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THE DARBOUX PROBLEM INVOLVING THE DISTRIBUTIONAL HENSTOCK–KURZWEIL INTEGRAL

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Abstract In this paper, using the Schauder Fixed Point Theorem and the Vidossich Theorem, we study the existence of solutions and the structure of the set of solutions of the Darboux problem involving the distributional Henstock–Kurzweil integral. The two theorems presented in this paper are extensions of the previous results of Deblasi and Myjak and of Bugajewski and Szufla.

Keywords: Henstock–Kurzweil integral; Schauder Fixed Point Theorem; Vidossich Theorem; distribution; Darboux problem; distributional Henstock–Kurzweil integral

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1. Introduction

In this paper we prove the existence of solutions and investigate the topological characterization of the set of solutions of the following Darboux problem:

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= f(x,y,z), \\ z(x,0) &= 0, \quad 0 \leqslant x \leqslant d_1, \\ z(0,y) &= 0, \quad 0 \leqslant y \leqslant d_2, \end{aligned}\right\} \tag{1.1}$$

where $K = \{(x,y)\colon 0 \leqslant x \leqslant d_1,\ 0 \leqslant y \leqslant d_2\}$, $B = \{z \in C(K)\colon \|z\|_\infty \leqslant b\}$, $d_1, d_2, b > 0$, and f is a function on $K \times B$. As usual, $C(K)$ denotes the space of all continuous functions $z\colon K \to \mathbb{R}$ with the uniform norm $\|\cdot\|_\infty$, $\partial^2 z/\partial x \partial y$ denotes the mixed distributional derivative of z. The function f in the above Darboux problem is distributionally Henstock–Kurzweil integrable on K.

The Darboux problem has been studied by many authors (see, for example, [**1**,**4**,**6**–**8**]). In particular, in [**4**] Bugajewski and Szufla used the approximate derivatives instead of the usual derivatives to consider the corresponding problem and obtained some interesting results.

In this paper, instead of the approximate derivatives we use the distributional derivatives to study the Darboux problem (1.1). It is known that the notion of a distributional derivative is very general, including, for example, ordinary derivatives and approximate derivatives. Firstly, we define the distributional Henstock–Kurzweil integral (D_{HK}-integral) on Q, where Q denotes the open rectangle $(a, b) \times (c, d)$ on $\mathbb{R}^2$. Secondly, we prove the existence of solutions of the Darboux problem (1.1). Thirdly, we investigate the structure of the set of solutions. The theorems obtained extend the results of [**4**, **7**].

We say that a distribution f is D_{HK}-integrable on Q if there is a continuous function $F \in B_C$ on $\bar{Q}$ whose second-order mixed distributional derivative is f (see § 2 for the definition of B_C). The space of D_{HK}-integrable distributions on Q is a separable Banach space under an appropriative norm.

From the definition of the D_{HK}-integral, we know that the D_{HK}-integral contains the Riemann integral, Lebesgue integral, Henstock–Kurzweil integral and the wide Denjoy integral (for details, see [**5**, **9**–**11**, **13**, **15**–**17**]).

This paper is organized as follows. In § 2, we present the preliminary concepts and the properties of the D_{HK}-integral, including the Fubini Theorem, Dominated Convergence Theorem and some lemmas of the D_{HK}-integral on Q. In § 3, we apply the Schauder Fixed Point Theorem in order to discuss the existence of solutions of (1.1). We also show that the set of solutions of equation (1.1) is an R_δ by using the Vidossich Theorem stated in [**14**]. Here, R_δ is the intersection of a decreasing sequence of compact absolute retracts [**3**]. In § 4, we give an example to show that our results generalize the corresponding results of [**4**, **7**] substantially.

2. Preliminaries

We denote by Q the open rectangle $(a, b) \times (c, d)$ in the plane $\mathbb{R}^2$, and by $\mathcal{D}(Q)$ the subset of $C^\infty(Q)$ such that every $\phi \in \mathcal{D}(Q)$ has a compact support in Q. A distribution on Q is defined to be a continuous linear functional on $\mathcal{D}(Q)$. The space of all distributions on Q, denoted by $\mathcal{D}'(Q)$, is the dual space of $\mathcal{D}(Q)$.

