Canad. Math. Bull. Vol. **45** (3), 2002 pp. 417–421

On Deformations of the Complex Structure on the Moduli Space of Spatial Polygons

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Abstract. For an integer $n \geq 3$, let M_n be the moduli space of spatial polygons with n edges. We consider the case of odd n. Then M_n is a Fano manifold of complex dimension $n-3$. Let Θ_{M_n} be the sheaf of germs of holomorphic sections of the tangent bundle TM_n. In this paper, we prove $H^q(M_n, \Theta_{M_n}) = 0$ for all $q \geq 0$ and all odd n. In particular, we see that the moduli space of deformations of the complex structure on M_n consists of a point. Thus the complex structure on M_n is locally rigid.

1 Introduction

For an integer $n \geq 3$, let M_n be the moduli space of spatial polygons $P = (a_1, a_2, \dots, a_n)$ whose edges are vectors $a_i \in \mathbf{R}^3$ of length $|a_i| = 1$ $(1 \leq i \leq n)$. Two polygons are identified if they differ only by motions in $\mathbf{R}^3$. The sum of the vectors is assumed to be zero. Thus:

$$M_n = \{P = (a_1, \dots, a_n) \in (S^2)^n : a_1 + \dots + a_n = 0\} / \mathrm{SO}(3). \tag{1.1}$$

For odd n or $n = 4$, M_n has no singular points. In fact, this is a Fano manifold (*i.e.* the anticanonical bundle is ample) of complex dimension $n - 3$ [8]. On the other hand, for even $n \geq 6$, M_n has cone-like singular points [5].

In this paper, we assume n to be odd. Since $M_3 = \{\text{point}\}$, we assume that $n \geq 5$. Then many topological properties of M_n are already known. For example, the cohomology ring $H^*(M_n, \mathbf{R})$ is known in [1], [3], [7], and the intersection pairings $\int_{M_n} \alpha \cdot \beta \big(\alpha, \beta \in H^*(M_n, \mathbf{R})\big)$ are known in [4].

We consider the following problem: Is it possible to deform the complex structure on M_n? Let V be a complex manifold and let Θ_V be the sheaf of germs of holomorphic sections of the tangent bundle TV. Then it is well-known that deformations of the complex structure on V are parametrized by a subspace of the cohomology group $H^1(V, \Theta_V)$ (see [9]). In particular if $H^1(V, \Theta_V) = 0$, then the moduli space of deformations of the complex structure on V consists of a point. Thus we cannot deform the complex structure on V. We shall prove that the cohomology $H^*(V, \Theta_V)$ is special when $V = M_n$. Let Θ_{M_n} be the sheaf of germs of holomorphic sections of the tangent bundle TM_n. Then our main result is the following theorem.

Theorem A *For all $q \geq 0$ and all odd n, we have*

$$H^q(M_n, \Theta_{M_n}) = 0.$$

Received by the editors January 1, 2000; revised April 1, 2000.
AMS subject classification: Primary: 14D20; secondary: 32C35.
Keywords: polygon space, complex structure.

https://doi.org/10.4153/CMB-2002-043-8 Published online by Cambridge University Press

In particular, the fact $H^1(M_n, \Theta_{M_n}) = 0$ tells us the following:

Theorem B *For all odd n, the moduli space of deformations of the complex structure on M_n consists of a point. Thus the complex structure on M_n is locally rigid.*

Remark 1.2 When $n = 5$, Theorem A is already known. (See Section 3 for detail.)

This paper is organized as follows. In Section 2, we prove Theorem A except the cases $(n, q) = (5, 0), (5, 1)$ or $(7, 1)$. In Section 3, we study these cases.

2 Proof of Theorem A for General Cases

In this section, we prove the following:

Theorem 2.1

(i) For all odd $n \geq 7$, we have $H^0(M_n, \Theta_{M_n}) = 0$.
(ii) For all odd $n \geq 9$, we have $H^1(M_n, \Theta_{M_n}) = 0$.
(iii) For all $q \geq 2$ and all odd $n \geq 5$, we have $H^q(M_n, \Theta_{M_n}) = 0$.

