecke algebra \mathcal{H}_q over $q \in Q_n$; and write H_K, H_{KI}, H_{IK}, H_I for the (nonderived) algebras and bimodules of Section 6.5, but tensoring over all $q \in Q_n$. All of these will be taken with \mathbb{Z}/p^n coefficients.

Note that the action of $H^*(T_n, \mathbb{Z}/p^n)$ on $H^*(Y_0(Q_n), \mathbb{Z}/p^n)$ factors through the action of \mathcal{H}_I (for example, see the Remark in Section 2.10). So $H^*(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}, \text{Frob}^T}$ is generated in degree q over \mathcal{H}_I . Taking the sum over all possible lifts Frob^T , as in (98), we see that $H^*(Y_0^*(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}}$ is also generated in degree q over the derived \mathcal{H}_I .

Now, each of the following maps are surjective:

$$\begin{aligned} H^q(Y(1), \mathbb{Z}/p^n)_{\mathfrak{m}} \otimes H_{KI} &\twoheadrightarrow H^q(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}}, \\ H^q(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}} \otimes \mathcal{H}_I^j &\twoheadrightarrow H^{q+j}(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}}, \\ H^{q+j}(Y_0(Q_n), \mathbb{Z}/p^n)_{\mathfrak{m}} \otimes H_{IK} &\twoheadrightarrow H^{q+j}(Y(1), \mathbb{Z}/p^n)_{\mathfrak{m}}, \end{aligned}$$

where the second statement is what we just proved, whereas the first and third statement come from Lemma 6.6. Also there is a map $(H_{KI} \otimes \mathcal{H}_I^j \otimes H_{IK}) \rightarrow \mathcal{H}_K^j$ compatible with the respective actions, just arising from composition of Exts. We get

$$H^q(Y(1), \mathbb{Z}/p^n)_{\mathfrak{m}} \otimes \mathcal{H}_K^j \twoheadrightarrow H^{q+j}(Y(1), \mathbb{Z}/p^n)_{\mathfrak{m}}.$$

Passing to the limit (as in the discussion of Section 2.13) concludes the proof. □

8. The reciprocity law

In Section 7, we proved, conditional under assumptions (Sections 6.1, 6.2) on the existence of Galois representations attached to modular forms and other assumptions that simplify the integral situation (Section 6.1), that the global derived Hecke algebra is ‘big enough,’ in the sense discussed around (2).

We now turn to the question mentioned in Section 1.3: we index elements of this global derived Hecke algebra by means of a certain dual Selmer group. This is achieved in Theorem 8.5. This Theorem is not an end in itself; rather, it just gives the correct language for us to formulate the central conjecture of the paper, Conjecture 8.8.

8.1. The coadjoint representations. We are interested in the coadjoint representation, that is, the dual of the representation of G^\vee on its Lie algebra \mathfrak{g}^\vee . Denoting by $\tilde{\mathfrak{g}}$ the \mathbf{Z} -dual to this Lie algebra, we obtain

$$\text{Ad}^* : G^\vee \rightarrow GL(\tilde{\mathfrak{g}})$$

which we regard as a morphism of algebraic groups over \mathbf{Z} . (Why the *coadjoint* representation rather than the adjoint? They are isomorphic for G semisimple, at least away from small characteristic. However, canonically what comes up for us is the coadjoint; for example, when one works with tori, as in Section 9.1, the difference is important.)

In particular, given a representation $\sigma : G_{\mathbf{Q}} \rightarrow G^\vee(R)$ we denote by $\text{Ad}^*\sigma : G_{\mathbf{Q}} \rightarrow GL(R \otimes_{\mathbf{Z}} \tilde{\mathfrak{g}})$ the composition of σ with the coadjoint representation. When σ is valued in \mathbf{Z}_p , we write $\text{Ad}^*\sigma_n$ for the reduction of $\text{Ad}^*\sigma$ modulo p^n .

8.2. Galois cohomology. We freely use the theory of Fontaine and Laffaille which (in good circumstances) parameterizes crystalline representations of $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, even with torsion coefficients. For a summary, see Section 4 of [2].

Fix once and for all the interval $[-\frac{p-3}{2}, \frac{p-3}{2}] \subset \mathbb{N}$ of Hodge weights. We say that a representation of $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on a finitely generated \mathbf{Z}_p -module is ‘crystalline’ if it is isomorphic to a subquotient of a crystalline representation with Hodge weights in $[-\frac{p-3}{2}, \frac{p-3}{2}]$. This indexing of Hodge weights is useful for adjoint representations which have weights symmetric around 0.

Recall that for any p -torsion crystalline M we can define

$$H_f^1(\mathbf{Q}_p, M) \subset H^1(\mathbf{Q}_p, M)$$

which classifies those extensions $M \rightarrow ? \rightarrow 1$ which are crystalline; it is in fact a submodule and it is identified (by Fontaine–Laffaille theory) with a

corresponding Ext-group in the category of filtered Dieudonné modules. In particular, this allows one to check that H_f^1 is isomorphic to the cokernel of the map

$$F^0 D(M) \xrightarrow{1-\text{Frob}} D(M), \tag{120}$$

where $D(M)$ is the associated filtered Dieudonné module. Also, the kernel of $1 - \text{Frob}$ in (120) is isomorphic to $H^0(\mathbf{Q}_p, M)$. In particular, for M finite we have

$$\#H_f^1(\mathbf{Q}_p, M) = \frac{|D(M)|}{|F^0 D(M)|} \cdot \#H^0(\mathbf{Q}_p, M), \tag{121}$$

which can be effectively used to compute the size of H_f^1 (note: the size of $D(M)$ and M coincide).

We need to know that the subspaces $H_f^1(\mathbf{Q}_p, M)$ and $H_f^1(\mathbf{Q}_p, M^*)$ (with $M^* := \text{Hom}(M, \mu_{p^\infty})$) are each other’s annihilators under the local duality pairing $H^1(\mathbf{Q}_p, M) \times H^1(\mathbf{Q}_p, M^*) \simeq \mathbf{Q}/\mathbf{Z}$. This follows from the fact they annihilate each other (their product would come from an Ext^2 in the category of Fontaine–Laffaille modules, but the relevant Ext^2 vanishes by [2, Lemma 4.4]) and a size computation using (121).

8.3. Selmer groups. Let \mathbf{Q}_S be the largest extension of \mathbf{Q} unramified outside S , and let M be a module for the Galois group of \mathbf{Q}_S/\mathbf{Q} ; thus M defines an étale sheaf on $\mathbf{Z}[\frac{1}{S}]$. We write

$$H^1\left(\mathbf{Z}\left[\frac{1}{S}\right], M\right) \supset H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], M\right)$$

for (respectively) the étale cohomology of M (equivalently the group cohomology of $\text{Gal}(\mathbf{Q}_S/\mathbf{Q})$ with coefficients in M), and the subset of this group consisting of classes that are crystalline at p , that is, classes whose image in $H^1(\mathbf{Q}_p, M)$ lies in the subgroup $H_f^1(\mathbf{Q}_p, M)$ defined above.

Note that we impose *no* local condition on classes in $H_f^1(\mathbf{Z}[\frac{1}{S}], M)$ except for the crystalline condition at p .

We write $H_f^1(\mathbf{Q}, M)$ for the usual Bloch–Kato Selmer group: this is the subgroup of classes in $H^1(\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), M)$ which are unramified away from p , and crystalline at p . In general, we have an inclusion $H_f^1(\mathbf{Q}, M) \subset H_f^1(\mathbf{Z}[\frac{1}{S}], M)$; the former is more restrictive, requiring that the cohomology class be unramified at places of $S - \{p\}$. However, in our applications, M will be a module such that $H^1(\mathbf{Q}_v, M)$ vanishes for $v \in S - \{p\}$, and so $H_f^1(\mathbf{Q}, M) = H_f^1(\mathbf{Z}[\frac{1}{S}], M)$.

8.4. We follow the notation of the previous section, described in Section 6.1; in particular, we have an arithmetic manifold $Y(1) = Y(K_0)$, an automorphic representation Π , corresponding to a maximal ideal \mathfrak{m} of the Hecke algebra; and an associated Galois representation $\rho : G_{\mathbb{Q}} \rightarrow G^{\vee}(\mathbf{Z}_p)$. We let S be the set of ramified primes for ρ or K_0 , together with p .

Put

$$V := H_f^1(\mathbf{Z} \left[\frac{1}{S} \right], \text{Ad}^* \rho(1))^{\vee}, \tag{122}$$

where we wrote $-^{\vee}$ for $\text{Hom}(-, \mathbf{Z}_p)$. We prove in Lemma 8.9 that both the H_f^1 above and V are (under our assumptions) free \mathbf{Z}_p -modules of rank δ .

We produce an action of V on $H^*(Y(K), \mathbf{Z}_p)_{\mathfrak{m}}$. To explain it, fix \mathbf{A} a maximal torus of \mathbf{G} and let q be a Taylor–Wiles prime of level n , equipped with an element of $T^{\vee}(k)$ conjugate to Frobenius at q . Let

$$T_q = \mathbf{A}(\mathbf{F}_q)/p^n.$$

From this data we construct:

- A natural embedding (Section 8.17) of

$$\iota_{q,n} : H^1(T_q, \mathbf{Z}/p^n) \hookrightarrow \left(\mathcal{H}_{q, \mathbf{Z}/p^n}^{(1)} \right)_{\mathfrak{m}} \tag{123}$$

into the degree 1 component $\mathcal{H}^{(1)}$ of the local, full level, derived Hecke algebra $\mathcal{H}_{q, \mathbf{Z}/p^n}$. (More precisely, we use its completion at the maximal ideal \mathfrak{m} .)

- A map

$$f_{q,n} : H^1(T_q, \mathbf{Z}/p^n) \rightarrow V/p^n. \tag{124}$$

We have already explained, in a special case, the construction of $f_{q,n}$ in (16). We briefly outline the general case: Given $\alpha \in H^1(T_q, \mathbf{Z}/p^n)$, we obtain, by (139), an element α' in the quotient of $H^1(\mathbf{Q}_q, \text{Ad}\rho_n)$ by unramified classes; now, we associate to α the functional sending $\beta \in H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho(1))$ to the local pairing $\langle \alpha', \beta_q \rangle_q \in \mathbf{Z}/p^n$, where β_q is the restriction of β to $H^1(\mathbf{Q}_q, \text{Ad}^* \rho(1))$.

Finally recall that under our assumptions (7(a) of Section 6.1), $H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$ is torsion-free; its reduction modulo p^n coincides with $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$.

THEOREM 8.5. *Let notation and assumptions be as established in Section 6 (in particular, Sections 6.1, 6.2). Let V be as in (122).*

There exists a function $a : \mathbf{Z}_{\geq 1} \rightarrow \mathbf{Z}_{\geq 1}$ and an action of \mathbf{V} on $H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$ by endomorphisms of degree +1 with the following property:

(*) For any $n \geq 1$ and any prime $q \equiv 1$ modulo $p^{a(n)}$, equipped with a strongly regular element of $T^\vee(k)$ conjugate to $\bar{\rho}(\text{Frob}_q)$, the two actions of $H^1(T_q, \mathbf{Z}/p^n)$ on $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$ coincide: one via $f_{q,n}$ and one via $t_{q,n}$.

The property (*) uniquely characterizes the \mathbf{V} action (this is true for any function a).

In particular, the strict global derived Hecke algebra contains \mathbf{V} , and so also the exterior algebra freely generated by \mathbf{V} (the induced map from $\wedge^* \mathbf{V}$ to endomorphisms of $H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$ is injective).

Note that the uniqueness part of the statement is straightforward, because the condition pins down the action of \mathbf{V}/p^r for arbitrarily large r : by Chebotarev, the images $f_{q,n}(H^1(T_q, \mathbf{Z}/p^n))$ generate \mathbf{V}/p^n even when restricted to primes $q \equiv 1$ modulo $p^{a(n)}$; this basically follows from the existence of Taylor–Wiles data (see in particular, (142)).

Finally, under a further ‘multiplicity one’ assumption, this result is sufficient to force the whole derived Hecke algebra to be graded-commutative. We separate this result from the main analysis because it is inessential to our main goals and it requires this additional multiplicity one assumption.

Before we state the proposition, we note that, by the argument of (37) together with [21, Lemma 6.20], the action of global Hecke algebra $\tilde{\mathbb{T}}$ on cohomology induces an action of $\tilde{\mathbb{T}}$ on $H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$.

PROPOSITION 8.6. *If $H^q(Y(1), \mathbf{Z}_p)_{\mathfrak{m}} = \mathbf{Z}_p$, then the image of the full global derived Hecke algebra $\tilde{\mathbb{T}}$ inside $\text{End } H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$ in fact precisely coincides with the exterior algebra generated by \mathbf{V} , and, in particular, the global derived Hecke algebra (acting on \mathfrak{m} -part of cohomology) is graded-commutative.*

This same conclusion of graded commutativity holds, more generally, when there exists a semisimple \mathbf{Q}_p -algebra \mathbf{S} of (degree-preserving) endomorphisms of $H^(Y(1), \mathbf{Q}_p)$ commuting with $\tilde{\mathbb{T}}$ and such that $H^q(Y(1), \mathbf{Z}_p)_{\mathfrak{m}} \otimes \mathbf{Q}_p$ has multiplicity one as a \mathbf{S} -module.*

REMARK.

- A natural choice for the commuting subalgebra \mathbf{S} in Proposition 8.6 is the \mathbf{Q}_p -algebra generated by the local undervived Hecke algebras at all primes that are not good. However, the multiplicity one condition is only realistic when there is a *unique* tempered representation π_∞ of $\mathbf{G}(\mathbf{R})$ with nontrivial $(\mathfrak{g}, \mathbf{K})$ -

cohomology. If this uniqueness condition is satisfied, then we would likely expect the multiplicity one condition to be valid in the great majority of cases.

Under the assumptions we are currently working (simply connected group, and working over \mathbf{Q}) only SL_{2n+1} has this property. However, this unicity of π_∞ in fact applies whenever we work over a CM base field, or for PGL_n over \mathbf{Q} . (Note that, for PGL_n with n even, one should enlarge \mathbf{S} using the action of the component group of the archimedean maximal compact.) The analysis of Sections 6–8 would extend to those cases with only some notational changes.

- That we get an *integral* isomorphism of $\tilde{\mathbb{T}}$ and \wedge^*V , in the first statement of the Proposition, is an artifact of our simplifying hypotheses. We do not expect $\tilde{\mathbb{T}}$ to be an integral exterior algebra in general, but corresponding statements should remain valid $\otimes \mathbf{Q}$.

In general, we would expect the $\tilde{\mathbb{T}}$ to be $\mathbb{T} \otimes \wedge^*V$, where \mathbb{T} is the usual Hecke algebra, after tensoring with \mathbf{Q} . But here our assumptions mean that \mathbb{T} is just \mathbf{Z}_p , and moreover that the conclusion is true integrally.

8.7. Formulation of the conjecture. We are now ready to state the conjecture, the formulation of which is the main point of this paper. (Although the trip was fun too.)

The formulation of the conjecture itself rests on the conjecture of Langlands that associates to Π a motive, or more precisely a system of motives indexed by representations of the dual group. Unfortunately it is difficult to find a comprehensive account of this conjecture in print; the reader may consult the brief remarks in [22] or the appendix of [25].

Continue with notation as in Theorem 8.5. As in the discussion of Sections 1.2 and 1.3, let M_{coad} be the motive with \mathbf{Q} coefficients associated to Π and the coadjoint representation of G^\vee , if it exists. *A priori*, one may not always be able to descend the coefficients of M_{coad} to \mathbf{Q} , although we expect this is possible in most if not all cases. (See discussion in [25, Appendix, A.3]). In what follows we assume that M_{coad} can indeed be descended to \mathbf{Q} coefficients; if not one can simply reformulate the conjecture by replacing \mathbf{Q} by a field extension.

