On the Determination of Centres of Curvature.—In *Mathematical Notes* for May 1915, Dr J. M'Whan gives a note on the determination of centres of curvature, stating that "the only assumption involved is, therefore, as stated, that of the definition of the centre of curvature as the limiting position of the intersection of these normals."

Unfortunately, however, there are many applications of the notion of curvature to geometry, statics, dynamics, and other subjects which are not based directly on this definition, but which depend much more intimately on the conception of the circle of curvature as the limiting position of a circle passing through three points on the curve which ultimately coincide. This conception gives rise to an entirely independent definition of curvature. Most text-books assume that the two definitions are equivalent, but this is far from obvious, and, indeed, objections might be raised to that assumption based on mere considerations of common sense.

When two normals to a curve, say at two points P and Q intersect in O, their lengths OP and OQ are in general unequal. It is therefore impossible to describe a circle with centre O passing through both P and Q, and it thus appears impossible that the point O defined as the intersection of consecutive normals can be the centre of a circle satisfying the alternative definition of the circle of curvature.

I do not like worrying my students with rigorous proofs, but the difficulty seems to me so great that I now always keep my class taking down notes while I dictate the following proof in front of the blackboard (failing anything better).

To prove that the intersection of consecutive normals to a curve is the centre of curvature.

Let P be any point on a curve. Take the tangent and normal at P as axes of x and y. With these axes let the equation of the curve be $y=f(x)$, and let $f(x)$ be expanded in powers of x by Taylor's Theorem. Since the axis of x is tangent at the origin, $f(0)$ and $f'(0)$ vanish, and therefore the equation assumes the form

$$y = ax^2 + bx^3 + cx^4 + \dots$$

This gives

$$\frac{dy}{dx} = 2ax + 3bx^2 + 4cx^3 + \dots.$$

The equation of the normal at a near point $Q\,(x'\,y')$ is

$$(y-y')\,(2ax' + 3bx'^2 + \dots) + (x-x') = 0.$$

Putting $x = 0$ and dividing by x', we find that the point of intersection of the normals at P and Q is given by

$$(y - y')(2a + 3bx' + \ldots) = 1.$$

Putting x' and y' both zero, this gives in the limit

$$y = \frac{1}{2a}.$$

Now, according to Newton's treatment, the radius of curvature is given by

$$Lt\ \frac{y}{x^2} = \frac{1}{2\rho} \quad \text{whence} \quad \frac{1}{2a} = \rho,$$

and it follows that $y = \rho$ according to this definition.

The Newtonian method treats the circle of curvature at P as the limiting position of a circle *touching* the curve at P and passing through a point Q which ultimately coincides with P.

This method is a little less general than the definition of the circle of curvature as the limiting position of a circle through three consecutive points. But it is a necessary condition for the existence of a circle of curvature that the circle through three points of a curve should tend to a unique limiting position, whatever be the law according to which the points approach coincidence, and this covers the case, contemplated in the Newtonian method, in which two of the points coincide before the third point coincides with them.

Personally, I have always regarded it as pure "fudge" to treat the definition quoted by Dr M'Whan as equivalent to that based on considerations of the circle through three consecutive points of a curve. If such an assumption can be justified on any grounds, I should be interested to hear of them; meanwhile I have my doubts.

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On the Solutions of the Differential Equation

$$\frac{d^n x}{dt^n} + P_1 \frac{d^{n-1}x}{dt^{n-1}} + \ldots + P_{n-1}\frac{dx}{dt} + P_n x = 0.$$

The following is a direct proof that any $n + 1$ particular integrals of the differential equation

$$x_n + P_1 x_{n-1} + P_2 x_{n-2} + \ldots + P_{n-1} x_1 + P_n x = 0,$$