First we prove Theorem 2.1(iii). Recall that M_n is a Fano manifold [8]. That is, the anticanonical bundle $K^* = \Lambda^{n-3}TM_n$ is ample, where we write the canonical bundle by K. Since $H^q(M_n, \Theta_{M_n}) \cong H^q(M_n, \Omega^{n-4}K^*)$, we have the result by the Kodaira-Nakano vanishing theorem [2].

In order to prove Theorem 2.1(i) and (ii), we identify M_n with the moduli space of stable points on $\mathbf{C}P^1$. In what follows, we fix odd n and set $n = 2m + 1$. Let $X = (\mathbf{C}P^1)^n$ and $G = \mathrm{PSL}(2, \mathbf{C})$. Then the group G acts diagonally on X. A n-tuple $(x_1, \dots, x_n) \in X$ is called *stable* if it contains no point of $\mathbf{C}P^1$ with multiplicity $> m$. Let X^s be the open subset of X consisting of all stable points. Then X^s is G-stable, the quotient $p\colon X^s \to Y$ exists and is a principal G-bundle, and Y is biholomorphic to M_n. In particular, p is an affine morphism and satisfies $p^G_* \mathcal{O}_{X^s} = \mathcal{O}_Y$, where p^G_* denotes the invariant direct image.

Let $\mathfrak{g}$ be the Lie algebra of G; let TX (resp. TY) be the tangent bundle of X (resp. Y), and let Θ_X (resp. Θ_Y) be its sheaf of germs of holomorphic sections. As p is a principal G-bundle, the differential $dp\colon TX^s \to p^*TY$ fits into an exact sequence of vector bundles over X^s:

$$0 \to \underline{\mathfrak{g}} \to TX^s \to p^*TY \to 0, \tag{2.2}$$

where $\underline{\mathfrak{g}}$ denotes the trivial bundle $X^s \times \mathfrak{g}$ over X^s.

The long exact sequence of cohomology defined by (2.2) begins with

$$0 \to H^0(X^s, \mathcal{O}_{X^s}) \otimes \mathfrak{g} \to H^0(X^s, \Theta_{X^s}) \to H^0(X^s, p^*\Theta_Y) \to$$

$$\to H^1(X^s, \mathcal{O}_{X^s}) \otimes \mathfrak{g} \to H^1(X^s, \Theta_{X^s}) \to H^1(X^s, p^*\Theta_Y) \to H^2(X^s, \mathcal{O}_{X^s}) \otimes \mathfrak{g}.$$

Take G-invariants in this exact sequence. Since p is affine and $p^G_* \mathcal{O}_{X^s} = \mathcal{O}_Y$, we have $H^q(X^s, p^*\Theta_Y)^G = H^q(Y, \Theta_Y)$ for all q. Thus, we have an exact sequence

$$0 \to \left(H^0(X^s, \mathcal{O}_{X^s}) \otimes \mathfrak{g}\right)^G \to H^0(X^s, \Theta_{X^s})^G \to H^0(Y, \Theta_Y) \to \tag{2.3}$$

$$\to \left(H^1(X^s, \mathcal{O}_{X^s}) \otimes \mathfrak{g}\right)^G \to H^1(X^s, \Theta_{X^s})^G \to H^1(Y, \Theta_Y) \to \left(H^2(X^s, \mathcal{O}_{X^s}) \otimes \mathfrak{g}\right)^G.$$

Proposition 2.4 *The restriction maps $H^q(X, \mathcal{O}_X) \to H^q(X^s, \mathcal{O}_{X^s})$ and $H^q(X, \Theta_X) \to H^q(X^s, \Theta_{X^s})$ are isomorphisms for $q \le m-2$.*