Thus there is an identification of Galois modules

$$\text{étale realization of } M_{\text{coad}} \simeq \text{Ad}^* \rho \otimes \mathbf{Q}_p$$

and there is a regulator map from the motivic cohomology

$$H_{\text{mot}}^1(\mathbf{Q}, M_{\text{coad}, \mathbf{Z}}(1)) \longrightarrow H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^* \rho(1)\right) \otimes \mathbf{Q}_p.$$

As in Section 1.2.2, the motivic cohomology group on the left-hand side has been restricted to classes that extend to an integral model. We assume that this regulator map is an isomorphism. (In general, the appropriate conjecture is that the regulator, taken to the group of classes that are crystalline at p and unramified at all primes in S , is an isomorphism. However, by virtue of our assumptions in Section 6.2, part (e), it is not necessary to explicitly impose ‘unramified at primes in S .’) Let $V_{\mathbf{Q}_p} = V \otimes \mathbf{Q}_p$, and let $V_{\mathbf{Q}}$ be those classes in $V_{\mathbf{Q}_p}$ whose pairing with motivic cohomology lies in \mathbf{Q} .

Write $H^*(Y(1), -)_{\Pi}$ for the Hecke eigenspace for the character $\mathbb{T}_{K_0} \rightarrow \mathbf{Z}$ associated with Π (see (73)). Our assumptions imply that $H^*(Y(1), \mathbf{Z}_p)_{\Pi} = H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$.

CONJECTURE 8.8. *Notation as above. With reference to the action*

$$\wedge^* V_{\mathbf{Q}_p} \hookrightarrow H^*(Y(1), \mathbf{Q}_p)_{\Pi}$$

furnished by Theorem 8.5, the action of $V_{\mathbf{Q}}$ preserves $H^(Y(1), \mathbf{Q})_{\Pi}$.*

Some rather scant evidence is discussed in the next section (Section 9). As we have mentioned in the introduction, much more compelling is that we have been able to obtain numerical evidence for a coherent analog of the conjecture, in a joint work with Michael Harris.

Recall (Section 6.2) we assume that

$$H^0(\mathbf{Q}_p, \text{Ad}^* \bar{\rho}) = H^2(\mathbf{Q}_p, \text{Ad}^* \bar{\rho}) = 0 \quad (125)$$

which implies the same conclusions for $\text{Ad}^* \bar{\rho}(1)$. In particular, $H^1(\mathbf{Q}_p, \text{Ad}^* \rho(1))$ is torsion-free and surjects onto $H^1(\mathbf{Q}_p, \text{Ad}^* \bar{\rho}(1))$. Finally, $H_f^1(\mathbf{Q}_p, \text{Ad}^* \rho(1))$ is a saturated submodule of $H^1(\mathbf{Q}_p, \text{Ad}^* \rho(1))$ and we have an equality of ranks

$$\text{rank}_{\mathbf{Z}_p} H_f^1(\mathbf{Q}_p, \text{Ad}^* \rho(1)) = \text{rank}_{\mathbf{F}_p} H_f^1(\mathbf{Q}_p, \text{Ad}^* \bar{\rho}(1)),$$

as follows from explicit computation. In particular, $H_f^1(\mathbf{Q}_p, \text{Ad}^* \rho(1))$ surjects onto $H_f^1(\mathbf{Q}_p, \text{Ad}^* \bar{\rho}(1))$. Also observe that, because of the assumed ‘big image’ (Section 6.2 assumption (b)) of $\bar{\rho}$, we have

$$H^0\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^* \bar{\rho}(1)\right) = 0, \quad (126)$$

and so $H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho(1))$ is torsion-free.

LEMMA 8.9. *Both $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho(1))$ and V are free \mathbf{Z}_p -modules of rank δ .*

Proof. First of all, because the Taylor–Wiles method in this case implies an $R = T$ theorem (see (116)) and we are assuming that the Hecke algebra is isomorphic to \mathbf{Z}_p (see (75)) we get from a tangent space computation that $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}\bar{\rho}) = 0$. We now apply Tate global duality to this statement. It implies both the surjectivity of

$$H^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^*\bar{\rho}(1)\right) \twoheadrightarrow \frac{H^1(\mathbf{Q}_p, \text{Ad}^*\bar{\rho}(1))}{H_f^1(\mathbf{Q}_p, \text{Ad}^*\bar{\rho}(1))}, \tag{127}$$

and the injectivity of

$$H^2\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^*\bar{\rho}(1)\right) \hookrightarrow \underbrace{\prod_{v \in S} H^2(\mathbf{Q}_v, \text{Ad}^*\bar{\rho}(1))}_{=0 \text{ by Section 6.2}} \tag{128}$$

so in fact $H^2(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\bar{\rho}(1)) = 0$.

The surjectivity (127) holds also for $\text{Ad}^*\rho(1)$, not just the mod p reduction. This follows because $H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\rho(1))$ surjects onto $H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\bar{\rho}(1))$, by (128); and the induced map

$$\frac{H^1(\mathbf{Q}_p, \text{Ad}^*\rho(1))}{H_f^1(\mathbf{Q}_p, \text{Ad}^*\rho(1))} / P \rightarrow \frac{H^1(\mathbf{Q}_p, \text{Ad}^*\bar{\rho}(1))}{H_f^1(\mathbf{Q}_p, \text{Ad}^*\bar{\rho}(1))}$$

is an isomorphism.

The Euler characteristic formula, taken together with (126), (127), and (128), allows one to compute

$$\dim H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^*\bar{\rho}(1)\right) = \delta. \tag{129}$$

Now examine the short exact sequences

$$\begin{array}{ccccc} H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\rho(1)) & \xrightarrow{p} & H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\rho(1)) & \longrightarrow & H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\bar{\rho}(1)) \\ \downarrow j & & \downarrow & & \downarrow \\ \frac{H^1(\mathbf{Q}_p, \text{Ad}^*\rho(1))}{H_f^1(\mathbf{Q}_p, \text{Ad}^*\rho(1))} & \xrightarrow{p} & \frac{H^1(\mathbf{Q}_p, \text{Ad}^*\rho(1))}{H_f^1(\mathbf{Q}_p, \text{Ad}^*\rho(1))} & \longrightarrow & \frac{H^1(\mathbf{Q}_p, \text{Ad}^*\bar{\rho}(1))}{H_f^1(\mathbf{Q}_p, \text{Ad}^*\bar{\rho}(1))} \end{array} \tag{130}$$

Since we have seen that j is onto, it follows that the induced maps of vertical kernels is a short exact sequence; that and (129) imply that

$$H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^*\rho(1)\right) \simeq \mathbf{Z}_p^\delta,$$

as claimed. □

8.10. Cohomological vanishing in the Taylor–Wiles method. In the Taylor–Wiles method we choose a set of primes Q such that, with $SQ = S \cup Q$, we have the following properties:

- (a) Q is a Taylor–Wiles datum of some level, and
- (b) The map $H^1(\mathbf{Z}[\frac{1}{SQ}], \text{Ad}\bar{\rho}) \rightarrow \frac{H^1(\mathbf{Q}_p, \text{Ad}\bar{\rho})}{H_f^1(\mathbf{Q}_p, \text{Ad}\bar{\rho})}$ is surjective, and the map $H^2(\mathbf{Z}[\frac{1}{SQ}], \text{Ad}\bar{\rho}) \rightarrow \prod_{v \in Q} H^2(\mathbf{Q}_p, \text{Ad}\bar{\rho})$ is injective.

(Recall that our local assumptions at S mean there is no local cohomology there: Section 6.2, assumption (e)).

Observe also that if Q is such a set of primes, and Q' is a further set satisfying (a) and (b), then certainly $Q \cup Q'$ satisfies (a) and (b) too. Indeed, the cohomological criteria of (b) are equivalent to asking that

$$H_f^1\left(\mathbf{Z}\left[\frac{1}{SQ}\right], \text{Ad}^*\bar{\rho}(1)\right) \rightarrow \prod_{v \in Q} H^1(\mathbf{Q}_v, \text{Ad}^*\bar{\rho}(1)) \tag{131}$$

is injective, and this is stable under enlarging Q (it is equivalent to the same injectivity on $H_f^1(\mathbf{Z}[\frac{1}{S}], -)$, since anything in the kernel would be unramified at Q).

Now, if we choose a system of such data Q_n of level n , we can (by passing to a subsequence and reindexing, for example, as in [14, Section 13.10]) achieve a new sequence Q_n which satisfy the ‘limit properties’ of Section 7.3.

DEFINITION 8.11. A sequence of Taylor–Wiles data Q_n of level n is called *convergent* if:

- Q_n have the cohomological properties stated in (b) above and,
- One can pass to the limit in the sense of Section 7.3, that is, there exists data \mathbf{R} , \mathbf{S} , \mathbf{C} , f_n , g_n , and so on, satisfying all the properties enumerated in Section 7.3.

In particular, any sequence of Taylor–Wiles data has a convergent subsequence, after reindexing the subsequence.

8.12. The tangent spaces to \mathbf{R} and \mathbf{S} . As in Section 7.4 both \mathbf{R} and \mathbf{S} are augmented to \mathbf{Z}_p :

$$\mathbf{S} \rightarrow \mathbf{R} \rightarrow \mathbf{Z}_p,$$

and the composite $S \rightarrow \mathbf{Z}_p$ is the standard augmentation of S . The kernel of these augmentations are denoted by $J \subset R$ and $I \subset S$.

First of all, we set let t_R be the ‘tangent space’ to R over \mathbf{Z}_p , that is to say

$$t_R \simeq \text{Hom}_*(R, \mathbf{Z}_p[\varepsilon]/\varepsilon^2),$$

where the subscript $*$ means that the homomorphism lifts the natural augmentation $R \rightarrow \mathbf{Z}_p$. Equivalently, R is the derivations of R/\mathbf{Z}_p into \mathbf{Z}_p , or the \mathbf{Z}_p -linear dual of J/J^2 .

We can make exactly the same definition for S . The surjection $S \twoheadrightarrow R$ induces a surjection $I/I^2 \twoheadrightarrow J/J^2$ and thus a natural injection $t_R \rightarrow t_S$ with saturated image (that is, split). We write W for the cokernel of the map on tangent spaces, so we have an exact sequence

$$t_R \hookrightarrow t_S \rightarrow W. \tag{132}$$

Then W is a free \mathbf{Z}_p -module of rank δ .

8.13. Tangent spaces to \bar{R}_n and S_n . We suppose now that Q_n are a convergent sequence (Section 8.10) of Taylor–Wiles data.

Recall that S, R are defined as ‘limits,’ roughly speaking, of rings $S_n \rightarrow \bar{R}_n$ that occur at level Q_n in the Taylor–Wiles process. We recall that \bar{R}_n is not the full (crystalline at p) deformation ring R_{Q_n} at level Q_n , but is a ‘very deep’ Artinian quotient of it.

These rings are also compatibly augmented over \mathbf{Z}/p^n , that is,

$$S_n \longrightarrow \bar{R}_n \longrightarrow \mathbf{Z}/p^n$$

(see the bottom row of (113), and compose with the reduction of the map $R_{\bar{\rho}} \rightarrow \mathbf{Z}_p$ which arises from our fixed automorphic representation Π).

LEMMA 8.14. *The map $R_{Q_n} \rightarrow \bar{R}_n$ induces an isomorphism upon applying $\text{Hom}_*(-, \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2)$, where Hom_* means that the map lifts the natural augmentations to \mathbf{Z}/p^n .*

Proof. Write $A := \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2$. A map $R_{Q_n} \rightarrow A$ gives rise to a deformation $\tilde{\rho}_n : G_{\mathbf{Q}} \rightarrow G^\vee(\mathbf{Z}/p^n)$ that lifts the modulo p^n reduction of ρ .

We want to show that $R_{Q_n} \rightarrow A$ must factor through \bar{R}_n . To do so we must show (see (112)) that the map dies on the $K(n)$ th power of the maximal ideal $\mathfrak{m}_{R_{Q_n}}$, and also that ρ_n automatically has inertial level $\leq n$ at primes in Q_n (see page 68 for definition).

Note that the maximal ideal (p, ε) of $\mathbf{Z}/p^n[\varepsilon]/\varepsilon^2$ satisfies $(p, \varepsilon)^{n+1} = 0$. Since $K(n) \geq n + 1$ by assumption, we only need verify that $\tilde{\rho}_n$ has inertial

level $\leq n$ at all Taylor–Wiles primes. After suitably conjugating, we can suppose $\tilde{\rho}_n|_{\mathbf{Q}_q}$ to have image in T^\vee (see Lemma 6.12). Restricted to inertia it takes image inside the kernel of $T^\vee(\mathbf{Z}/p^n[\varepsilon]/\varepsilon^2) \rightarrow T^\vee(\mathbf{Z}/p^n)$; this group has exponent p^n , and so $\tilde{\rho}_n$ has inertial conductor $\leq n$. \square

We also define

$$t_{R_n} = \text{Hom}_*(R_{Q_n}, \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2).$$

It does not matter whether we use R_{Q_n} or \bar{R}_n in this definition, as we just showed. Similarly we define

$$t_{S_n} = \text{Hom}_*(S_n, \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2).$$

There is a natural map $t_{R_n} \rightarrow t_{S_n}$ induced by $S_n \rightarrow \bar{R}_n$.

Finally define

$$W_n = \text{cokernel}(t_{R_n} \rightarrow t_{S_n}). \tag{133}$$

The maps $\mathbf{R} \rightarrow \bar{R}_n$ and $\mathbf{S} \rightarrow S_n$ give rise to an isomorphism of short exact sequences as below:

$$\begin{array}{ccccccc} 0 & \longrightarrow & t_{R_n} & \longrightarrow & t_{S_n} & \longrightarrow & W_n \longrightarrow 0 \\ & & \downarrow \alpha, \sim & & \downarrow \beta, \sim & & \downarrow \gamma, \sim \\ 0 & \longrightarrow & t_{\mathbf{R}}/p^n & \longrightarrow & t_{\mathbf{S}}/p^n & \longrightarrow & \mathbf{W}/p^n \longrightarrow 0 \end{array} \tag{134}$$

This requires some explanation.

First of all, we explain the maps. Note first of all that there is a natural map $t_{\mathbf{S}}/p^n \simeq \text{Hom}_*(\mathbf{S}, \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2)$, which is an isomorphism. Similarly for \mathbf{R} . This means that there are maps

$$\alpha : t_{S_n} \rightarrow t_{\mathbf{S}}/p^n, \quad \beta : t_{R_n} \rightarrow t_{\mathbf{R}}/p^n \tag{135}$$

that are induced by the projections $\mathbf{S} \rightarrow S_n$ and $\mathbf{R} \rightarrow \bar{R}_n$. This explains α, β ; and the map γ is the induced map on cokernels.

Next we see that α, β, γ are isomorphisms.

For S this is the assertion that homomorphism $\mathbf{S} \rightarrow \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2$ factors through S_n . Indeed, referring to the coordinate presentation (84) and (85), each element x_i must go to $a\varepsilon$ for some $a \in \mathbf{Z}/p^n$, and then $(1 + x_i)^{p^n}$ is carried to $(1 + a\varepsilon)^{p^n} = 1$. For R , we use the fact (114) that we can identify \bar{R}_n with $\mathbf{R} \otimes_{\mathbf{S}} S_n$. As above, any homomorphism $\mathbf{S} \rightarrow \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2$ (lifting the augmentation) factors through S_n , and in particular any homomorphism $\mathbf{R} \rightarrow \mathbf{Z}/p^n$, lifting the natural one, factors through \bar{R}_n .

Now, the bottom row is exact, by definition and the freeness of \mathbf{W} . The top row is exact at the left because the vertical maps α, β are isomorphisms, and exact at the right by definition. Then it follows that γ is an isomorphism too.

This concludes the explanation of diagram (134).

8.15. Tangent spaces to S_n reinterpreted. Let us reinterpret the tangent space to S_n in a few different (canonical) ways.