For simplicity, we write $\partial = \partial_{xy} = \partial_{yx}$ to denote the mixed distributional derivative. We denote by ∂_1 and ∂_2 the distributional derivatives with respect to x and y, respectively, and by '$\int$' the D_{HK}-integral.

We introduce the definition of B_C.

Definition 2.1. $B_C = \{F \in C(\bar{Q})\colon F(a, y) = F(x, c) = 0 \text{ for } x \in [a, b],\ y \in [c, d]\}$, where $\bar{Q}$ is the closure of Q.

It can be verified that B_C is a closed subspace of $C(\bar{Q})$ with the uniform norm $\|F\|_\infty = \max\{|F(x, y)|\colon (x, y) \in \bar{Q},\ F(x, y) \in C(\bar{Q})\}$.

Now we are able to define the distributional Henstock–Kurzweil integral:

$$D_{\mathrm{HK}}(Q) = \{f \in \mathcal{D}'(Q) \mid f = \partial F,\ F \in B_C\}.$$

It is easy to see that if $f \in D_{\mathrm{HK}}(Q)$, the corresponding continuous function $F \in B_C$ satisfying $\partial F = f$ is unique.

Definition 2.2. A distribution f is D_{HK}-integrable on Q if $f \in D_{\mathrm{HK}}(Q)$.

The D_{HK}-integral of f on Q is given by

$$\int_Q f = F_f(b, d).$$

Remark 2.3. If we consider only the D_{HK}-integral in a one-dimensional interval, then the D_{HK}-integral is the A_C-integral of [**13**]. That is, f is D_{HK}-integrable on $[a, b] \subset \mathbb{R}^1$ if there exists a continuous function F on $[a, b]$ with $F(a) = 0$ whose distributional derivative is f (see [**13**] for details).

We consider the structure of space $D_{\mathrm{HK}}(Q)$. For $f \in D_{\mathrm{HK}}(Q)$, the norm is defined by

$$\|f\| = \sup\biggl\{\biggl|\int_{(a,x)\times(c,y)} f\biggr| : (x, y) \in \bar{Q}\biggr\}.$$

Lemma 2.4 (Ang *et al.* [2, Theorem 1]). *The normed space $(D_{\mathrm{HK}}(Q), \|\cdot\|)$ is complete, separable and isomorphic to $(B_C, \|\cdot\|_\infty)$.*

We will state a Fubini-type theorem for the D_{HK}-integral which will be used later. At first we introduce some definitions.

Definition 2.5. Let $f \in D_{\mathrm{HK}}(Q), x \in [a, b], y \in [c, d]$. We define

$$\begin{aligned}\int_a^x f(\xi, \cdot)\,\mathrm{d}\xi &= \partial_2 F_f(x, \cdot) \quad \text{in } \mathcal{D}'((c, d)),\\ \int_c^y f(\cdot, \eta)\,\mathrm{d}\eta &= \partial_1 F_f(\cdot, y) \quad \text{in } \mathcal{D}'((a, b)).\end{aligned}$$

It is clear that

$$\int_a^x f(s, \cdot)\,\mathrm{d}s \in D_{\mathrm{HK}}((c, d)), \qquad \int_c^y f(\cdot, t)\,\mathrm{d}t \in D_{\mathrm{HK}}((a, b)),$$

where $D_{\mathrm{HK}}((a, b))$ and $D_{\mathrm{HK}}((c, d))$ are respectively the spaces of D_{HK}-integrable distributions on (a, b) and (c, d), i.e.

$$D_{\mathrm{HK}}((a, b)) = \{f \in \mathcal{D}'(a, b) \mid f = \partial F,\ F \in B_C\},$$

where $B_C = \{F \in C([a, b]) \colon F(a) = 0\}$, and f is the distributional derivative of F.