Proof We use (b) $\Rightarrow$ (d) of [10, p. 36, Theorem (1.14)]. For X, A and q in the theorem, we take $X = (\mathbf{C}P^1)^n$, $A = X - X^s$ and $q = m-1$. (Recall that we set $n = 2m+1$.) Let $\mathcal{F}$ be a locally free sheaf on X and we consider (b) of the theorem. In [10, p. 26], a subvariety of X is defined by $S_{m+k}(\mathcal{F}) = \{x \in X : \operatorname{codh}_{\mathcal{O}_x} \mathcal{F}_x \le m+k\}$. Here $\operatorname{codh}_{\mathcal{O}_x} \mathcal{F}_x$ denotes the homological codimension of $\mathcal{F}_x$ over $\mathcal{O}_x$ (see [10, p. 22]). Now it is easy to see that $S_{m+k}(\mathcal{F}) = \begin{cases} \varnothing & 0 \le k \le m \\ X & k \ge m+1. \end{cases}$ Hence we have $\dim\bigl(A \cap S_{m+k}(\mathcal{F})\bigr) \le k$ for all k. Thus (b) is satisfied in our situation. Then (d) of the theorem holds. Thus the restriction maps $H^q(X, \mathcal{F}) \to H^q(X^s, \mathcal{F})$ are bijective for $q \le m-2$ and injective for $q = m-1$. This completes the proof of Proposition 2.4. ■

Now we apply Proposition 2.4 to (2.3). Since $H^0(X, \mathcal{O}_X) = \mathbf{C}$, $H^0(X, \Theta_X) = \mathfrak{g}^n$ and $H^q(X, \mathcal{O}_X) = 0 = H^q(X, \Theta_X)$ if $q \ge 1$, we obtain for $m \ge 3$:

$$0 \to \mathfrak{g}^G \to (\mathfrak{g}^n)^G \to H^0(Y, \Theta_Y) \to 0 \quad \text{and} \quad H^1(Y, \Theta_Y) \subseteq \bigl(H^2(X^s, \mathcal{O}_{X^s}) \otimes \mathfrak{g}\bigr)^G.$$

Since $\mathfrak{g}^G = (\mathfrak{g}^n)^G = 0$, we have $H^0(Y, \Theta_Y) = 0$. Hence Theorem 2.1(i) holds.

Similarly, one obtains $H^1(Y, \Theta_Y) = 0$ for $m \ge 4$. Hence Theorem 2.1(ii) holds.

3 Proof of Theorem A For $n = 5$ and 7

By Theorem 2.1, it suffices to study $H^q(M_n, \Theta_{M_n})$ with $(n, q) = (5, 0), (5, 1)$ or $(7, 1)$ in order to complete the proof of Theorem A. First we study the case $n = 5$. For $r \le 6$, let S_r be the surface obtained from $\mathbf{C}P^2$ by blowing up r points in general position (the so called Del Pezzo surface of degree $9 - r$). Then M_5 is biholomorphic to S_4. (See [8, p. 74].) The cohomology $H^*(S_r, \Theta_{S_r})$ was determined in [9, p. 225–226] as follows. $\dim H^0(S_r, \Theta_{S_r}) = \begin{cases} 8-2r & r \le 3 \\ 0 & r \ge 4, \end{cases}$ $\dim H^1(S_r, \Theta_{S_r}) = \begin{cases} 0 & r \le 4 \\ 2r-8 & r \ge 5, \end{cases}$ and $\dim H^2(S_r, \Theta_{S_r}) = 0$. In particular, $H^*(S_r, \Theta_{S_r}) = 0$ if and only if $r = 4$.

Now the remaining case is $H^1(M_7, \Theta_{M_7})$. By Theorem 2.1(i) and (iii), it suffices to prove $\chi(M_7, \Theta_{M_7}) = 0$. We shall prove this in more general form.

Theorem 3.1 *For all odd n, we have*

$$\chi(M_n, \Theta_{M_n}) = 0.$$

In what follows, we prove this theorem using the Hirzebruch-Riemann-Roch formula. As in Section 2, we fix odd n and set $n = 2m+1$. First we recall the structure of $H^*(M_n, \mathbf{R})$. For $i \in \{1, \ldots, n\}$, we define $A_{n,i} \subset (\mathbf{R}^3)^n$ by

$$A_{n,i} = \left\{ P = (a_1, \ldots, a_n) \in (S^2)^n : a_1 + \cdots + a_n = 0 \text{ and } a_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

https://doi.org/10.4153/CMB-2002-043-8 Published online by Cambridge University Press