By (88) we have an isomorphism

$$\mathfrak{t}_{S_n} \simeq \text{Ext}_{S_n}^1(\mathbf{Z}/p^n, \mathbf{Z}/p^n). \tag{136}$$

Now S_n was, by definition (Section 6.4), the \mathbf{Z}/p^n -group algebra of the group T_n ; thus, from the above equation, we get a canonical isomorphism

$$\mathfrak{t}_{S_n} \simeq H^1(T_n, \mathbf{Z}/p^n). \tag{137}$$

Next we connect \mathfrak{t}_{S_n} to Galois cohomology. Recall from Lemma 6.12 that, for any $q \in Q_n$, a deformation of $\rho|_{G_q}$ can be conjugated to lie in the torus, and in particular factors through the profinite completion of \mathbf{Q}_q^\times . Now we can identify $H^1(G_q, \text{Ad } \rho_n)$ with the set of lifts of ρ_n to $G^\vee(\mathbf{Z}/p^n[\varepsilon]/\varepsilon^2)$, modulo conjugacy. This lift sends tame inertia to the kernel of reduction modulo ε . In particular, having fixed an element of $T^\vee(k)$ conjugate to $\bar{\rho}(\text{Frob}_q)$, we get a canonical isomorphism

$$\frac{H^1(\mathbf{Q}_q, \text{Ad } \rho_n)}{H_{\text{ur}}^1(\mathbf{Q}_q, \text{Ad } \rho_n)} \simeq \text{Hom}(\mathbf{F}_q^\times, \text{Lie}(T^\vee) \otimes \mathbf{Z}/p^n). \tag{138}$$

Identifying the Lie algebra with $X_*(T^\vee)$, we get

$$\begin{aligned} \frac{H^1(\mathbf{Q}_q, \text{Ad } \rho_n)}{H_{\text{ur}}^1(\mathbf{Q}_q, \text{Ad } \rho_n)} &\simeq \text{Hom}(\mathbf{F}_q^\times/p^n, \mathbf{Z}/p^n) \otimes X_*(T^\vee) \\ &\simeq \text{Hom}(\underbrace{X_*(\mathbf{A}) \otimes \mathbf{F}_q^\times/p^n}_{\mathbf{A}(\mathbf{F}_q)/p^n}, \mathbf{Z}/p^n) \end{aligned} \tag{139}$$

(here the subscript ‘ur’ means unramified) and thus, from (81) and (137)

$$\mathfrak{t}_{S_n} \simeq \bigoplus_{q \in Q_n} \frac{H^1(\mathbf{Q}_q, \text{Ad } \rho_n)}{H_{\text{ur}}^1(\mathbf{Q}_q, \text{Ad } \rho_n)} \tag{140}$$

where we emphasize that the isomorphism depends on the choice of an element of $T^\vee(k)$ conjugate to Frobenius at q , for each $q \in Q_n$.

There is an isomorphism similar to (136) for \mathfrak{t}_S ; in particular, $\mathfrak{t}_S \simeq \text{Ext}_S^1(\mathbf{Z}_p, \mathbf{Z}_p)$ and more usefully

$$\mathfrak{t}_S/p^n \simeq \text{Ext}_S^1(\mathbf{Z}_p, \mathbf{Z}/p^n).$$

The composite

$$\text{Ext}_{S_n}^1(\mathbf{Z}/p^n, \mathbf{Z}/p^n) \longrightarrow \text{Ext}_S^1(\mathbf{Z}/p^n, \mathbf{Z}/p^n) \longrightarrow \text{Ext}_S^1(\mathbf{Z}_p, \mathbf{Z}/p^n). \tag{141}$$

Gives the natural identification—map β from (134)—of $\text{Ext}_{S_n}^1(\mathbf{Z}/p^n, \mathbf{Z}/p^n) = \mathfrak{t}_{S_n}$ with $\text{Ext}_S^1(\mathbf{Z}_p, \mathbf{Z}/p^n) = \mathfrak{t}_S/p^n$.

8.16. V_n and Galois cohomology. We exhibit now a canonical surjection

$$t_{S_n} \rightarrow V/p^n. \tag{142}$$

In fact, this surjection uses no more than the fact that Q_n is a Taylor–Wiles datum. If we suppose that Q_n are a convergent sequence (Section 8.10) of Taylor–Wiles data, we see that this actually descends to an isomorphism

$$W_n \simeq V/p^n, \tag{143}$$

that is, W_n (defined in (133)) is isomorphic to $\text{Hom}(H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho(1)), \mathbf{Z}/p^n)$.

As before, let SQ_n be the union of the set S with the set Q_n . Examine:

$$\begin{array}{ccccc} \text{Hom}_*(R_{Q_n}, \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2) & \longrightarrow & \text{Hom}_*(S_n, \mathbf{Z}/p^n[\varepsilon]/\varepsilon^2) & \longrightarrow & W_n \\ \downarrow \sim & & \downarrow \sim & & \downarrow = \\ H_f^1(\mathbf{Z}[\frac{1}{SQ_n}], \text{Ad} \rho_n) & \xrightarrow{\varphi} & \prod_{v \in Q_n} \frac{H^1(\mathbf{Q}_v, \text{Ad} \rho_n)}{H_{\text{ur}}^1(\mathbf{Q}_v, \text{Ad} \rho_n)} & \longrightarrow & W_n \end{array} \tag{144}$$

The first vertical map is just the computation of tangent spaces to deformation rings (working over \mathbf{Z}/p^n rather than a field), the second vertical map is (140), and φ is restriction in Galois cohomology.

There is now a natural pairing:

$$H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^* \rho_n(1)\right) \times W_n \longrightarrow \mathbf{Z}/p^n \tag{145}$$

which, we emphasize again, depends on the choice of toral elements conjugate to Frobenius at each prime in Q_n . To be explicit, an element of W_n is represented by a collection

$$(\beta_v) \in \prod_{v \in Q_n} \frac{H^1(\mathbf{Q}_v, \text{Ad} \rho_n)}{H_{\text{ur}}^1(\mathbf{Q}_v, \text{Ad} \rho_n)},$$

modulo image(φ); to pair $\alpha \in H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_n(1))$ with $(\beta_v)_{v \in Q_n}$ we take the sum of local pairings

$$(\alpha, (\beta_v)_{v \in Q_n}) \mapsto \sum (\alpha_v, \beta_v)_v \tag{146}$$

where the local pairing is defined by restricting α to \mathbf{Q}_v and using local reciprocity. This pairing (146) is well defined because each α_v , that is, the restriction of α to \mathbf{Q}_v , is actually unramified. Moreover, if the collection (β_v) come from a class in $H_f^1(\mathbf{Z}[\frac{1}{SQ_n}], \text{Ad} \rho_n)$, the value of (146) is zero by global reciprocity: our local assumptions means that the local pairings for $v \in S$

vanishes. Thus the pairing (146) descends to the quotient of $\prod_{v \in Q_n} H^1$ by $\text{image}(\varphi)$. This concludes our discussion of (145).

We have also seen (Section 8.13, and similar arguments to Lemma 8.9) that both W_n and $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_n(1))$ are free \mathbf{Z}/p^n -modules of dimension δ . Let us check that (145) is a perfect pairing of \mathbf{Z}/p^n modules, that is, the map

$$W_n \rightarrow H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^* \rho_n(1)\right)^\vee$$

is an isomorphism, where \vee means homomorphisms to \mathbf{Z}/p^n . Since both sides have the same size, it is enough to check that the map is surjective, and thus enough to show that the induced map

$$t_{S_n} \rightarrow H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^* \bar{\rho}(1)\right)^\vee \tag{147}$$

is surjective, where \vee now means homomorphisms to \mathbf{Z}/p .

Now the Taylor–Wiles set Q_n is chosen (131) so that $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \bar{\rho}(1)) \hookrightarrow \prod_{v \in Q_n} H^1(\mathbf{Q}_v, \text{Ad}^* \bar{\rho}(1))$. The image of this map consists of classes unramified at Q_n , so we also have

$$H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^* \bar{\rho}(1)\right) \hookrightarrow \prod_{v \in Q_n} H^1(\mathbf{Q}_v, \text{Ad}^* \bar{\rho}(1))^{\text{ur}}.$$

When we dualize this, and apply local duality at primes in Q_n , we get the surjectivity of (147).

8.17. The injection $H^1(T_q) \hookrightarrow \mathcal{H}_q^{(1)}$. In this section, we suppose that Q_n is a Taylor–Wiles datum of level n , but do not assume that it is part of a convergent sequence (Section 8.10) of Taylor–Wiles data. Let $q \in Q_n$, thus equipped with $\text{Frob}_q^T \in T^\vee(k)$. We work exclusively with $S = \mathbf{Z}/p^n$ coefficients. As before, \mathcal{H}_q denotes the local derived Hecke algebra at q , and $\mathcal{H}_q^{(1)}$ denotes its degree 1 component. Let $T_q = \mathbf{A}(\mathbf{F}_q)/p^n$.

We are going to describe the map (123), which is necessary for the formulation of the Theorem. More precisely, we are going to describe a map

$$\theta : H^1(T_q) \rightarrow (\mathcal{H}_q^{(1)})_{\mathfrak{m}}, \tag{148}$$

where the subscript means that we complete at the ideal of the Hecke algebra (that is, the degree zero component of \mathcal{H}_q) corresponding to \mathfrak{m} .

The easiest way to think about θ is probably through the following property: for each $\alpha \in H^1(T_q)$, the action of $\theta(\alpha)$ on $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$ is thus:

Pullback to $Y_0(q)$, project to Frob_q^T -eigenspace, cup with α , pushdown to $Y(1)$, (149)

where the projection is done with reference to the splitting of Corollary 6.7.

The formal definition of the map θ is given in (154), and the validity of (149) will follow from the Lemma below.

The Satake isomorphism of Section 3 gives

$$\mathcal{H}_q \xrightarrow{\sim} (S[X_*] \otimes H^*(\mathbf{A}(\mathbf{F}_q)))^W$$

and, in particular, with \mathbf{Z}/p^n coefficients the map $H^1(T_q) \rightarrow H^1(\mathbf{A}(\mathbf{F}_q))$ is an isomorphism, so

$$\mathcal{H}_q^{(1)} \xrightarrow{\sim} (S[X_*] \otimes H^1(T_q))^W.$$

Now $\text{Frob}_q^T \in T^\vee(k)$ gives a map $X_*(\mathbf{A}) = X^*(T^\vee) \rightarrow k^\times$, that is, it gives rise to a character $\chi_{\text{Frob}_q^T} : S[X_*] \rightarrow k$. The pullback of this to $S[X_*]^W$ defines the maximal ideal \mathfrak{m} (using the Satake isomorphism). Let us denote by $\tilde{\mathfrak{m}}$ the extension of the ideal \mathfrak{m} back to $S[X_*]$; we caution that it is no longer maximal, and rather it cuts out Frob_q^T together with all its W -conjugates. We have an identification of completions

$$S[X_*]_{\tilde{\mathfrak{m}}} \cong \bigoplus_{w \in W} S[X_*]_{w\chi}, \tag{150}$$

where we have denoted by $S[X_*]_{w\chi}$ the completion of $S[X_*]$ at the maximal ideal that is the kernel of $w\chi$.

Next the natural inclusion

$$(S[X_*] \otimes H^*(T_q))^W \hookrightarrow S[X_*] \otimes H^*(T_q) \tag{151}$$

induces the first map of

$$(S[X_*] \otimes H^*(T_q))_{\mathfrak{m}}^W \xrightarrow{(151)} S[X_*]_{\tilde{\mathfrak{m}}} \otimes H^*(T_q) \xrightarrow{(150)} S[X_*]_{\chi} \otimes H^*(T_q). \tag{152}$$

The composite map of (152) is an isomorphism, and thus by composing with the Satake isomorphism we get an isomorphism

$$(\mathcal{H}_q^{(1)})_{\mathfrak{m}} \xrightarrow{\sim} S[X_*]_{\chi} \otimes H^1(T_q). \tag{153}$$

We then define the map $\theta : H^1(T_q) \rightarrow (\mathcal{H}_q^{(1)})_{\mathfrak{m}}$ by the rule

$$\theta : h \in H^1(T_q) \mapsto 1 \otimes h \in S[X_*]_{\chi} \otimes H^1(T_q) \xrightarrow{(153)^{-1}} (\mathcal{H}_q^{(1)})_{\mathfrak{m}} \tag{154}$$

where the superscript 1 refers to cohomological degree, and the final map is the inverse of (153). Note that this embedding depends on the choice of Frob_q^T ; if we replace Frob_q^T by $w\text{Frob}_q^T$ then the embedding is modified by means of the natural action of w on T_q .

This concludes the description of the map $H^1(T_q) \rightarrow \mathcal{H}_q^{(1)}$. We now want to justify (149). For this we describe an explicit preimage of $\theta(\alpha)$ under the map

$$(\mathbf{H}_{KI} \otimes \mathcal{H}_I \otimes \mathbf{H}_{IK}) \rightarrow \mathcal{H}_K \tag{155}$$

in the case of the group $\mathbf{G}(\mathbf{Q}_q)$, where notation is as before (see, for example, the proof of Theorem 7.6); note that \mathcal{H}_K coincides with what was previously called \mathcal{H}_q . Observe that everything here is a compatibly a module under the center of \mathbf{H}_{II} , which is identified (Section 4.3) with $S[X_*]^W$. In particular, it makes sense to complete at \mathfrak{m} .

Now let $e_\lambda \in \mathbf{H}_{II}$ correspond to the characteristic function of $I_q \lambda I_q$, where $\lambda \in X_*$. Then $\lambda \mapsto e_\lambda$ defines an embedding $S[X_*] \rightarrow \mathbf{H}_{II}$, and, completing, an embedding

$$S[X_*]_{\widehat{\mathfrak{m}}} \rightarrow (\mathbf{H}_{II})_{\mathfrak{m}}. \tag{156}$$

As before (Section 4.2) we let e_K be the characteristic function of K_q , divided by its measure; if we write e_w for the characteristic function of $I_q w I_q$, we have $e_K = \frac{1}{|W|} \sum_{w \in W} e_w$. It can be considered, as in Section 4.2, as an element of \mathbf{H}_{KI} or \mathbf{H}_{IK} . Also $|W|e_K e_\lambda \in \mathbf{H}_{KI}$ (that is, the product of $|W|e_K \in \mathbf{H}_{KI}$ with $e_\lambda \in \mathbf{H}_{II}$) corresponds to the characteristic function of $I_q \lambda K_q$.

LEMMA 8.18. *Let $\Theta \in S[X_*]_{\widehat{\mathfrak{m}}}$ be chosen so that it projects under (150) to the identity in $S[X_*]_{\mathfrak{X}}$ and to zero in all $S[X_*]_{w\mathfrak{X}}$, for $w \in W$ not the identity. We use the same letter for its image $\Theta \in (\mathbf{H}_{II})_{\mathfrak{m}}$ under (156).*

For $h \in H^1(T_q)$ let $\langle h \rangle$ be the associated element of $\mathcal{H}_I^{(1)}$, that is, $\langle h \rangle$ is supported on the identity double coset of I , and the associated cohomology class is obtained from h by means of the restriction isomorphism $H^1(I) \xrightarrow{\sim} H^1(T_q)$.

Then the \mathfrak{m} -completion of (155) sends $|W|e_K \Theta \otimes \langle h \rangle \otimes e_K$ to $\theta(h)$.

As above, the product $e_K \Theta$ is understood as the product of $e_K \in \mathbf{H}_{KI}$ with $\Theta \in \mathbf{H}_{II}$. In words, this amounts precisely to the description (149) for θ , taking into account that Θ realizes precisely the projection onto the Frob_q^T component for the splitting (98).

Proof. Consider the map (no completions, at the moment)

$$(\mathbf{H}_{KI} \otimes \mathcal{H}_I^{(1)} \otimes \mathbf{H}_{IK}) \rightarrow \mathcal{H}_K^{(1)} \rightarrow (S[X_*] \otimes H^1(T_q))^W. \tag{157}$$

We show it sends $A := e_K e_\lambda \otimes \langle h \rangle \otimes e_K$, for λ dominant, to the ‘ W -average’ of $\lambda \otimes h$, that is, $|W|^{-1} \sum_{w \in W} w \cdot (\lambda \otimes h)$.