Lemma 2.6 (Fubini Theorem [2, Theorem 4]). *For all $f \in D_{\mathrm{HK}}(Q)$, we have*

$$\int_Q f = \int_a^b \biggl(\int_c^d f(\cdot, \eta)\,\mathrm{d}\eta\biggr) = \int_c^d \biggl(\int_a^b f(\xi, \cdot)\,\mathrm{d}\xi\biggr).$$

Since the space L^1 of Lebesgue integrable functions is a Banach space and there exist excellent convergence theorems, the Lebesgue integral has many important applications. It follows from Lemma 2.4 that $D_{\mathrm{HK}}(Q)$ is also a separable Banach space. In fact, the D_{HK}-integral also has significant convergence theorems.

Definition 2.7. A sequence $\{f_n\colon f_n \in D_{\mathrm{HK}}(Q)\}$ is said to converge strongly to $f \in D_{\mathrm{HK}}(Q)$ if $\|f_n - f\| \to 0$ as $n \to \infty$.

Now we introduce an order in the space $D_{\mathrm{HK}}(Q)$. For $f, g \in D_{\mathrm{HK}}(Q)$, we say that $f \succeq g$ (or $g \preceq f$) if and only if $f - g$ is a positive measure on Q. In [**2**] it is shown that the D_{HK}-integral has the order-preserving property, i.e.

$$\int_Q f \geqslant \int_Q g, \tag{2.1}$$

whenever $f \succeq g$, $f, g \in D_{\mathrm{HK}}(Q)$ [**2**, p. 360].

For a sequence $\{f_n\} \subset D_{\mathrm{HK}}(Q)$, an important question is whether the convergence of $f_n \to f$ with $f \in D_{\mathrm{HK}}(Q)$ implies

$$\lim_{n\to\infty} \int_Q f_n = \int_Q f.$$

The following convergence theorem gives the answer.

Lemma 2.8 (Dominated Convergence Theorem of the D_{HK}-integral [2, Corollary 5]). *Let $\{f_n\} \subset D_{\mathrm{HK}}(Q)$ such that $f_n \to f$ in $\mathcal{D}'(Q)$. Suppose there exist $g, h \in D_{\mathrm{HK}}(Q)$ satisfying $g \preceq f_n \preceq h$ for all $n \in N$. Then $f \in D_{\mathrm{HK}}(Q)$ and*

$$\lim_{n\to\infty} \int_Q f_n = \int_Q f.$$

Lemma 2.9 (Ang *et al.* [2, Theorem 2]).

(i) *Let $f \in D_{\mathrm{HK}}(Q)$ and $Q' = (a', b') \times (c', d') \subset Q$. Then $f|_{Q'} \in D_{\mathrm{HK}}(Q')$.*

(ii) *For $a \leqslant m \leqslant b$ and $c \leqslant n \leqslant d$, let $Q_1 = (a, m) \times (c, n)$, $Q_2 = (m, b) \times (c, n)$, $Q_3 = (a, m) \times (n, d)$, $Q_4 = (m, b) \times (n, d)$. Then, for each $(f_1, f_2, f_3, f_4) \in \prod_{i=1}^{4} D_{\mathrm{HK}}(Q_i)$, there exists a unique $f \in D_{\mathrm{HK}}(Q)$ such that $f|_{Q_i} = f_i$, $1 \leqslant i \leqslant 4$. Moreover,*

$$\int_Q f = \sum_{i=1}^{4} \int_{Q_i} f_i.$$

(iii) *Let $a < m < m' < b$ and $f \in \mathcal{D}'(Q)$. Then $f \in D_{\mathrm{HK}}(Q)$ if and only if $f|_{(a,m')\times(c,d)} \in D_{\mathrm{HK}}((a, m') \times (c, d))$ and $f|_{(m,b)\times(c,d)} \in D_{\mathrm{HK}}((m, b) \times (c, d))$.*

3. Main results and proofs

In this section, we consider the Darboux problem (1.1), where $K = \{(x, y)\colon 0 \leqslant x \leqslant d_1,\ 0 \leqslant y \leqslant d_2\}$, $B = \{z \in C(K)\colon \|z\|_\infty \leqslant b\}$ and $d_1, d_2, b > 0$, f is a function on $K \times B$. As before, $\partial^2 z/\partial x \partial y$ denotes the mixed distributional derivative of z.

