PRIME POWER REPRESENTATIONS OF FINITE LINEAR GROUPS

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1. Introduction. There are five well-known, two-parameter families of simple finite groups: the unimodular projective group, the symplectic group,[1] the unitary group,[2] and the first and second orthogonal groups, each group acting on a vector space of a finite number of elements **(2; 3)**. If k is the dimension of this space, we denote these groups by $\mathfrak{L}_k$, $\mathfrak{S}_k$, $\mathfrak{U}_k$, $\mathfrak{O}_k$ and $\mathfrak{O}_k'$, respectively. By analogy, groups $\mathfrak{O}_2$, $\mathfrak{O}_4$ and $\mathfrak{O}_2'$ (which are not simple) can be defined. Our main conern then is the proof of the following result:

THEOREM. *Let $\mathfrak{G}$ be one of the groups $\mathfrak{L}_k$, $\mathfrak{S}_k$, $\mathfrak{U}_k$, $\mathfrak{O}_k$ or $\mathfrak{O}_k'$ with $k \geqslant 2$. Let p be the characteristic of the base field, let d be the order of a p-Sylow subgroup $\mathfrak{P}$ of $\mathfrak{G}$, and let m be the index of the normalizer $\mathfrak{P}$ in $\mathfrak{G}$. Let Σ be any vector space of dimension d over a field of characteristic 0 or prime to m. Then $\mathfrak{G}$ has an irreducible representation of degree d with Σ as the representation space.*

The special case $\mathfrak{G} = \mathfrak{L}_2$ was proved by Jordan **(7)** and Schur **(12)**, independently; the case $\mathfrak{G} = \mathfrak{L}_3$ by Brinckmann **(1)**; and the case $\mathfrak{G} = \mathfrak{L}_n$ first by the present author **(13)** and then later by Green **(5)**. In **(4)**, Frame proved the theorem when $\mathfrak{G} = \mathfrak{U}_3$. All of these authors dealt only with the character of the representation, not with the representation itself. The methods of the present paper are constructive and yield the representation space and the representing matrices explicitly. It is hoped that the geometric ideas introduced in this construction may be of independent interest.

In §§2, 3, 4, and 5, the group $\mathfrak{L}_n$ is dealt with. In §6, the other groups are considered. In §7, a few observations are added.

As a general reference to the definitions and properties of the spaces and groups to be considered, we cite **(2)** and **(3)**.

2. Preliminary definitions and notations. Throughout §§2, 3, 4 and 5, V denotes a vector space of dimension n over a field of q elements and of characteristic p. The symbol S^r denotes an r-dimensional subspace of V; if the superscript is omitted, the dimension is to be taken as 1; subscripts are used to distinguish subspaces of the same dimension. The symbol $\{S^i, S^j, \ldots\}$ denotes the subspace spanned by $S^i, S^j, \ldots$.

Definition 1. An *r-simplex* is an ordered set of r linearly independent 1-spaces: $[S_1, S_2, \ldots, S_r]$. Each S_j is called a vertex of the simplex. An n-simplex is more briefly called a simplex.

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[1]Sometimes called the abelian group.

[2]Sometimes called the hyperorthogonal group.

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Definition 2. A *composition sequence* (abbreviated to c.s.) is a sequence of n subspaces $[S^1, S^2, \dots, S^n]$ such that $S^j \subset S^{j+1} (j = 1, 2, \dots, n-1)$.

Definition 3. Let $\Delta = [S_1, S_2, \dots, S_n]$ be a simplex and $\nabla = [S^1, S^2, \dots, S^n]$ a c.s. Suppose that there exists a permutation σ of the numbers $1, 2, \dots, n$ such that

$$S^j = \{S_{\sigma(1)}, S_{\sigma(2)}, \dots, S_{\sigma(j)}\} \qquad (j = 1, 2, \dots, n). \tag{1}$$

Then ∇ is called a *face* of Δ: a positively or negatively oriented face according as σ is even or odd. Each of the $n!$ faces of Δ determines an opposite face $\nabla_1 = [S_1^1, S_1^2, \dots, S_1^n]$ defined by

$$S_1^j = \{S_{\sigma(n)}, S_{\sigma(n-1)}, \dots, S_{\sigma(n-j+1)}\} \qquad (j = 1, 2, \dots, n). \tag{2}$$

Our first result is a useful characterization of opposite faces:

LEMMA 1. *If* $\nabla = [S^1, S^2, \dots, S^n]$ *and* $\nabla_1 = [S_1^1, S_1^2, \dots, S_1^n]$ *are two faces of a simplex* $\Delta = [S_1, S_2, \dots, S_n]$ *then a necessary and sufficient condition for* ∇ *and* ∇_1 *to be opposite is that*

$$S^j \cap S_1^{n-j} = 0 \qquad (j = 1, 2, \dots, n-1). \tag{3}$$

If ∇ *and* ∇_1 *are two c.s. for which* (3) *holds, there is a simplex* Δ, *uniquely determined to within an ordering of its vertices, which has* ∇ *and* ∇_1 *as* (*opposite*) *faces.*