The claimed result will follow easily from this: Consider an element $\Theta' = \sum c_\lambda \lambda \in S[X_*]$ with the property that c_λ is nonzero only for λ dominant, and also $\Theta \equiv \Theta'$ modulo some high power of \tilde{m} . Our claim implies that $|W|e_K \Theta' \otimes \langle h \rangle \otimes e_K$ is sent to the sum of Weyl translates of $\sum c_\lambda (\lambda \otimes h)$. The image of $\sum c_\lambda (\lambda \otimes h)$ in $\bigoplus_{w \in W} S[X_*]_{w\chi} \otimes H^1(T_q)$ is very close to $1 \otimes h$ in the $w = 1$ factor, and very close to zero in the other factors; after summing over W , its projection to the $w = 1$ factor remains very close to $1 \otimes h$. Here ‘very close’ is taken in the topology of the complete local rings $S[X_*]_{w\chi}$. In other words, $\theta(h)$ and the image of $|W|e_K \Theta' \otimes \langle h \rangle \otimes e_K$ under (155) and (153) are very close; in the limit, this shows the desired result.

We now consider everything in the ‘function model’ of Section 2.3. Let $a_1 \in H_{KI}, a_2 \in H_{IK}$ be the images of $e_K e_\lambda, e_K$ in the function model. Then a_1 corresponds to the function sending (xK, yI) to $|W|^{-1}$ precisely when $Iy^{-1}xK = I\lambda K$, equivalently $Kx^{-1}yI = K\bar{\lambda}I$, where we write $\bar{\lambda} \in X_*$ for the negative of λ . Also a_2 corresponds to the function (xI, yK) which is 1 exactly when $xK = yK$. Moreover, the function $\langle h \rangle$ is supported on the diagonal in G/I and sends (I, I) to $h \in H^1(T_q) \simeq H^1(I)$.

The second map of (157) is given by restricting arguments to the torus, and restricting cohomology classes to T_q . We can compute this restriction using the ‘localization’ results of Section 4.6; these results assert that restriction to the torus actually preserves multiplication.

We restrict to the torus. Suppose that $\mu \in X_*$ is dominant, with negative $\bar{\mu} \in X_*$. We compute:

$$A(K, \bar{\mu}K) = \sum_{y \in \bar{W}} a_1(K, yI) \langle h \rangle (yI, yI) a_2(yI, \bar{\mu}K).$$

The first term is nonzero only for those y satisfying $KyI = K\bar{\lambda}I$, that is, $y \in \bar{W} \cap K\bar{\lambda}I$. This implies that $y = w\bar{\lambda}$, with $w \in W$. So this equals

$$= |W|^{-1} \sum_{w \in W} \langle h \rangle (w\bar{\lambda}I, w\bar{\lambda}I) a_2(w\bar{\lambda}I, \bar{\mu}K).$$

The final term is nonzero (and equals 1) exactly when $\bar{\mu}K = w\bar{\lambda}K$. Recall that both λ and μ are dominant. Thus this only happens if $\lambda = \mu$ and $w \in W_\mu$:

$$A(K, \bar{\mu}K) = |W|^{-1} \delta_{\lambda\mu} \sum_{w \in W_\mu} \langle h \rangle (w\bar{\lambda}I, w\bar{\lambda}I). \tag{158}$$

On the right, $\langle h \rangle (w\lambda I, w\lambda I) \in H^1(T_q)$ equals $w \cdot h \in H^1(T_q)$.

Now, the image of A under the derived Satake map is the W -invariant element of the derived Hecke algebra of the torus defined by

$$\mu \mapsto \text{restriction to } T_q \text{ of } A(\mu K, K).$$

As we have just computed, for μ dominant, $A(\mu K, K) = A(K, \bar{\mu}K)$ is nonvanishing only when $\mu = \lambda$, where its value is $|W|^{-1} \sum_{w \in W_\mu} wh$. Therefore the image of A under (157) is the W -average of $\lambda \otimes h$ as claimed. \square

8.19. Producing an action of V/p^n on the cohomology at level 1. Let Q_n be a convergent sequence of Taylor–Wiles data (Section 8.10). For each integer n we produce an action of V/p^n on automorphic cohomology $H^*(Y(1), \mathbf{Z}/p^n)_\mathfrak{m}$. *A priori* these actions will not be guaranteed to be compatible with one another; later we see at least that they ‘converge as $n \rightarrow \infty$ ’ to give an action of V on $H^*(Y(1), \mathbf{Z}_p)_\mathfrak{m}$.

More exactly, we begin by constructing an action of \mathfrak{t}_{S_n} , then prove (the Lemma below) that it factors through W_n , and finally we have identified $W_n \simeq V/p^n$ in (143). This gives the desired action, and we discuss the ‘convergence as $n \rightarrow \infty$ ’ in the next section.

Thus, let Q_n be a convergent sequence of Taylor–Wiles data.

We have

$$\mathfrak{t}_{S_n} \stackrel{(137)}{\cong} H^1(T_n, \mathbf{Z}/p^n) \stackrel{(148)}{\hookrightarrow} \text{degree 1 component of } \bigotimes_{q \in Q_n} (\mathcal{H}_q)_\mathfrak{m}, \quad (159)$$

where the Hecke algebras are taken with \mathbf{Z}/p^n coefficients. The composite embedding will be denoted

$$\iota_{Q_n} : \mathfrak{t}_{S_n} \rightarrow \bigotimes_{q \in Q_n} (\mathcal{H}_q)_\mathfrak{m}. \quad (160)$$

This gives rise to an action of \mathfrak{t}_{S_n} by degree +1 endomorphisms of automorphic cohomology $H^*(Y(1), \mathbf{Z}/p^n)_\mathfrak{m}$, whose explicit description is essentially that already given in (149), just replacing the role of one prime by many.

The embedding (137) and so also this action depends on the choice of elements $\text{Frob}_q^T \in T^\vee(k)$ for each prime $q \in Q_n$. Should we modify Frob_q^T by an element $w_q \in W$, the Weyl group, the action of \mathfrak{t}_{S_n} is also modified (see comments after (154)) by the action of w_q in the obvious way.

LEMMA 8.20. *Let Q_n be a convergent sequence of Taylor–Wiles data, as in Definition 8.11. Then, for each n , the action of \mathfrak{t}_{S_n} on $H^*(Y(1), \mathbf{Z}/p^n)_\mathfrak{m}$ (via ι_{Q_n}) is trivial on the image of \mathfrak{t}_{R_n} , and thus factors through the map $\mathfrak{t}_{S_n} \rightarrow W_n$.*

Proof. Consider the diagrams (118) of our previous story, now with identifications with \mathfrak{t}_S and \mathfrak{t}_{S_n} included:

$$\begin{array}{ccc}
 H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}_p)) & \times \underbrace{\mathrm{Ext}_S^1(\mathbf{Z}_p, \mathbf{Z}_p)}_{\mathfrak{t}_S} \ni \tilde{v} & \longrightarrow H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}_p)) \\
 \downarrow = & \downarrow & \downarrow \\
 \tilde{x} \in H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}_p)) & \times \mathrm{Ext}_S^1(\mathbf{Z}_p, \mathbf{Z}/p^n) \ni v & \longrightarrow H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}/p^n)) \\
 \downarrow v & \uparrow U & \downarrow = \\
 x \in H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}/p^n)) & \times (\mathrm{Ext}_S^1(\mathbf{Z}/p^n, \mathbf{Z}/p^n)) & \longrightarrow H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}/p^n)) \\
 \uparrow \sim & \uparrow & \uparrow \sim \\
 H^*(\mathrm{Hom}_{S_n}(\mathbf{C} \otimes_S S_n, \mathbf{Z}/p^n)) & \times \underbrace{(\mathrm{Ext}_{S_n}^1(\mathbf{Z}/p^n, \mathbf{Z}/p^n))}_{\mathfrak{t}_{S_n}} & \longrightarrow H^*(\mathrm{Hom}_{S_n}(\mathbf{C} \otimes_S S_n, \mathbf{Z}/p^n)).
 \end{array}
 \tag{161}$$

In particular, let $x \in H^*(\mathrm{Hom}_{S_n}(\mathbf{C} \otimes_S S_n, \mathbf{Z}/p^n))$ be liftable to $\tilde{x} \in H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}_p))$; let $v \in \mathfrak{t}_{S_n} \simeq \mathfrak{t}_{S/p^n}$ be lifted to $\tilde{v} \in \mathfrak{t}_S$. Then the image of (x, v) in the bottom row is obtained from projecting the image of (\tilde{x}, \tilde{v}) at the top row.

Let us recall from Section 7.3 part (g) that \mathbf{C} is quasi-isomorphic to \mathbf{R} as an S -module in a single degree. Thus we can explicitly compute what goes on in the top row. This explicit computation (see Lemma B.1 in Appendix Section Appendix B) shows that any element $\tilde{v} \in \mathfrak{t}_S$ that lies in the image of \mathfrak{t}_R acts *trivially* on $H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}_p))$.

For later use, note that these explicit computations also show that

$$H^*(\mathrm{Hom}_S(\mathbf{C}, \mathbf{Z}_p)) \text{ is free over } \wedge^* \mathfrak{t}_S / \mathfrak{t}_R.
 \tag{162}$$

From (134) it then follows that $\mathrm{image}(\mathfrak{t}_{R_n} \rightarrow \mathfrak{t}_{S_n})$ acts trivially on

$$H^*(\mathrm{Hom}_{S_n}(\mathbf{C} \otimes_S S_n, \mathbf{Z}/p^n)) \simeq H^*(Y_0(Q_n), \mathbf{Z}/p^n)_{\mathfrak{m}, \mathrm{Frob}^T},$$

where the action of $\mathfrak{t}_{S_n} = H^1(T_n, \mathbf{Z}/p^n)$ is by cup product, as in (119).

By (149) this means that the action of $\iota_{Q_n}(\mathfrak{t}_{S_n})$ on $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$ is trivial on $\mathrm{image}(\mathfrak{t}_{R_n} \rightarrow \mathfrak{t}_{S_n})$. Thus, this action of \mathfrak{t}_{S_n} on $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$ factors through W_n as claimed. \square

8.21. Summary. Let us summarize more carefully what we have said to date:

For any Taylor–Wiles datum Q_n we have an action of $\mathfrak{t}_{S_n} = \mathrm{Hom}(T_n, \mathbf{Z}/p^n)$ on $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$ constructed via an embedding

$$\iota_{Q_n} : \mathrm{Hom}(T_n, \mathbf{Z}/p^n) \hookrightarrow \text{derived Hecke algebra}$$

(see (159)). On the other hand, we have a surjective morphism (see Section 8.16)

$$f_{Q_n} : \text{Hom}(T_n, \mathbf{Z}/p^n) \twoheadrightarrow \mathbf{V}/p^n.$$

These constructions, for a given n , depend only on Q_n ; they do not involve the Taylor–Wiles limit process.

DEFINITION 8.22. We say that a Taylor–Wiles datum Q_n of level n is *strict of level n* (or just *strict*) if the map

$$\mathfrak{t}_{S_n} \xrightarrow{\iota_{Q_n}} \text{End } H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$$

factors through f_{Q_n} . Thus, a strict Taylor–Wiles datum of level n gives rise to an action of \mathbf{V}/p^n on $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$.

What we have proved, then, amounts to the following:

LEMMA 8.23. *If the Q_n are a convergent sequence of Taylor–Wiles data (Definition 8.11), then each Q_n is strict, in the sense of Definition 8.22.*

Note we do not know that the resulting actions of \mathbf{V}/p^n are compatible for different n , in any sense.

8.24. Dependence of our construction on choices. We now study dependence on choices. Using the results of this Section, we conclude the proof of Theorem 8.5 in Section 8.26.

First we discuss a minor point, the choice of Frob_q^T s: Suppose we choose two different such choices for a given set Q_n , differing by the action of $w \in W_s$. (Recall from Section 6.3 that W_s is just a product of copies of the Weyl group, one copy for each prime in Q_n). Then the actions of \mathfrak{t}_{S_n} on cohomology differ by the action of $w \in W_s$ (comment after (154)). Also, $w : \mathfrak{t}_{S_n} \rightarrow \mathfrak{t}_{S_n}$ is compatible with the pairings previously constructed, that is, this diagram commutes

$$\begin{array}{ccc} \mathfrak{t}_{S_n} & \times & H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_n(1)) \xrightarrow{\text{Frob}_q^T} \mathbf{Z}/p^n & (163) \\ \downarrow w & & \downarrow = & \\ \mathfrak{t}_{S_n} & \times & H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_n(1)) \xrightarrow{w \text{Frob}_q^T} \mathbf{Z}/p^n & \end{array}$$

This shows that the action of \mathbf{V}/p^n on $H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}}$ did not depend on the choice of Frob_q^T s for $q \in Q_n$.

We now discuss the more serious issue of choice of Taylor–Wiles data.

LEMMA 8.25. *Given two sequences Q_n, Q'_n of strict Taylor–Wiles data, there is a subsequence \mathcal{J} of the integers with following property:*

For each $k \geq 1$, there is j_0 such that, for each $j \in \mathcal{J}, j \geq j_0$, the two actions V on $H^(Y(1), \mathbf{Z}/p^k)_m$ —arising from reducing modulo p^k the ‘ Q_j -action’ and the ‘ Q'_j -action’—coincide with one another. (Recall that we are supposing that $H^*(Y(1), \mathbf{Z}_p)$ is free over \mathbf{Z}_p .)*

Proof. It will be convenient to relabel the sequences of strict Taylor–Wiles data as $Q_n^{(1)}, Q_n^{(2)}$. It will be harmless to suppose that the sets of primes underlying $Q_n^{(1)}$ and $Q_n^{(2)}$ are disjoint (otherwise, we can, for example, just compare both of them with a third set, disjoint from both of them).

We compare them both to $Q_n := Q_n^{(1)} \amalg Q_n^{(2)}$ (with the obvious choice of Frob_q^T for $q \in Q_n$). Of course Q_n is bigger than either $Q_n^{(1)}$ or $Q_n^{(2)}$. However it still a sequence of Taylor–Wiles data. Let $T_n, T_n^{(1)}, T_n^{(2)}$ be the analogs of T_n for $Q_n, Q_n^{(1)}, Q_n^{(2)}$ respectively; then $T_n = T_n^{(1)} \times T_n^{(2)}$, and correspondingly $t_{S_n} = t_{S_n}^{(1)} \oplus t_{S_n}^{(2)}$, where $t_{S_n} = \text{Hom}(T_n, \mathbf{Z}/p^n)$, and so on.

We have a diagram

$$\begin{array}{ccc}
 t_{S_n}^{(1)} & \xrightarrow{\quad} & t_{S_n} \xleftarrow{\quad} t_{S_n}^{(2)} \\
 & \searrow^{\alpha^{(1)}} & \downarrow \alpha \\
 & & H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_n(1))^\vee
 \end{array} \tag{164}$$

where all the α -maps are as in Section 8.16.

The upper maps are compatible for the actions on cohomology previously defined (Section 8.19), and everything maps compatibly to the bottom group $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_n(1))^\vee$. Moreover the action of $t_{S_n}^{(1)}$ and $t_{S_n}^{(2)}$ on mod p^n cohomology factors through the bottom row by the assumed strictness. However, we do not know that the action of t_{S_n} factors through α .