Let $SO(2)$ act on $\mathbf{R}^3$ by rotation about the z-axis. Then for odd n, the diagonal $SO(2)$-action on $(\mathbf{R}^3)^n$ is free on $A_{n,i}$ and we have $M_n = A_{n,i}/\,SO(2)$. (See (1.1).) Therefore, $A_{n,i} \to M_n$ is a principal $SO(2)$-bundle. Let $\xi_i \to M_n$ be the holomorphic line bundle associated to $A_{n,i} \to M_n$: $\xi_i = (A_{n,i} \times \mathbf{C})/S^1$, where we identify $SO(2)$ with S^1 and let S^1 act on $A_{n,i} \times \mathbf{C}$ by $(P,\alpha)\cdot g = (Pg, \alpha g)$ $\big((P,\alpha) \in A_{n,i} \times \mathbf{C}, g \in S^1\big)$. We define $z_i \in H^2(M_n, \mathbf{R})$ to be the first Chern class of the line bundle ξ_i: $z_i = c_1(\xi_i)$ $(1 \le i \le n)$. Now we have the following theorem.

Theorem 3.2 (**[1]**, **[3]**, **[7]**) *When $n = 2m+1$, the algebra $H^*(M_n, \mathbf{R})$ is generated by $z_1, \dots, z_n$ with the relations:*

(i) $z_1^2 = \cdots = z_n^2$.
(ii) $\prod_{j\in J}(z_i + z_j) = 0$, for all $i \in \{1,\dots,n\}$ and $J \subseteq \{1,\dots,n\}$ such that $i \notin J$ and $|J| = m$, where $|J|$ denotes the cardinal number.

Next we study the intersection pairings. For a sequence $(d_1,\dots,d_n)$ of nonnegative integers with $\sum_{i=1}^n d_i = n-3$, we set $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle = \int_{M_n} z_1^{d_1}\cdots z_n^{d_n}$. In particular for $0 \le k \le m-1$, we set $\langle \rho_{n,2k}\rangle = \int_{M_n} z_1^{2k} z_2 \cdots z_{n-2k-2}$. By Theorem 3.2(i) and the action of the symmetric group Σ_n on M_n, it suffices to determine $\langle \rho_{n,2k}\rangle$ for $0 \le k \le m-1$ in order to determine $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle$ for all sequences. Concerning this, we have the following:

Theorem 3.3 (**[4]**) *When $n = 2m+1$, the numbers $\langle \rho_{n,2k}\rangle$ $(0 \le k \le m-1)$ are given as follows.*

$$\langle \rho_{n,2k}\rangle = (-1)^k \frac{\binom{m-1}{k}\binom{2m-1}{m}}{\binom{2m-1}{2k+1}}.$$

Finally we recall the description of the total Chern class $c(TM_n)$.

Theorem 3.4 (**[3]**) *We have*

$$c(TM_n) = (1 - z_1^2)^{-1}\prod_{i=1}^{n}(1 + z_i).$$

Recall that we have holomorphic line bundles $\xi_i \to M_n$ $(1 \le i \le n)$. Using the Hirzebruch-Riemann-Roch formula [2], it is easy to prove the following proposition from Theorems 3.3 and 3.4.

Proposition 3.5 *For $1 \le i \le n$, we have*

(i) $\chi(M_n, \xi_i) = 0$.
(ii) $\chi(M_n, \xi_i^*) = -1$.

Now we prove Theorem 3.1. By Theorem 3.4, we have $\mathrm{ch}(TM_n) = -1 - e^{z_1} - e^{-z_1} + \sum_{i=1}^n e^{z_i}$. Using the Hirzebruch-Riemann-Roch formula, we have $\chi(M_n, \Theta_{M_n}) = -\chi(M_n, \mathcal{O}_{M_n}) - \chi(M_n, \xi_1) - \chi(M_n, \xi_1^*) + \sum_{i=1}^n \chi(M_n, \xi_i)$. By [6], [8], we have $\chi(M_n, \mathcal{O}_{M_n}) = 1$. Then we see by Proposition 3.5 that $\chi(M_n, \Theta_{M_n}) = -1 - 0 - (-1) + n \cdot 0 = 0$. This completes the proof of Theorem 3.1.

Acknowledgments The authors wish to thank John Millson for many useful comments.

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https://doi.org/10.4153/CMB-2002-043-8 Published online by Cambridge University Press