Definition 3.1. A function $z \in C(J)$ with $J \subset K$ is a solution of equation (1.1) if it satisfies equation (1.1) with $z(x, y) \in B$ for $(x, y) \in J$.

To prove the existence of solutions of equation (1.1), we need the following well-known Schauder Fixed Point Theorem.

Lemma 3.2 (Schechter [12, Theorem 6.15]). *Let M be a convex, closed subset of a normed vector space X. Let T be a continuous map of M into a compact subset K of M. Then T has a fixed point.*

In what follows, we assume that f satisfies the following assumptions:

(D_1) f is D_{HK}-integrable on K for every $z \in B$;

(D_2) $z \to f(x, y, z)$ is continuous for almost every $(x, y) \in K$;

(D_3) there exist $g, h \in D_{\mathrm{HK}}$ satisfying $g(\cdot,\cdot) \preceq f(\cdot,\cdot,z) \preceq h(\cdot,\cdot)$ for every $z \in B$.

We now present the first main result of this paper.

Theorem 3.3. *Under the assumptions* (D_1)–(D_3), *there exists at least one solution to equation (1.1) on J, for $J \subset K$.*

Proof. We give the proof in three steps.

Step 1. Obviously, equation (1.1) is equivalent to the following integral equation:

$$z(x, y) = \int_{[0,x]\times[0,y]} f(\xi, \eta, z(\xi, \eta))\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.1}$$

Since $g, h \in D_{\mathrm{HK}}$, the primitives of g, h are continuous on K.
Let

$$b = \max_{K' \subseteq K} \left\{ \left| \int_{K'} g \right|, \left| \int_{K'} h \right| \right\}.$$

Choose positive numbers d'_1, d'_2 in such a way that

$$-b \leqslant \int_{[0,x]\times[0,y]} g \leqslant b \quad \text{and} \quad -b \leqslant \int_{[0,x]\times[0,y]} h \leqslant b,$$

for $0 \leqslant x \leqslant d'_1$, $0 \leqslant y \leqslant d'_2$, $J = \{(x, y)\colon 0 \leqslant x \leqslant d'_1,\ 0 \leqslant y \leqslant d'_2\}$.
By (2.1) and the condition (D_3), we have

$$\int_{[0,x]\times[0,y]} f(\cdot,\cdot,z) \leqslant \int_{[0,x]\times[0,y]} h \leqslant b,$$

$$\int_{[0,x]\times[0,y]} f(\cdot,\cdot,z) \geqslant \int_{[0,x]\times[0,y]} g \geqslant -b.$$

Define

$$F(z)(x, y) = \int_{[0,x]\times[0,y]} f(\xi, \eta, z(\xi, \eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

for $z \in \tilde{B}, (x, y) \in J$, where $\tilde{B} = \{z \in C(J)\colon \|z\|_\infty \leqslant b\}$. Then

$$\|F(z)\|_\infty = \left\| \int_{[0,x]\times[0,y]} f \right\|_\infty = \max_{(x,y)\in J} \left| \int_{[0,x]\times[0,y]} f(\cdot,\cdot,z(\cdot,\cdot)) \right| \leqslant b$$

for $z \in \tilde{B}$. So $F(z)(x, y) \in \tilde{B}$ and therefore $F(\tilde{B}) \subset \tilde{B}$.

Step 2. We verify that the family $F(\tilde{B})$ is equi-uniformly continuous.

$$F(z)(x_1,y_1) - F(z)(x_2,y_2) = \int_{[0,x_1]\times[0,y_1]} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta - \int_{[0,x_2]\times[0,y_2]} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

for $(x_1,y_2),(x_2,y_2) \in J$, $z \in \tilde{B}$.