What is missing is control of the deformation ring after adding level Q_n . To obtain this, we must run now the Taylor–Wiles limit process for Q_n . That involves passing to a subsequence. In other words, all we are guaranteed is that there is a subsequence n_j such that (Q_{n_j}, j) form a convergent sequence of Taylor–Wiles data of level j . It is possible that n_j is very much larger than j . Our prior analysis of convergent data (Lemma 8.23) implies that the action of $\text{Hom}(T_{n_j}, \mathbf{Z}/p^j)$ on mod p^j cohomology of $Y(1)$ factors through V/p^j , or to say it explicitly:

(*) The action of $t_{S_{n_j}}$ on $H^*(Y(1), \mathbf{Z}/p^j)_m$, via its embedding

ι_{Q_n} into the derived Hecke algebra, followed by reduction to \mathbf{Z}/p^j coefficients, factors through the map

$$f_{Q_n} : \mathfrak{t}_{S_n} \rightarrow H_f^1\left(\mathbf{Z}\left[\frac{1}{S}\right], \text{Ad}^* \rho_j(1)\right)^\vee.$$

The proof of the Lemma easily follows. We take \mathcal{J} to be the subsequence of n_j s. Let k be as in the Lemma. Take $n = n_j$, for any $j \geq k$, and take $w^{(i)} \in \mathfrak{t}_{S_n}^{(i)}$ that have the same image in $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_n(1))^\vee$. The images of $w^{(i)}$ in \mathfrak{t}_{S_n} have the same image in $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^* \rho_j(1))^\vee$, (we are using the fact that the map $H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad} \rho_n(1)) \rightarrow H^1(\mathbf{Z}[\frac{1}{S}], \text{Ad} \rho_j(1))$ is surjective, by discussion before (126)) and therefore they act the same way on mod p^k cohomology by (*) above. □

8.26. Conclusion of the proof of Theorem 8.5. Let us call a sequence of Taylor–Wiles data Q_n of level n (where we do not require n to vary through all the integers, but possibly some subsequence thereof) **V-convergent** if:

- Each Q_n is strict (Definition 8.22) thus giving an action of \mathbf{V} on $H^*(Y(1), \mathbf{Z}/p^n)_\mathfrak{m}$.
- The actions converge to an action of \mathbf{V} on $H^*(Y(1), \mathbf{Z}_p)_\mathfrak{m}$. In other words, if we fix k , the action of \mathbf{V} on $H^*(Y(1), \mathbf{Z}/p^k)_\mathfrak{m}$ arising from reducing the Q_n -action is eventually constant.

By Lemma 8.23 and passing to a further subsequence, we see that **V**-convergent sequences exist. By Lemma 8.25, if Q, Q' are two **V**-convergent sequences, the resulting actions of \mathbf{V} on cohomology coincide. Thus at this point we have defined an action of \mathbf{V} on cohomology that is independent of choices, namely, the action arising from any **V**-convergent sequence. This action has the following property:

(†): For any sequence Q_n of Taylor–Wiles data, there is a subsequence Q_{n_r} such that (for every r) the following two actions of $\mathfrak{t}_{S_{n_r}}$ on $H^*(Y(1), \mathbf{Z}/p^r)_\mathfrak{m}$ coincide:

- The action via $\iota : \mathfrak{t}_{S_{n_r}} \rightarrow$ derived Hecke algebra with \mathbf{Z}/p^r coefficients (see (160)).
- The action obtained from the **V**-action, via $f : \mathfrak{t}_{S_{n_r}} \twoheadrightarrow \mathbf{V}/p^r$ (see (142)).

To see this, we first pass from Q_n to a convergent subsequence (Q_{m_r}, r) , where we regard Q_{m_r} as having level r ; by Lemma 8.23 this means that Q_{m_r} is a strict datum of level r . We then pass to a further subsequence m'_r to extract a V -convergent sequence; this gives the assertions above, but with \mathbf{Z}/p^r and V/p^r replaced by $\mathbf{Z}/p^{k(r)}$, $V/p^{k(r)}$ where $k(r) \rightarrow \infty$ with r . Passing to a further subsequence gives the desired result.

Proof. (of Theorem 8.5, using (†)). We have already constructed an action of V ; let us prove, by contradiction, that it has property (*) from the theorem. Suppose that there is an integer A and an infinite sequence of primes $q_n \equiv 1$ modulo p^n such that the pullback of the action via $f_{q_n, A} : H^1(T_{q_n}, \mathbf{Z}/p^A) \rightarrow V/p^A$ fails to coincide with the action of $H^1(T_{q_n}, \mathbf{Z}/p^A)$ via the embedding $\iota_{q_n, A}$ into the derived Hecke algebra with \mathbf{Z}/p^A coefficients. We can choose a Taylor–Wiles system Q_n containing q_n and then get a contradiction to (†) as soon as $r > A$. This proves (*).

Now let us show that the image of $\wedge^* V$ in endomorphisms of cohomology coincides with the global derived Hecke algebra.

Refer to the diagram (161), constructed with a convergent sequence of Taylor–Wiles data Q_n . We only use a subsequence of n s which is V -convergent. Consider for $n \geq k$ the map

$$W = \mathfrak{t}_S/\mathfrak{t}_R \xrightarrow{(134)} \mathfrak{t}_{S_n}/\mathfrak{t}_{R_n} \xrightarrow{(142)} V/p^k. \tag{165}$$

By (134) and the discussion after (142), the composite actually gives an isomorphism

$$W/p^k \simeq V/p^k. \tag{166}$$

For fixed k and large n , the map (165) is independent of n : Choose $\tilde{v} \in \mathfrak{t}_S$. Let v_n, v_m be its image in $\mathfrak{t}_{S_n}, \mathfrak{t}_{S_m}$. As we saw in the diagram (161), the actions of v_n, v_m on $\text{mod } p^k$ cohomology must coincide, because both can be computed by means of the lift \tilde{v} . For this we are implicitly using (115) to see that the composite map

$$\begin{aligned} H^*(\text{Hom}_S(\mathbf{C}, \mathbf{Z}_p)) &\rightarrow H^*(\text{Hom}_{S_n}(\mathbf{C} \otimes_S S_n, \mathbf{Z}/p^n)) \\ &\xrightarrow{\sim} H^*(Y(1), \mathbf{Z}/p^n)_{\mathfrak{m}} \rightarrow H^*(Y(1), \mathbf{Z}/p^k)_{\mathfrak{m}} \end{aligned} \tag{167}$$

is independent of n , for $n \geq k$.

So the images of v_n, v_m in V/p^k have the same actions on $\text{mod } p^k$ cohomology. The V/p^k action on $\text{mod } p^k$ cohomology is faithful (by (166) and (162)) so this forces the image of v_n, v_m in V/p^k to coincide as claimed.

Therefore, passing to the limit over n , we get a map $W/p^k \rightarrow V/p^k$, which is easily seen to be compatible as we increase k ; thus the inverse limit over k defines an isomorphism

$$W \xrightarrow{\sim} V. \tag{168}$$

Next, referring to (161) the action of $\text{Ext}_{\mathbb{S}}^*(\mathbb{S}/I, \mathbb{S}/I) \simeq \wedge^* t_{\mathbb{S}}$ on $H^*(\text{Hom}_{\mathbb{S}}(\mathbb{C}, \mathbb{Z}_p))$ certainly factors through $\wedge^* W$; this action of $\wedge^* W$ on $H^*(\text{Hom}_{\mathbb{S}}(\mathbb{C}, \mathbb{Z}_p))$ is compatible under the identifications (115) and (168) with the action of $\wedge^* V$ on $H^*(Y(1), \mathbb{Z}_p)_{\mathfrak{m}}$.

Therefore, by (162), V freely generates an exterior algebra inside $\text{End}(H^*)$, and H^* is freely generated over $\wedge^* V$ by $H^q(Y(1), \mathbb{Z}_p)_{\mathfrak{m}}$, its minimal degree component. On the other hand, the image of $\wedge^* V$ in $\text{End}(H^*)$ is contained in the global derived Hecke algebra—the action of an element V is, by definition, a limit of actions of elements in the derived Hecke algebra. Indeed, V lies inside the strict global derived Hecke algebra. \square

8.27. Proof of Proposition 8.6. In our current situation, we have an inclusion

$$\wedge^* V \subset \tilde{\mathbb{T}} \tag{169}$$

and we have seen that $H^*(Y(1), \mathbb{Z}_p)_{\mathfrak{m}}$ is generated by its lowest nonvanishing degree H^q as a $\wedge^* V$ -module. If we know that $H^*(Y(1), \mathbb{Z}_p)_{\mathfrak{m}} = \mathbb{Z}_p$, we argue that (169) is an equality just as in the corresponding argument after (56).

If we are given \mathbb{S} as in the statement of the Proposition, let $M = H^q(Y(1), \mathbb{Z}_p)_{\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as a \mathbb{S} -module. It is semisimple by assumption; let \mathbb{S}' be the commutant of \mathbb{S} inside \mathbb{Q}_p -linear endomorphisms of M . The natural map

$$M \otimes \wedge^* V_{\mathbb{Q}_p} \rightarrow H^*(Y(1), \mathbb{Z}_p)_{\mathfrak{m}} \otimes \mathbb{Q}_p$$

is now an isomorphism of $\mathbb{S} \otimes \wedge^* V_{\mathbb{Q}_p}$ modules. However,

$$(\text{graded}) \text{ commutant of } \mathbb{S} \otimes \wedge^* V_{\mathbb{Q}_p} \text{ on } M \otimes \wedge^* V_{\mathbb{Q}_p} = \mathbb{S}' \otimes \wedge^* V_{\mathbb{Q}_p},$$

as can be verified by computing in steps: the commutant of \mathbb{S} alone equals $\mathbb{S}' \otimes \text{End}(\wedge^* V_{\mathbb{Q}_p})$, and then inside here the (graded) commutant of $\wedge^* V_{\mathbb{Q}_p}$ is $\mathbb{S}' \otimes \wedge^* V_{\mathbb{Q}_p}$.

Since this commutant contains the image of $\tilde{\mathbb{T}}$ acting on \mathbb{Q}_p -cohomology, the latter is also graded-commutative, since \mathbb{S}' is commutative by the multiplicity one hypothesis. Note that the action on \mathbb{Q}_p cohomology is faithful because of our torsion freeness assumption, see 7(a) of Section 6.1. \square

8.28. The action of Hecke operators. To conclude, let us translate what we have proved into a more concrete assertion about the action of a derived Hecke operator.

Let q be a unramified prime for ρ , with $q \equiv 1$ modulo p^n , and such that $\bar{\rho}(\text{Frob}_q)$ is conjugate to a strongly regular element of $T^\vee(k)$. Let $\nu \in X_*(\mathbf{A})^+$ be strictly dominant and let

$$\alpha \in H^1(\mathbf{A}(\mathbf{F}_q), \mathbf{Z}/p^n).$$

To this we can associate in a natural way (see below) a derived Hecke operator $T_{q,\nu,\alpha}$ as well as an element $[q, \nu, \alpha] \in \mathbf{V}/p^n$; we prove that the actions of these are compatible (see Lemma below), justifying the assertions made in Section 1.5.

First of all, a small piece of linear algebra. Let k be a field. Suppose given a fixed character $\psi \in X^*(T^\vee)$. Let $g \in G^\vee(k)$ be regular semisimple, with centralizer Z_g ; this data allows us to construct a homomorphism of k -vector spaces

$$e_{\psi,g} : \text{Lie}(T^\vee) \rightarrow \text{Lie}(Z_g),$$

$$e_{\psi,g} : \sum_{\phi: T^\vee \xrightarrow{\sim} Z_g} \langle \psi, \phi^{-1}(g) \rangle \cdot d\phi$$

where the sum is taken over all conjugations of T^\vee to Z_g over \bar{k} ; the morphism is nonetheless defined over k .

Example: if $G^\vee = \text{SL}_2$, take T^\vee, B^\vee in the standard way to be the diagonal subgroup and upper triangular matrices, and take $\psi : \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mapsto x$. Then $e_{\psi,g}$ sends $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{Lie}(T^\vee)$ to the element $2g - \text{trace}(g) \in M_2(k)$: it is enough to check this for $g \in T^\vee$, where the result is clear.

Let q, ν, α be as described at the start of this subsection. We can then construct a class

$$[q, \nu, \alpha] \in \mathbf{V}/p^n$$

in the following way: regarding ν as a character of T^\vee , and use the linear algebra construction mentioned with $k = \mathbf{Q}_p$ to make the first map of

$$\text{Lie}(T^\vee) \xrightarrow{e_{\nu, \text{Frob}_q}} \text{Lie}(Z_{\rho(\text{Frob}_q)}) \hookrightarrow \text{Lie}(G^\vee) \tag{170}$$

(at first we get this $\otimes \mathbf{Q}_p$ but then it preserves the integral structures, with reference to the natural \mathbf{Z}_p -models of the three groups above). The resulting embedding $\text{Lie}(T^\vee) \rightarrow \text{Ad}\rho$ is a morphism of $\text{Gal}(\overline{\mathbf{Q}}_q/\mathbf{Q}_q)$ -modules, where $\text{Lie}(T^\vee)$ is taken to have the trivial action.

As before, we may identify

$$H^1(\mathbf{A}(\mathbf{F}_q), \mathbf{Z}/p^n) = H^1(X_*(\mathbf{A}) \otimes \mathbf{F}_q^\times, \mathbf{Z}/p^n) = \text{Hom}(\mathbf{F}_q^\times, X_*(T^\vee)/p^n)$$

and so from α we obtain a class

$$\alpha' \in \frac{H^1(\mathbf{Q}_q, \text{Lie}(T^\vee)/p^n)}{H_{\text{ur}}^1(\mathbf{Q}_q, \text{Lie}(T^\vee)/p^n)}.$$

Here $\text{Lie}(T^\vee)$ is taken as a trivial Galois module. We can then form

$$\text{pushforward of } \alpha' \text{ via (170)} \in \frac{H^1(\mathbf{Q}_q, \text{Ad}\rho_n)}{H_{\text{ur}}^1(\mathbf{Q}_q, \text{Ad}\rho_n)},$$

and, as usual, this can be paired with $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\rho(1))$ by means of local reciprocity. In this way we obtain a functional $H_f^1(\mathbf{Z}[\frac{1}{S}], \text{Ad}^*\rho(1)) \rightarrow \mathbf{Z}/p^n$, which we denote as

$$[q, v, \alpha] \in \mathbf{V}/p^n.$$

LEMMA 8.29. *Let q, v, α be as above. Let $[q, v, \alpha] \in \mathbf{V}/p^n$ be as defined above.*

Let $T_{q,v,\alpha}$ be the derived Hecke operator with \mathbf{Z}/p^n coefficients which is supported on the G_q -orbit of (vK_q, K_q) and whose value at (vK_q, K_q) which corresponds to α under the cohomology isomorphism $H^(K_q \cap vK_q v^{-1}, \mathbf{Z}/p^n) \cong H^*(\mathbf{A}(\mathbf{F}_q), \mathbf{Z}/p^n)$.*

Then $T_{q,v,\alpha}$ corresponds to $[q, v, \alpha] \in \mathbf{V}/p^n$, in the following asymptotic sense:

There is $N_0(m)$ such that for q, v, α as above with $q \equiv 1$ modulo $p^{N_0(m)}$, the actions of $T_{q,v,\alpha}$ and $[q, v, \alpha]$ on $H^(Y(1), \mathbf{Z}/p^m)_{\mathfrak{m}}$ coincide.*

Proof. Under the derived Satake isomorphism, $T_{q,v,\alpha}$ is sent to

$$\sum_w wv \otimes w\alpha \in (S[X_*] \otimes H^1(\mathbf{A}(\mathbf{F}_q)))^W.$$

With notation as in Section 8.17, let $\Theta_v \in S[X_*]_{\mathfrak{m}}^W$ be defined so that its image under $S[X_*]_{\mathfrak{m}}^W \hookrightarrow S[X_*]_{\mathfrak{m}} \rightarrow S[X_*]_{\mathcal{X}}$ is equal to v . Here, we regard $v \in X_* \hookrightarrow S[X_*]$.

Then, after completing at \mathfrak{m} , we have an equality

$$\text{Satake}(T_{q,v,\alpha}) = \sum_{w \in W} \Theta_{wv} \cdot \text{Satake}(\theta(w\alpha)) \in (S[X_*] \otimes H^1(\mathbf{A}(\mathbf{F}_q)))_{\mathfrak{m}}^W$$

where θ is as in (154). (We can check this using the isomorphism (152): it gives an isomorphism of the target group with $S[X_*]_{\mathcal{X}} \otimes H^1(T_q)$, under which $\theta(w\alpha)$ is, by its very definition, sent to $1 \otimes w\alpha$; under the same isomorphism Θ_{wv} is sent to $wv \otimes 1$, and the result follows.)

As before we have fixed $\text{Frob}_q^T \in T^\vee(\mathbf{F}_p)$ an element conjugate to $\bar{\rho}(\text{Frob}_q)$; fix a lift $t_q \in T^\vee(\mathbf{Z}_p)$ that is conjugate to $\rho(\text{Frob}_q)$. Then Θ_{wv} (more exactly, its preimage under Satake) acts on $H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$ by $\langle wv, t_q \rangle$ (this makes sense: $wv \in X^*(T^\vee)$ and $t_q \in T^\vee(\mathbf{Z}_p)$, so they can be paired to get an element of \mathbf{Z}_p^\times). Using Theorem 8.5, we see that the action of $T_{q,v,\alpha}$ on $H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$

corresponds (in the sense of the lemma statement) to the element

$$\sum_{w \in W} \langle wv, t_q \rangle \cdot f_{q,n}(w\alpha) = \sum_{w \in W} \langle v, w^{-1}t_q \rangle f_{q,n}(w\alpha) \in V/p^n.$$

Winding through the definitions, this element of V/p^n is exactly $[q, v, \alpha]$. □

9. Some very poor evidence for the main conjecture: Tori and the trivial representation

We verify that the main conjecture (Conjecture 8.8) holds in the case when \mathbf{G} is an anisotropic torus. This is straightforward, but still slightly comforting.

One may also verify that a certain analogous statement to Conjecture 8.8 holds in the situation studied in Section 5, but there we do not understand the situation clearly at present—hopefully it will eventually prove to be a specialization of the general conjecture to the nontempered case.

9.1. Setup. Let \mathbf{T} be an anisotropic F -torus; let \mathcal{O} be the ring of integers of F . Let us fix a finite set of places S such that \mathbf{T} admits a smooth model over $\mathcal{O}[\frac{1}{S}]$. We assume it contains all places \wp above the rational prime p .

The associated symmetric space

$$S = \mathbf{T}(F \otimes \mathbf{R})/\text{maximal compact}$$

has q, δ invariants (see (18)):

$$q = 0, \quad \delta = \dim(S).$$

The arithmetic manifold $Y(K)$ associated to a level structure K is a disjoint union of copies of S/Δ , where Δ is a congruence subgroup of $\mathbf{T}(\mathcal{O})$. Moreover the quotient S/Δ is a union of compact tori, and thus the rank of Δ , that is, $\dim_{\mathbf{Q}}(\Delta \otimes \mathbf{Q})$, equals δ . We suppose that K is chosen so small that Δ is free of p -torsion.

9.2. Galois cohomology. Let M be the motive associated to the first homology group $H_1(\mathbf{T})$ of \mathbf{T} .

Let $X_*(\mathbf{T})$ be the cocharacter group of \mathbf{T} . It carries an action of $\text{Gal}(\overline{F}/F)$. Coming from $X_* \otimes_{\mathbf{Z}} \mathbb{G}_m \xrightarrow{\sim} \mathbf{T}$, we get an isomorphism of $\text{Gal}(\overline{F}/F)$ -modules

$$M_p := p\text{-adic realization of } M = X_*(\mathbf{T}) \otimes_{\mathbf{Z}} \mathbf{Z}_p(1) \simeq \varprojlim \mathbf{T}[p^n],$$

the p -adic Tate module of \mathbf{T} . Computing with the Kummer sequence,

$$H^1\left(\mathcal{O}\left[\frac{1}{S}\right], M_p\right) \simeq \mathbf{T}\left(\mathcal{O}\left[\frac{1}{S}\right]\right) \otimes \mathbf{Z}_p. \tag{171}$$

Inside the left-hand side, we can consider those classes that are crystalline at p and unramified at *all primes away from p , including primes in S* . We denote this subgroup by $H^1_{f,\text{ur}}(F, M_p)$. (For a precise definition we refer to [2]: the group we have called $H^1_{f,\text{ur}}$ is the group denoted by $H^1_{f,U}$ in [2, Definition 5.1] with U taken to be all finite places. This group is often simply written $H^1_f(F, -)$ in the literature.)

The subgroup on the right-hand side corresponding to $H^1_{f,\text{ur}}(F, M_p)$, call it

$$\Delta' \subset \mathbf{T}\left(\mathcal{O}\left[\frac{1}{S}\right]\right) \otimes \mathbf{Z}_p,$$

is commensurable to the image of $\Delta \otimes \mathbf{Z}_p$ in the right-hand side of (171). Indeed, the constraint that a point $t \in \mathbf{T}(\mathcal{O}[\frac{1}{S}])$ have cohomology class in $H^1_{f,\text{ur}}$ is equivalent to forcing t to belong to a suitable open compact subgroup of $\mathbf{T}(F_v)$ for each $v \in S$; see [18, Theorem 2.3.1] for this statement in the trickier case when v is above p . We suppose (shrinking Δ a little if necessary) that $\Delta \otimes \mathbf{Z}_p \subset \Delta'$.

Also, the motivic cohomology $H^1_{\text{mot}}(M, \mathbf{Q}(1))$ is identified with the $\mathbf{T}(\mathbf{Q}) \otimes \mathbf{Q}$, as we may see by first passing to an extension that trivializes the torus \mathbf{T} . Presumably the following is valid, but I did not try to check it:

Assumption: The subgroup of ‘integral classes’ $H^1_{\text{mot}}(M_{\mathbf{Z}}, \mathbf{Q}(1))$ (see discussion after (7)) is identified with the image of $\Delta \otimes \mathbf{Q}$ inside $\mathbf{T}(\mathbf{Q}) \otimes \mathbf{Q}$.

Now a cohomological automorphic form Π for \mathbf{T} is trivial on the connected component of $\mathbf{T}(F \otimes \mathbf{R})$, that is, they are the *finite order* idèle class characters of \mathbf{T} . However, the associated coadjoint motive (see Section 1.2) does not depend on which idele class character: we have simply

$$(\text{coadjoint motive for } \Pi)(1) \simeq M,$$

the motive M described above. (Indeed, the \mathbf{Z}_p -linear dual of \hat{T} is identified with

$$\text{Lie}(T^\vee)^\vee \simeq (\text{Lie}(\mathbf{G}_m) \otimes X_*(T^\vee))^\vee \simeq X^*(T^\vee) \otimes \mathbf{Z}_p \simeq X_*(\mathbf{T}) \otimes \mathbf{Z}_p$$

where we have fixed an isomorphism $\mathbf{Z} \simeq \text{Lie}(\mathbf{G}_m)$.)

Now let us examine Conjecture 8.8 in this case. Put

$$V = H_{f,ur}^1(F, M_p)^\vee,$$

where M_p is the p -adic étale realization, and \vee denotes \mathbf{Z}_p -linear dual; thus $V = \text{Hom}(\Delta', \mathbf{Z}_p)$. The notation $H_{f,ur}^1$ was defined after (171).

There is a natural action of V on $H^*(Y, \mathbf{Z}_p)$, obtained from the maps

$$V = H^1(\Delta', \mathbf{Z}_p) \rightarrow H^1(\Delta, \mathbf{Z}_p). \quad (172)$$

Moreover, motivic cohomology gives a lattice in $V \otimes \mathbf{Q} = H^1(\Delta', \mathbf{Q}_p)$ (the classes which are \mathbf{Q} -valued on $\Delta \subset \Delta'$) and obviously this lattice indeed preserves $H^*(Y, \mathbf{Q})$, in the \mathbf{Q} -linear extension of the action of V .

The only point to be discussed is that the action (172) is indeed that resulting from the same formalism as Section 8. We describe this only briefly. Let v be a finite place not belonging to S , so that \mathbf{T} extends to a smooth torus over \mathcal{O}_v . As usual we have an injection,

$$\text{Hom}(\mathbf{T}(\mathcal{O}_v), \mathbf{Z}/p^n) \hookrightarrow \text{derived Hecke algebra at } v,$$

and thus an action of the left-hand group on the cohomology of $H^*(Y, \mathbf{Z}/p^n)$; explicitly, this action is obtained by pulling back cohomology classes via $\Delta \rightarrow \mathbf{T}(\mathcal{O}_v)$, and cup product. By just the same procedure as that described in (124), we can construct a map

$$\text{Hom}(\mathbf{T}(\mathcal{O}_v), \mathbf{Z}/p^n) \rightarrow \underbrace{V/p^n}_{\simeq \text{Hom}(\Delta', \mathbf{Z}/p^n)},$$

and one verifies this is the map induced by $\Delta' \rightarrow \mathbf{T}(\mathcal{O}_v) \otimes \mathbf{Z}_p$. Then the action of V/p^n on $H^*(Y, \mathbf{Z}/p^n)$ is compatible with the ‘derived Hecke’ action of $\text{Hom}(\mathbf{T}(\mathcal{O}_v), \mathbf{Z}/p^n)$ for all v ; and in fact this compatibility determines the action of V/p^n .

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Gunnar Carlsson pointed out to me that my original definition (which was the one presented below in Section 2.3) should be equivalent to the much more familiar definition with Ext-groups given in the introduction. The definition with Ext-groups, or rather a differential graded version thereof, was already defined by Schneider around 2008 and published in [28]; I was unaware of Schneider’s work at the time. In any case there is little overlap between our paper and [28]. (See also [24]).

Frank Calegari explained many ideas related to his paper [9] with Geraghty, and, more generally, taught me (over several years!) about Galois representations and their deformations. He also pointed out several typos and mistakes in the manuscript. The joint paper [10] influenced the ideas here, for example, Section 2.7.

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Appendix A. Remedial algebra

In this section, we fill in some ‘intuitively obvious’ claims in the text in grotesque detail, in particular, the identifications between various different models of the derived Hecke algebra. (The word ‘remedial’ in the title of this section refers to my own lack of fluency with homological algebra.)

An action of a topological group will be called *smooth* if the stabilizer of every point is an open subgroup. We fix a finite ring S of cardinality prime to p . A ‘smooth representation of G ,’ in this section, will be a smooth action of G on an S -module. We write SG (or occasionally $S[G]$ when typography requires) for the group algebra of G with coefficients in S . We write Hom_{SG} for homomorphisms of SG -modules; in Section A.11 we abbreviate this to Hom_G because other notation becomes very dense.

Note that the usage of U and K in this section do not precisely match their usage in the main text.

A.1. Let K be a profinite group, which admits a pro- p open normal compact subgroup U . Then the category \mathcal{C} of smooth representations of K is an abelian category with enough projectives. In fact, if Q is a projective K/U module, then considering Q as a smooth K -module \tilde{Q} it remains projective: $\text{Hom}_{SK}(\tilde{Q}, V) =$

$\text{Hom}_{S[K/U]}(Q, V^U)$ and $V \mapsto V^U$ is exact by the hypothesis on U . (One can lift U -invariants under a surjection $V_1 \twoheadrightarrow V_2$ by averaging.)

In this situation, restriction to a finite index subgroup $K' \subset K$ preserves projectivity. Indeed $\text{Hom}_{SK'}(\text{Res}_{K'}^K A, B) = \text{Hom}_{SK}(A, \text{Ind}_{K'}^K B)$ and the induction functor is exact.

A.2. Now let $G = G_v$ be the points of a reductive group over a p -adic field, or any open subgroup thereof. Then the category of smooth representations of G_v is an abelian category and it has enough projectives.

Indeed, consider $W = S[G/U]$ for an open pro- p compact $U \subset G$. Then

$$\text{Hom}_{SG}(W, V) \simeq V^U,$$

which is obviously exact in V . So W is projective, and now given any other V we choose generators v_i for V , open pro- p compact subgroups U_i fixing v_i , corresponding projectives W_i , and then get $\bigoplus W_i \twoheadrightarrow V$.

Throughout the remainder of this section, we suppose that G is as above, that K is an open compact subgroup of G (in particular, K is profinite), and that $U \subset K$ is a pro- p open normal compact subgroup of K .

A.3. Fix a resolution of the trivial K -representation by projective smooth K -modules:

$$\mathbf{Q} : \cdots \rightarrow Q_i \rightarrow \cdots \rightarrow Q_1 \rightarrow S. \tag{173}$$

(Here, and in what follows, we use boldface letters to denote complexes.) To be explicit, let us take \mathbf{Q} to be the standard ‘bar’ resolution of S by free $S[K/U]$ -modules, considered as a complex of smooth K -representations.

Then $\text{Hom}(\mathbf{Q}, \mathbf{Q})$ computes $H^*(K, S)$, the continuous cohomology of the profinite group K with S coefficients: indeed, the cohomology of $\text{Hom}(\mathbf{Q}, \mathbf{Q})$ is identified with $H^*(K/U, S)$, which is identified by pullback with the continuous cohomology $H^*(K, S)$.

The complex $\text{Hom}(\mathbf{Q}, \mathbf{Q})$ has the structure of differential graded algebra arising from composition, and the resulting multiplication on $H^*(K, S)$ coincides with the cup product (this reduces to a corresponding statement for a finite group; for that see [37]).

If $K' \subset K$ is a finite index subgroup, then $\text{Hom}_{SK'}(\mathbf{Q}, \mathbf{Q})$ still computes $H^*(K', S)$ (see remarks above). Moreover, the averaging operator $\sum_{K/K'}$ – that is to say, the map sending

$$f \in \text{Hom}_{SK'}(\mathbf{Q}, \mathbf{Q}) \mapsto \sum_{\kappa} \kappa f \kappa^{-1} \in \text{Hom}_{SK}(\dots)$$

realizes the corestriction map $H^*(K', S) \rightarrow H^*(K, S)$, where the κ sum is taken over a set of coset representatives for K/K' in K .

A.4. Induction and Frobenius reciprocity. We use the word ‘induction’ for the functor from K -modules to G modules

$$M \rightsquigarrow SG \otimes_{SK} M. \tag{174}$$

This is isomorphic to the usual ‘compact’ induction, namely space of functions

$$\text{ind}_K^G(M) := \{f : G \rightarrow M : f(gk) = k^{-1}f(g), \tag{175}$$

$$f \text{ is supported on finitely many left translates of } K\}$$

where the action of $h \in G$ is by left translation, that is, $l_h f(x) = f(h^{-1}x)$. We drop the word ‘compact’ and simply refer to (175) or (174) as ‘induction’; we refer to the model (175) for induction as the ‘function model.’

We can define inverse isomorphisms between (175) and (174) as follows: define $\text{ind}_K^G(M) \rightarrow SG \otimes_{SK} M$ via

$$f \mapsto \sum_{x \in (G/K)} g_x \otimes f(g_x),$$

where $g_x \in G$ is a representative for $x \in G/K$; in the other direction, we send $g \otimes m$ to the function supported on gK whose value on g equals m .

A.5. Frobenius reciprocity. We have Frobenius reciprocity

$$\text{Hom}_{SG}(\text{ind}_K^G M, N) \simeq \text{Hom}_{SK}(M, N)$$

and therefore induction carries projective K -modules to projective G -modules. Explicitly an SK -homomorphism $f : M \rightarrow N$ is sent to its obvious G -linear extension $SG \otimes_{SK} M \rightarrow N$.

If $G \supset K$ has finite index, we also have the reverse adjointness (since ‘compact induction’ and ‘induction’ coincide): to give a K -map $f : M \rightarrow M'$ is the same as giving a map $F_f : M \rightarrow \text{ind}_K^G M'$. Explicitly, in the function model for the induced representation, F_f is characterized by the property $F_f(m)(e) = f(m)$, and so

$$F_f(m)(g) = l_{g^{-1}} F_f(m)(e) = F_f(g^{-1}m)(e) = f(g^{-1}m)$$

and thus in the tensor product model

$$F_f(m) = \sum_{x \in (G/K)} g_x \otimes f(g_x^{-1}m). \tag{176}$$

A.6. Restriction of induced representations. Let Q be a smooth representation of K . The restriction of $\text{ind}_K^G Q$ to K is isomorphic to

$$\bigoplus_{x \in K \backslash G / K} S[Kg_x K] \otimes_{SK} Q \simeq \bigoplus_K S[K] \otimes_{SK_x} Q_x \simeq \bigoplus_x \text{ind}_{K_x}^K Q_x \tag{177}$$

where $x = g_x K$ runs through a set of representatives for K -orbits on G/K , and we write $K_x = K \cap g_x K g_x^{-1}$; moreover for a K -module Q , we denote by Q_x the K_x -module whose underlying space is Q , but for which the action $*$ of K_x on Q is defined thus:

$$k * q = (\text{Ad}(g_x^{-1})k) q. \tag{178}$$

The first map of (177) is given explicitly by

$$k_1 g_x k_2 \otimes q = k_1 g_x \otimes k_2 q \mapsto k_1 \otimes k_2 q \tag{179}$$

and the inverse map sends

$$k \otimes q \mapsto k g_x \otimes q. \tag{180}$$

In the function model of the induced representations, the composite map of (177) sends $F : Kg_x K \rightarrow Q$ to the function $F' : k \mapsto F(kg_x)$. In the reverse direction, given a function F' in the function model of $\text{ind}_{K_x}^K Q_x$, the inverse of (177) sends it to

$$\sum_{k \in K / K_x} k g_x \otimes f(k) \in S[Kg_x K] \otimes_{SK} Q. \tag{181}$$

A.7. Derived Hecke algebra. The *derived Hecke algebra* for the pair (G, K) with coefficients in S is defined as

$$\bigoplus_i \text{Ext}_{SG}^i(S[G/K], S[G/K])$$

where the Ext-groups are taken in the category of smooth S -representations.