According to the Fubini theorem of the D_{HK}-integral,

$$\begin{aligned} F(z)(x_1,y_1) &- F(z)(x_2,y_2) \\ &= \int_0^{x_2}\left(\int_0^{y_1} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\right)\mathrm{d}\eta + \int_{x_2}^{x_1}\left(\int_0^{y_1} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\right)\mathrm{d}\eta \\ &\quad - \int_0^{y_1}\left(\int_0^{x_2} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\right)\mathrm{d}\eta + \int_{y_2}^{y_1}\left(\int_0^{x_2} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\right)\mathrm{d}\eta \\ &= \int_{x_2}^{x_1}\left(\int_0^{y_1} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\right)\mathrm{d}\eta + \int_{y_2}^{y_1}\left(\int_0^{x_2} f(\xi,\eta,z(\xi,\eta))\,\mathrm{d}\xi\right)\mathrm{d}\eta. \end{aligned}$$

Further, by the condition (D_3), we have the following inequalities:

$$\begin{aligned} \int_{x_2}^{x_1}\left(\int_0^{y_1} g(\xi,\eta)\,\mathrm{d}\xi\right)\mathrm{d}\eta &+ \int_{y_2}^{y_1}\left(\int_0^{x_2} g(\xi,\eta)\,\mathrm{d}\xi\right)\mathrm{d}\eta \\ &\leqslant F(z)(x_1,y_1) - F(z)(x_2,y_2) \\ &\leqslant \int_{x_2}^{x_1}\left(\int_0^{y_1} h(\xi,\eta)\,\mathrm{d}\xi\right)\mathrm{d}\eta + \int_{y_2}^{y_1}\left(\int_0^{x_2} h(\xi,\eta)\,\mathrm{d}\xi\right)\mathrm{d}\eta. \end{aligned}$$

Hence,

$$\begin{aligned} &|F(z)(x_1,y_1) - F(z)(x_2,y_2)| \\ &\quad\leqslant \left|\int_{[x_1,x_2]\times[0,y_1]} g\right| + \left|\int_{[0,x_2]\times[y_2,y_1]} g\right| + \left|\int_{[x_1,x_2]\times[0,y_1]} h\right| + \left|\int_{[0,x_2]\times[y_2,y_1]} h\right|. \end{aligned}$$

Since $g,h \in D_{\mathrm{HK}}$, the primitives of g and h are continuous and so are uniformly continuous on J. Hence, by the above inequality, the family $F(\tilde{B})$ is equi-uniformly continuous. In view of the Ascoli–Arzelà Theorem, $F(\tilde{B})$ is relatively compact.

Step 3. Lastly, we verify that the map $F\colon \tilde{B} \to \tilde{B}$ is continuous. Let $z_0 \in \tilde{B}$ and let $\{z_m\}$ be a sequence such that $z_m \in \tilde{B}$ for $m \in N$ and $z_m \to z_0$ as $m \to \infty$. According to condition (D_2), $f(\cdot,\cdot,z_m) \to f(\cdot,\cdot,z_0)$ as $m \to \infty$.

By Lemma 2.8 and condition (D_3),

$$\lim_{m\to\infty} \int_{\tilde{B}} f(\cdot,\cdot,z_m)\,\mathrm{d}s = \int_{\tilde{B}} f(\cdot,\cdot,z_0)\,\mathrm{d}s.$$

Hence, $\lim_{m\to\infty} F(z_m)(\cdot) = F(z_0)(\cdot)$, which implies that F is continuous.

Thus, F is a compact mapping. By Lemma 3.2, F has at least one fixed point, which completes the proof. □

In order to give a topological characterization of the solution set of equation (1.1), we state the well-known Vidossich Theorem.

Let $C_u(K,Y)$ be the space of all continuous mappings $x\colon K \to Y$, where K is a compact convex subset of a normed space and Y is a metric space equipped with the topology of uniform convergence. Denote by $B(t_0,\varepsilon)$ the closed ball with centre t_0 and radius ε. Denote by $x|_A$ the restriction of the map x to A.

Lemma 3.4 (Vidossich Theorem [14, Corollary 1.2]). *Let K be a compact convex subset of a normed space, Y be a closed convex subset of a Banach space Y_0, and F be a compact map $C_u(K,Y) \to C_u(K,Y)$. If there are $t_0 \in K$ and $y_0 \in Y$ such that the following two conditions hold*

(i) $F(x)(t_0) = y_0, \quad x \in C(K,Y),$

(ii) *For every $\varepsilon > 0$,*

$$x|_{k_\varepsilon} = y|_{k_\varepsilon} \Rightarrow F(x)|_{k_\varepsilon} = F(y)|_{k_\varepsilon}, \quad x,y \in C(K,Y),$$

where $K_\varepsilon = B(t_0,\varepsilon) \cap K$.