We can construct an explicit model as follows. Let \mathbf{Q} be as in (173). Then $\mathbf{P} = \text{ind} \mathbf{Q}$ is a projective resolution of $S[G/K]$. In particular, $\text{Hom}_{SG}(\mathbf{P}, \mathbf{P})$ has the structure of a differential graded algebra and its cohomology gives the derived Hecke algebra.

A.8. We now explicitly describe the isomorphism (25) between the derived Hecke algebra and its ‘double coset model.’

Let \mathbf{P}, \mathbf{Q} be as in Section A.7.

We have

$$\text{Hom}_{SG}(\mathbf{P}, \mathbf{P}) = \text{Hom}_{SK}(\mathbf{Q}, \text{ind}_K^G \mathbf{Q}) \xleftarrow{(177)} \underbrace{\bigoplus_{x \in K \backslash G / K} \text{Hom}_{SK}(\mathbf{Q}, \text{ind}_{K_x}^K \mathbf{Q}_x)}_{= \text{Hom}_{SK_x}(\mathbf{Q}, \mathbf{Q}_x)}, \quad (182)$$

where x varies now through $K \backslash G / K$, again $g_x K$ is a representative for x and $K_x = K \cap \text{Ad}(g_x)K$, and the twist operation \mathbf{Q}_x is as described in (178).

Note that the last map induces a cohomology isomorphism. We must see that $H^*(\text{Hom}_{SK}(\mathbf{Q}, -))$ commutes with the infinite direct sum $\bigoplus_x \text{ind}_{K_x}^K \mathbf{Q}_x$. However, \mathbf{Q}_x is cohomologically concentrated in degree 0, and so the same is true for $\text{ind}_{K_x}^K \mathbf{Q}_x$; it is enough to show, then, for any K -modules M_i , the obvious map

$$\bigoplus_i \text{Hom}_{SK}(\mathbf{Q}, M_i) \longrightarrow \text{Hom}_{SK}\left(\mathbf{Q}, \bigoplus_i M_i\right)$$

is a quasi-isomorphism. But this follows from the fact that taking U -invariants commutes with infinite direct sum, as does the functor $H^*(K/U, -)$.

Note that the cohomology of $\text{Hom}_{SK_x}(\mathbf{Q}, \mathbf{Q}_x)$ is identified with $H^*(K_x, S)$, because \mathbf{Q} and \mathbf{Q}_x are resolutions of S , and moreover \mathbf{Q} is a complex of projective K_x -modules. Thus, (182) gives rise to an isomorphism:

$$H^*(\text{Hom}_{SG}(\mathbf{P}, \mathbf{P})) \simeq \bigoplus_x H^*(K_x, S). \quad (183)$$

For later use, we explicate the map of (182), going from right to left: An element $f \in \text{Hom}_{SK_x}(\mathbf{Q}, \mathbf{Q}_x)$ must satisfy $f(\kappa q) = (g_x^{-1} \kappa g_x) f(q)$ for $\kappa \in K_x$; the associated element of $\text{Hom}_{SK}(\mathbf{Q}, \text{ind}_K^G \mathbf{Q})$ is given in the tensor product model of the induced representation by the formula of (176):

$$q \in \mathbf{Q} \mapsto \sum_{k \in K / K_x} k g_x \otimes f(k^{-1} q)$$

which is well defined.

A.9. Action of derived Hecke algebra on derived invariants. Now suppose that \mathbf{M} is a complex of smooth G -representations. There is a natural action of $\text{End}_{SG}(\mathbf{P})$ on $\text{Hom}_{SG}(\mathbf{P}, \mathbf{M})$. Moreover, the latter complex computes the hypercohomology $\mathbb{H}^*(K, \mathbf{M})$ of K with coefficients in the complex \mathbf{M} .

Thus, because of (183), we get an action of $H^*(K_x, S)$ on $\mathbb{H}^*(K, \mathbf{M})$. Let us describe the action of $h_x \in H^*(K_x, S)$ on $\mathbb{H}^*(K, \mathbf{M})$ as explicitly as possible, in particular, justifying the claims of Section 2.10:

LEMMA A.10. *With notation as above, the action of h_x is given by the following composite:*

$$\begin{aligned} \mathbb{H}^*(K, \mathbf{M}) &\xrightarrow{\text{Ad}(g_x^{-1})^*} \mathbb{H}^*(K_x, \mathbf{M}_x) \xrightarrow{m \mapsto g_x^m} \mathbb{H}^*(K_x, \mathbf{M}) \\ &\xrightarrow{\cup h_x} \mathbb{H}^*(K_x, \mathbf{M}) \xrightarrow{\text{Cores}} \mathbb{H}^*(K, \mathbf{M}). \end{aligned}$$

Here the first map is the pullback induced by $\text{Ad}(g_x^{-1}) : K_x \hookrightarrow K$, which pulls back \mathbf{M} to \mathbf{M}_x .

Proof. Choose $h'_x \in \text{Hom}_{SK}(\mathbf{Q}, \text{ind}_{K_x}^K \mathbf{Q}_x)$ representing h_x . For $f \in \text{Hom}_{SG}(\mathbf{P}, \mathbf{M})$ we denote by $f_x \in \text{Hom}_{SK}(\text{ind}_{K_x}^K \mathbf{Q}_x, \mathbf{M})$ the restriction. We denote by $[h'_x] \in \text{Hom}_{SK_x}(\mathbf{Q}, \mathbf{Q}_x)$ and $[f_x] \in \text{Hom}_{SK_x}(\mathbf{Q}_x, \mathbf{M})$ the elements obtained from h'_x, f_x using Frobenius reciprocity (but using the two different versions of Frobenius reciprocity).

We want to compare the composition $f_x \circ h'_x$ and $[f_x] \circ [h'_x]$ that is,

$$\begin{array}{ccc} h'_x \in \text{Hom}_{SK}(\mathbf{Q}, \text{ind}_{K_x}^K \mathbf{Q}_x) & \times & f_x \in \text{Hom}_{SK}(\text{ind}_{K_x}^K \mathbf{Q}_x, \mathbf{M}) \longrightarrow \text{Hom}_{SK}(\mathbf{Q}, \mathbf{M}) \\ \uparrow \sim & & \uparrow \sim \\ [h'_x] \in \text{Hom}_{SK_x}(\mathbf{Q}, \mathbf{Q}_x) & \times & [f_x] \in \text{Hom}_{SK_x}(\mathbf{Q}_x, \mathbf{M}) \longrightarrow \text{Hom}_{SK_x}(\mathbf{Q}, \mathbf{M}) \end{array} \tag{184}$$

We compute

$$f_x \circ h'_x(q) \stackrel{(176)}{=} f_x \left(\sum_{k_i \in K/K_x} k_i \otimes [h'_x](k_i^{-1}q) \right) = \sum_{k_i} k_i [f_x] \circ [h'_x](k_i^{-1}q)$$

that is, this is what we get by averaging $[f_x] \circ [h'_x]$ over the action of K/K_x . The cohomology class of the composition $[f_x] \circ [h'_x]$ simply amounts (at the level of cohomology) to the cup product of the class $[f_x] \in \mathbb{H}^*(K_x, \mathbf{M})$ (hypercohomology) with the class $[h'_x] \in H^*(K_x, S)$. So to prove the lemma it remains to show:

Subclaim: The class $[f_x] \in \mathbb{H}^*(K_x, \mathbf{M})$ is obtained from $[f] \in \mathbb{H}^*(K, \mathbf{M})$ via

$$\mathbb{H}^*(K, \mathbf{M}) \xrightarrow{\text{Ad}(g_x^{-1})^*} \mathbb{H}^*(K_x, \mathbf{M}_x) \xrightarrow{m \mapsto g_x^m} \mathbb{H}^*(K_x, \mathbf{M}).$$

At the chain level this map is given by the composite

$$\text{Hom}_{SK}(\mathbf{Q}, \mathbf{M}) \rightarrow \text{Hom}_{SK_x}(\mathbf{Q}_x, \mathbf{M}_x) \rightarrow \text{Hom}_{SK_x}(\mathbf{Q}_x, \mathbf{M})$$

where the first map is the trivial map, just considering maps of K modules as K_x -modules via $\text{Ad}(g_x^{-1}) : K_x \rightarrow K$; and the second map $\mathbf{M}_x \rightarrow \mathbf{M}$ is given

by $m \mapsto g_x m$. To check the subclaim, note that $[f_x] \in \text{Hom}_{S_{K_x}}(\mathbf{Q}_x, \mathbf{M})$ sends $q \in \mathbf{Q}_x$ to the value of f_x on the element $1 \otimes q \in \text{ind}_{K_x}^K \mathbf{Q}$, which is carried by the isomorphism inverse to (177) to $g_x \otimes q \in \text{ind}_K^G \mathbf{Q}$; thus,

$$[f_x] : q \mapsto g_x f(q)$$

where here, on the right-hand side, we regard f as a map $\mathbf{Q} \rightarrow \mathbf{M}$ by Frobenius reciprocity (that is, we pull it back via the obvious embedding $\mathbf{Q} \hookrightarrow \mathbf{P}, q \mapsto 1 \otimes q$). This concludes the justification of the subclaim. \square

A.11. Multiplication in the derived Hecke algebra. We finally analyze composition (that is, multiplication) in the derived Hecke algebra, explicating it with respect to the isomorphism (183), and thus justifying the description given in Section 2.3.

Let $\alpha, \beta, \gamma \in G/K$ with representatives $g_\alpha, g_\beta, g_\gamma \in G$. Suppose given $h_\alpha \in H^*(K_{g_\alpha})$ and similarly for β . We compute the product $h_\beta h_\alpha$ considered as elements of the derived Hecke algebra—or more precisely the $H^*(K_{g_\gamma})$ component of their product.

As in (182) we represent h_α by an element $h'_\alpha \in \text{Hom}_K(\mathbf{Q}, \text{ind}_K^G \mathbf{Q})$, and denote by $[h'_\alpha]$ the corresponding element of $\text{Hom}_{K_{g_\alpha}}(\mathbf{Q}, \mathbf{Q}_{g_\alpha})$ and similarly for h_β . By (176) we have the explicit formula

$$h'_\alpha : q \in \mathbf{Q} \mapsto \sum_{k \in K/K_\alpha} k g_\alpha \otimes [h'_\alpha](k^{-1}q) \in S[K g_\alpha K] \otimes_{S_K} \mathbf{Q},$$

where we make a modest abuse of notation by identifying K/K_α to a set of representatives for it in K . Now apply h'_β to the right-hand side, regarding $h'_\beta \in \text{Hom}_{S_G}(\mathbf{P}, \mathbf{P})$. The result is:

$$\sum_{k \in K/K_\alpha} \sum_{k' \in K/K_\beta} k g_\alpha k' g_\beta \otimes [h'_\beta] k'^{-1} [h'_\alpha] k^{-1} \in \text{Hom}_K(\mathbf{Q}, S G \otimes_{S_K} \mathbf{Q}). \tag{185}$$

The desired $H^*(K_{g_\gamma})$ component of the product $h_\beta \cdot h_\alpha$ is given by considering all k, k' for which $k g_\alpha k' g_\beta \in K g_\gamma K$, that is, it is represented by

$$\sum_{k g_\alpha k' g_\beta \in K g_\gamma K} k g_\alpha k' g_\beta \otimes [h'_\beta] k'^{-1} [h'_\alpha] k^{-1} \in \text{Hom}_K(\mathbf{Q}, S[K g_\gamma K] \otimes_{S_K} \mathbf{Q}). \tag{186}$$

By ‘dual’ Frobenius reciprocity (see before (176)) the right-hand side can be identified with $\text{Hom}_K(\mathbf{Q}, \text{ind}_{K_\gamma}^K \mathbf{Q}_\gamma) \simeq \text{Hom}_{K_\gamma}(\mathbf{Q}, \mathbf{Q}_\gamma)$. If we write

$kg_\alpha k' g_\beta = k_1 g_\gamma k_2$, an explicit formula for the corresponding element of $\text{Hom}_K(\mathbf{Q}, \text{ind}_{K_\gamma}^K \mathbf{Q}_\gamma)$ is given (179) by

$$q \mapsto \sum_{kg_\alpha k' g_\beta = k_1 g_\gamma k_2} \underbrace{k_1 \otimes k_2 [h'_\beta] k'^{-1} [h'_\alpha] k^{-1}(q)}_{\in SK \otimes_{SK_\gamma} \mathbf{Q}_\gamma}$$

where the right-hand sum is over the same k, k' as before, and we consider only those k, k' such that $kg_\alpha k' g_\beta \in K g_\gamma K$. Then the corresponding element of $\text{Hom}_{K_\gamma}(\mathbf{Q}, \mathbf{Q}_\gamma)$ is given (see (176)) by picking out those terms for which $k_1 \in K_\gamma$; in that case we can rewrite $k_1 g_\gamma k_2 = g_\gamma (\text{Ad}(g_\gamma)^{-1} k_1) k_2$ and so we may as well suppose that $k_1 = 1$. Thus, the desired result is

$$\sum_{k, k': kg_\alpha k' g_\beta = g_\gamma k''} k'' \underbrace{[h'_\beta] k'^{-1}}_{\text{Hom}_{K_{\text{Ad}(kg_\alpha)K'_k g_\beta}}(\mathbf{Q}_{kg_\alpha}, \mathbf{Q}_{kg_\alpha k' g_\beta})} \underbrace{[h'_\alpha] k^{-1}}_{\text{Hom}_{K_{kg_\alpha}}(\mathbf{Q}, \mathbf{Q}_{kg_\alpha})} \in \text{Hom}_{K_\gamma}(\mathbf{Q}, \mathbf{Q}_\gamma). \tag{187}$$

Here we observed that

$$[h'_\alpha] \circ k^{-1} \in \text{Hom}_{K_{kg_\alpha}}(\mathbf{Q}, \mathbf{Q}_{kg_\alpha}), [h'_\beta] \circ k'^{-1} \in \text{Hom}_{K_{k' g_\beta}}(\mathbf{Q}, \mathbf{Q}_{k' g_\beta}) \simeq \text{Hom}_{\text{Ad}(g)K_{k' g_\beta}}(\mathbf{Q}_g, \mathbf{Q}_{gk' g_\beta})$$

(the last isomorphism is the obvious one and we apply it to $g = kg_\alpha$). (For example, to check the first, note that for $z \in K_{kg_\alpha}$ and $q \in \mathbf{Q}$ we have

$$[h'_\alpha] \circ k^{-1}(zq) = [h'_\alpha]((k^{-1}zk)(k^{-1}q)) = (g_\alpha^{-1}k^{-1}zk g_\alpha)[h'_\alpha] \circ k^{-1}(q).$$

Indeed, $[h'_\alpha]$ represents the class in $H^*(K_{kg_\alpha})$, obtained by applying $\text{Ad}(k)$ to the class $h_\alpha \in H^*(K_{g_\alpha})$.) Returning to (187), set

$$x = kg_\alpha k' g_\beta K = g_\gamma K, \quad y = kg_\alpha K, \quad z = eK, \quad U = \text{stabilizer}(x, y, z).$$

Then x, y are in relative position β , and y, z are in relative position α , and x, z are in relative position γ .

Note also that

$$U = K_{kg_\alpha} \cap \text{Ad}(kg_\alpha)K_{k' g_\beta} = K \cap \text{Ad}(kg_\alpha)K \cap \text{Ad}(kg_\alpha k' g_\beta)K.$$

Therefore, the composite occurring in (187)

$$F = \underbrace{[h'_\beta] k'^{-1}}_{\text{Hom}_{K_{\text{Ad}(kg_\alpha)K'_k g_\beta}}(\mathbf{Q}_{kg_\alpha}, \mathbf{Q}_{kg_\alpha k' g_\beta})} \circ \underbrace{[h'_\alpha] k^{-1}}_{\text{Hom}_{K_{kg_\alpha}}(\mathbf{Q}, \mathbf{Q}_{kg_\alpha})}$$

actually belongs to $\text{Hom}_U(\mathbf{Q}, \mathbf{Q}_{kg_\alpha k'g_\beta})$; as such F defines a cohomology class for U . This cohomology class is given by taking the classes h_α, h_β , transporting them to classes in $H^*(K_{kg_\alpha})$ and $H^*(\text{Ad}(kg_\alpha)K'_{k'g_\beta})$, by means of $\text{Ad}(k) : K_{g_\alpha} \simeq K_{kg_\alpha}$ and $\text{Ad}(kg_\alpha k') : K_\beta \simeq \text{Ad}(kg_\alpha)K'_{k'g_\beta}$, restricting to U , and taking the cup product.

Said differently, let us think of h_α as a G -invariant association H_α from pairs $(u, v) \in G/K \times G/K$ to cohomology classes in $H^*(G_{uv})$ – the one whose value at $(g_\alpha K, e)$ is given by the original cohomology class in $H^*(K_{g_\alpha})$. Similarly for h_β . Then,

$$\text{cohomology class of } F = H_\beta(x, y) \cup H_\alpha(y, z) \in H^*(G_{xyz}).$$

Now $K_\gamma = G_{xz}$ acts on the set of (k, k', k'') as above, that is, satisfying $kg_\alpha k'g_\beta = g_\gamma k''$, via

$$\kappa : (k, k', k'') \mapsto (\kappa k, k', \text{Ad}(g_\gamma^{-1})\kappa k'').$$

For fixed (k, k', k'') the stabilizer of this K_γ -action is just U . The contribution of a single K_γ -orbit is given by

$$\sum_{\kappa \in K_\gamma/U} \text{Ad}(g_\gamma^{-1})\kappa \circ F \circ \kappa^{-1}$$

which is to say that it averages F , considered as an element of $\text{Hom}(\mathbf{Q}, \mathbf{Q}_\gamma)$, over the cosets of K_γ/U . (The $\text{Ad}g_\gamma^{-1}$ accounts for the twisted action on \mathbf{Q}_γ .) This precisely realizes the corestriction from U to K_γ .

In summary, we have recovered the description of multiplication in the derived Hecke algebra given in Section 2.3.

Appendix B. Koszul algebra; other odds and ends

Let B be a commutative ring with 1. Let

$$B[[x_1, \dots, x_r]] = S \xrightarrow{\iota} R = B[[y_1, \dots, y_{r-\delta}]]$$

where ι sends x_i to y_i for $i \leq r - \delta$, and x_i to zero for $i > r - \delta$.

Let \mathfrak{p}_S be the kernel of the natural augmentation $S \rightarrow B$, and similarly for R . Write \mathfrak{t}_S for the B -linear dual of $\mathfrak{p}_S/\mathfrak{p}_S^2$ and similarly for \mathfrak{t}_R . Just as in (88) we have a canonical identification

$$\text{Ext}_S^1(B, B) \simeq \mathfrak{t}_S.$$

We prove:

LEMMA B.1. $\text{Ext}_S^*(B, B)$ is a free-exterior algebra over its degree 1 component; thus we have $\text{Ext}_S^*(B, B) \simeq \wedge^* \mathfrak{t}_S$ as graded B -algebra. Moreover, there is an identification of $\text{Ext}_S^*(R, B)$ with $\wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)$ in such a way that the natural action of $\text{Ext}_S^*(B, B) \simeq \wedge^* \mathfrak{t}_S$ is the natural one obtained from the algebra map $\wedge^* \mathfrak{t}_S \rightarrow \wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)$.

This will follow from the computations of Section B.2 (more precisely, with the precisely analogous computations wherein one replaces the role of symmetric algebras by power series algebras).

B.2. Koszul algebra. Let W be a free module of rank e over a base ring B and consider the ring $R = \text{Sym}_B(W)$, that is, ‘the ring of functions on W^\vee .’ In what follows we omit the B subscript on Sym_B .

We have a resolution

$$\dots \rightarrow \underbrace{\text{Sym}(W) \otimes \wedge^2 W \rightarrow \text{Sym}(W) \otimes W \rightarrow \text{Sym}(W)}_{\mathbf{K}} \rightarrow B,$$

where the differential sends

$$r \otimes w_1 \wedge \dots \wedge w_r \in \text{Sym}(W) \otimes \wedge^i W \mapsto \sum_i (-1)^{i-1} r w_i \otimes w_1 \wedge \dots \widehat{w}_i \wedge \dots w_r.$$

There is a corresponding resolution where we replace $\text{Sym}(W)$ by its completion with respect to the augmentation $\text{Sym}(W) \rightarrow B$, that is, when we replace a symmetric algebra by a formal power series algebra.

In particular, we get

$$\text{Hom}_R(\mathbf{K}, B) \simeq (\wedge^i W)^\vee \text{ with zero differentials}$$

and thus an identification of $\text{Ext}_R^i(B, B)$ with $(\wedge^i W)^\vee$.

In fact, $\text{Ext}_R^*(B, B)$ is a free-exterior algebra, where the algebra structure on the Ext-groups arising from their identification with the cohomology of the differential graded algebra

$$\text{Hom}_R(\mathbf{K}, \mathbf{K}).$$

To see this, one verifies that each element of $w \in W^\vee$, considered as acting on \mathbf{K} by contractions, actually defines a degree -1 endomorphism of \mathbf{K} ; the resulting inclusion

$$\bigwedge^* (W^\vee) \hookrightarrow \text{Hom}_R(\mathbf{K}, \mathbf{K})$$

gives a quasi-isomorphism of differential graded algebras.

Suppose now that U is a free submodule of W such that W/U is also free. In this situation we have a quotient map

$$R = \text{Sym}(W) \rightarrow \text{Sym}(W/U) := \bar{R}.$$

We have a resolution of R -modules (where the differential is given by the same formula as before):

$$\underbrace{\cdots \rightarrow \text{Sym}(W) \otimes \wedge^2 U \rightarrow \text{Sym}(W) \otimes U \rightarrow \text{Sym}(W)}_{\mathbf{Q}} \xrightarrow{\sim} \bar{R},$$

and from this we identify $\text{Ext}_R^*(\bar{R}, B)$ with $(\wedge^* U)^\vee$.

LEMMA B.3. *The action of $\text{Ext}_R^*(B, B) \simeq (\wedge^* W)^\vee$ on this is the natural one that arises from the map $W^\vee \rightarrow U^\vee$.*

Proof. It is enough to check this for the action of $\text{Ext}_R^1(B, B)$. We have identifications:

$$\text{Ext}_R^*(\bar{R}, B) \simeq H^*(\text{Hom}_R(\mathbf{Q}, B)) = H^*(\text{Hom}_R(\mathbf{Q}, \mathbf{K})).$$

Explicitly, a class in $\omega_j \in (\wedge^j U)^\vee \simeq \text{Ext}_R^j(\bar{R}, B)$ is represented by a map of complexes $\mathbf{Q} \rightarrow \mathbf{K}$ as follows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & R \otimes \wedge^{j+1} U & \longrightarrow & R \otimes \wedge^j U & \longrightarrow & R \otimes \wedge^{j-1} U & (188) \\ & & \downarrow f & & \downarrow \omega_j \in \wedge^j U^\vee & & & \\ \cdots & \longrightarrow & K_1 = R \otimes W & \longrightarrow & K_0 = R & \longrightarrow & 0 \end{array}$$

(since this lifts the map $\mathbf{Q} \rightarrow B[j]$ associated to ω_j).

Fix a basis e_1, \dots, e_r for U and extend it to a basis e_1, \dots, e_e for W . For $I \subset \{1, \dots, e\}$ with cardinality r , we define $e_I \in \wedge^r W$ thus: write $I = \{i_1, \dots, i_r\}$ with $i_1 < \dots < i_r$ and put $e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_r}$. We may choose f to be given, explicitly, as

$$e_J \in \wedge^{j+1} U \mapsto \sum_{k \in J} (-1)^{[k]-1} \omega_j(e_{J-k}) \otimes e_k \in R \otimes W,$$

where $[k]$ means the position of k in J (that is, if J is ordered in increasing order, then 1 for the smallest element, two for the second smallest, and so forth).

To compute, now, the action of $\beta \in W^\vee = \text{Ext}_R^1(B, B)$ on the class ω_j , we just regard β as an R -module map $K_1 = R \otimes W \rightarrow R$, and then compose

$$\beta \circ f \in \text{Hom}_R(\wedge^{j+1} U \otimes R, R) = (\wedge^{j+1} U)^\vee.$$

Explicitly,

$$\beta \circ f : e_J \mapsto \sum_{k \in J} (-1)^{[k]-1} \bar{\beta}(e_k) \omega_j(e_{J-k}) = (\bar{\beta} \wedge \omega_j, e_J)$$

that is, $\bar{\beta} \wedge \omega_j$, where $\bar{\beta}$ is the image of β in U^\vee . This concludes the proof. \square

B.4. A result in topology. Suppose that $\pi : X \rightarrow Y$ is a covering of pointed Hausdorff topological spaces, with Galois group Δ . This covering is classified by a map $Y \rightarrow B\Delta$ from Y to the classifying space of Δ .

There are two natural actions of $H^*(\Delta, E)$ (with E a coefficient ring) on $H^*(Y, E)$:

- (a) The first arises from pullback of cohomology classes under $Y \rightarrow B\Delta$ together with cup product.
- (b) The second arises from the identification of the cochain complex of Y , with E coefficients, with

$$C^*(Y; E) \simeq \text{Hom}_{E\Delta}(C_*(X, E); E)$$

where $C_*(X; E)$ is the chain complex of X (or for example, the complex of a Δ -equivariant cell structure), thought of as a complex of $E\Delta$ -modules. Then one composes with self-maps of E in the derived category of $E\Delta$ -modules.

For lack of a reference, we prove the coincidence of these actions. For this we use the following standard Lemma concerning the coincidence of singular and sheaf cohomology (see [31] for a careful discussion; however this reference does not discuss the product structures):

LEMMA B.5. *For any locally contractible Hausdorff space M , and any E -module A , let \underline{A} be the constant sheaf on M with constant value A , considered as an object of the category of sheaves \mathcal{S} of E -modules on M .*

Then the complex of local chains $U \mapsto C^(U, A)$ defines a presheaf on M ; let C_A^* be its sheafification. Then $\underline{A} \rightarrow C_A^*$ is a flasque resolution of \underline{A} . Moreover, the natural maps*

$$C^*(M, A) \rightarrow \Gamma(M, C_A^*) = \text{Hom}_{\mathcal{S}}(\underline{A}, C_A^*)$$

induces, at the level of cohomology, an isomorphism

$$H^*(M, \underline{A}) \simeq \text{Ext}_{\mathcal{S}}^*(\underline{A}, \underline{A}) \tag{189}$$

which carries the cup product on the left to the Ext-product to the right.

Proof. (that (a) and (b) coincide). Observe, first of all, that every $E\Delta$ -module M gives a locally constant sheaf \underline{M} on Y , namely, the one represented by the covering $(X \times M)/\Delta \rightarrow Y$. The cochains $C^*(Y, \underline{M})$ are then given by $\text{Hom}_{E\Delta}(C_*(X, E), M)$.

Fix $\alpha_\Delta \in H^m(\Delta, E)$. It gives rise to a homomorphism $\alpha : E \rightarrow E[m]$ in the derived category of $E\Delta$ -modules, which can be represented by a diagram $E \xleftarrow{\sim} P \rightarrow E[m]$ where P is a complex of projective $E\Delta$ -modules. Thus we get a diagram of locally constant sheaves on Y :

$$\underline{\alpha} : \underline{E} \xleftarrow{\sim} \underline{P} \rightarrow \underline{E}[m].$$

This gives a map in the derived category of sheaves on Y , and thus an element of $\text{Ext}_S^m(\underline{E}, \underline{E})$; this element represents the pullback α_Y of α_Δ to Y . or rather its image under (189).

By the final sentence of the Lemma, the cup product with α_Y is given, at the level of cohomology, by the Ext-product, which is explicitly the composite:

$$H^*(Y, \underline{E}) \xleftarrow{\sim} H^*(Y, \underline{P}) \rightarrow H^*(Y, \underline{E}[m]).$$

By the lemma, these groups are naturally identified with the cohomology of the corresponding cochain groups; so the above composite coincides with

$$\text{Hom}_{E\Delta}(C_*(X, E), E) \xleftarrow{\sim} \text{Hom}_{E\Delta}(C_*(X, E), P) \rightarrow \text{Hom}_{E\Delta}(C_*(X, E), E)$$

where the middle term is now the Hom-complex between two complexes.

But this composite is also given by the Ext-product, in the category of $E\Delta$ -modules, with the class of α . This concludes the proof of the coincidence of (a) and (b). □

Index of symbols and important notation

- A_v , 29
- B^\vee , dual Borel, 14
- B_v , 29
- F , 13
- G_v , 14, 29
- H_f^1 , 76
- K_0 , 53
- K_v , 14
- M^* , dual of a Galois module, 77
- R , 54
- R_n , 69
- R_{Q_n} , 69
- S (set of ramified places), 55
- S_n , 59
- T , 29
- T (set of places), 54
- T^\vee , dual torus, 14
- T_n , 58
- T_q , 78
- W_n , 85
- W_s , 58
- X^* , 13, 29
- $Y(1)$, 53
- $Y(K)$, 14
- $Y_0(Q_n)$, 58
- $Y_0(q)$, 58
- $Y_1(q)$, 58
- $Y_1(q, n)$, 58
- $Y_1^*(Q_n)$, 58
- Ad^* , 76
- q , minimal degree of tempered cohomology, 3
- Frob_q^T , 57
- Π , 54
- S , 59
- V , 78
- V-convergent, 96
- χ_t , 63
- χ_{Frob^T} , 63
- $R_{\bar{\rho}}$, 57
- $\tilde{\mathbb{T}}$, 25
- $\tilde{\mathbb{T}}(V)$, 27
- $\tilde{\mathbb{T}}(V_0)$, 27
- $\iota_{q,n}$, 78
- \mathbb{H} (hypercohomology), 108
- $\mathbf{A}(F_v)^\circ$, 29
- \mathbf{G} , 13
- \mathcal{A} , 29
- \mathcal{B} , 29
- \mathcal{G} , 29
- \mathfrak{m} , 55
- \mathfrak{t}_{R_n} , 85
- \mathfrak{t}_{S_n} , 85
- $\mathfrak{t}_{\mathbb{R}}$, 84
- H_K , 36
- H_{II} , 36
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- H_{KI} , 36
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- \mathcal{H}^j , 16
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- ρ , 57
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- \tilde{W} , 36
- \bar{R}_n , 72
- $\tilde{\mathcal{X}}^*$, 64
- $f_{q,n}$, 78
- s (size of Taylor–Wiles datum), 57
- convergent (Taylor–Wiles data), 83
- derived Hecke algebra, 16
- global derived Hecke algebra, 25
- inertial level, 68
- level of K , 14
- strict global Hecke, 27
- strict Taylor–Wiles datum, 94
- strongly regular element, 57
- Taylor–Wiles prime, 57

Conflict of Interest: None.

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