Then the set of fixed points of F is an R_δ.

Recall that an R_δ is the intersection of a decreasing sequence of compact absolute retracts. Furthermore, Vidossich [**14**] pointed out that R_δ is a non-empty, compact and connected set.

Now we come to the second main result.

Theorem 3.5. *Under the above assumptions* (D_1)–(D_3), *there exists an interval $J \subset K$ such that the set S of all solutions of equation (1.1) defined on J is an R_δ.*

Proof. In the proof of Theorem 3.3, we proved that the operator F maps $\tilde{B}$ into $\tilde{B}$ and F is a compact mapping.

On the other hand, it is clear that F satisfies conditions (i) and (ii) of Lemma 3.4. Thus, F satisfies all conditions of Lemma 3.4 and therefore the set S is an R_δ, which completes the proof. □

Remarks 3.6.

(i) We note that if g is a Lebesgue integrable function, $h = -g$,

$$(L)\int_0^x \int_0^y f(\xi,\eta,z)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.2}$$

exists for every $z \in B$ and $\partial^2 z/\partial x \partial y$ denotes the mixed usual derivative of z, then Theorem 3.5 reduces to the classical Carathéodory existence theorem discussed in [**7**].

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(ii) If g and h are Denjoy integrable functions,

$$(D)\int_0^x \int_0^y f(\xi,\eta,z)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.3}$$

exists for every $z \in B$ and $\partial^2_{ap} z/\partial x \partial y$ denotes the mixed approximate derivative of z (see [**5**] for the definition), then Theorem 3.5 reduces to [**4**, Theorem 3].

Therefore, Theorem 3.5 is an extension of the corresponding results in [**4**, **7**].

4. An example

Example 4.1. Consider the following Darboux problem:

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= f(x,y,z) = g(x,y,z) + l(x,y), \\ z(x,0) &= 0, \quad 0 \leqslant x \leqslant 1, \\ z(0,y) &= 0, \quad 0 \leqslant y \leqslant 1, \end{aligned}\right\} \tag{4.1}$$

where $l(x,y)$ is distributionally Henstock–Kurzweil integrable on $[0,1] \times [0,1]$ but not necessarily Lebesgue integrable, and $g(x,y,z)$ satisfies the Carathéodory conditions:

(L_1) the map $z \mapsto g(x,y,z)$ is continuous for a.e. $(x,y) \in [0,1] \times [0,1]$;

(L_2) the map $(x,y) \mapsto g(x,y,z)$ is measurable for all $\|z\|_\infty \leqslant 1$;

(L_3) there exists $g_1(x,y) \in L^1([0,1] \times [0,1])$ such that $|g(x,y,z)| \leqslant g_1(x,y)$ for a.e. $(x,y) \in [0,1] \times [0,1]$ and $\|z\|_\infty \leqslant 1$.

We know the D_{HK}-integral contains the wide Denjoy integral. Note that $l(x,y)$ is distributionally Henstock–Kurzweil integrable, according to equation (4.1) and condition (L_3); thus, we have

$$l(x,y) - g_1(x,y) \leqslant f(x,y,z) \leqslant l(x,y) + g_1(x,y)$$

for $(x,y) \in [0,1] \times [0,1]$, $\|z\|_\infty \leqslant 1$.

Since $l(x,y) \in D_{\mathrm{HK}}$, $g_1(x,y) \in L^1$, we have $l \pm g_1 \in D_{\mathrm{HK}}$. Hence, the existence of a solution of equation (4.1) is guaranteed by Theorem 3.3. However, the existence theorems of [**4**, **7**] are ineffective for equation (4.1).

Furthermore, Theorem 3.5 shows that the set of solutions to problem (4.1) is an R_δ.

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