Proof. The assumption that ∇ and ∇_1 are opposite faces of Δ implies the existence of a permutation σ such that (1) and (2) hold. But then, since the S_j are linearly independent, (3) holds. If ∇ and ∇_1 are faces which are not opposite, there exists a permutation τ, different from σ, such that

$$S_1^j = \{S_{\tau(n)}, S_{\tau(n-1)}, \dots, S_{\tau(n-j+1)}\} \qquad (j = 1, 2, \dots, n).$$

If j is the first index such that $\sigma(j) \neq \tau(j)$, then $S_{\sigma(j)} \subset S^j \cap S_1^{n-j}$, contradicting (3). Suppose finally that ∇ and ∇_1 are c.s. for which (3) holds. Then it follows that

$$S_j = S^j \cap S_1^{n-j+1}$$

is 1-dimensional for each j. Thus these S_j are the only possible choices for vertices of a simplex Δ relative to which ∇ and ∇_1 are opposite faces. To complete the proof, we note that, for each j, $S_j \subset S^j$ but $S_j \not\subset S^{j-1}$. Thus the S_j are linearly independent so that $\Delta = [S_1, S_2, \dots, S_n]$ is a simplex and the equations (1) and (2) hold with σ the identity.

3. The spaces Σ and Σ^*. We proceed now to define representation spaces and to develop some of their properties. If Δ is a simplex and ∇ a c.s., we introduce an inner product (Δ, ∇), defined to be 1, -1 or 0 according as ∇ is a positive face, a negative face or not a face of Δ. If F is an arbitrary but fixed field, we can extend this inner product, by linearity, to linear combinations of simplexes and to linear combinations of c.s. over F. In this way, relative to

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this inner product, dual spaces Σ and Σ^* are determined. Thus an element of Σ (Σ^*) is a linear combination of simplexes (c.s.), and it is defined to be 0 if and only if it is orthogonal to all elements of Σ^* (Σ).

If $\epsilon(\sigma)$ is defined to be 1 or -1 according as σ is even or odd, an immediate consequence of the definitions is the following:

LEMMA 2. *If $[S_1, S_2, \ldots, S_n]$ is a simplex and σ a permutation, then*

$$[S_{\sigma(1)}, S_{\sigma(2)}, \ldots, S_{\sigma(n)}] = \epsilon(\sigma)[S_1, S_2, \ldots, S_n].$$

If Δ is an $(n-1)$-simplex and S a linearly independent 1-dimensional subspace, then $[S, \Delta]$ is used in our next result to denote the n-simplex whose vertices are obtained by taking first S and then the vertices of Δ in a positive order.

LEMMA 3. *Let $\{\Delta\}$ be a set of $(n-1)$-simplexes, all contained in one $(n-1)$-dimensional subspace S^{n-1} of V. Let S be a 1-space not in S^{n-1}. Then $\sum \Delta = 0$ implies $\sum [S, \Delta] = 0$.*

Proof. To each face $\nabla = [S^1, S^2, \ldots, S^{n-1}]$ of Δ we make correspond n faces $\nabla_1, \nabla_2, \ldots, \nabla_n$ of $[S, \Delta]$ defined by

$$\nabla_k = [S^1, S^2, \ldots, S^{k-1}, \{S, S^{k-1}\}, \{S, S^k\}, \ldots, \{S, S^{n-1}\}]; \; (k = 1, 2, \ldots, n).$$

Then one sees that, for each k, $([S, \Delta], \nabla_k) = (-1)^{k-1}(\Delta, \nabla)$. The required result now follows by summation on Δ with ∇ and ∇_k held fast.

LEMMA 4. *Let $S_1, S_2, \ldots, S_{n+1}$ be 1-spaces, and, for $k = 1, 2, \ldots, n+1$, let $\Delta_k = [S_1, S_2, \ldots, S_{k-1}, \tilde{S}_k, S_{k+1}, \ldots, S_{n+1}]$, where $\tilde{S}_k$ denotes that this vertex is to be omitted. Then*

$$\sum{}' (-1)^k \Delta_k = 0, \tag{4}$$

the summation being over those Δ_k, which are simplexes.

Proof. Suppose first that no n of the S_j are linearly dependent. Let $\nabla = [S^1, S_2, \ldots, S^n]$ be any face of Δ_{n+1}. Thus there is a permutation σ such that (1) holds. It is easy to see that

$$(\Delta_{n+1}, \nabla) = \epsilon(\sigma), (\Delta_{\sigma(n)}, \nabla) = (-1)^{n-\sigma(n)}\epsilon(\sigma), (\Delta_j, \nabla) = 0 \text{ if } j \neq n+1, \sigma(n).$$

Thus the left side of (4) is orthogonal to each face of Δ_{n+1}. Similarly, it is orthogonal to each face of $\Delta_1, \Delta_2, \ldots, \Delta_n$ (since an interchange of two S_j changes the sign of this sum); hence it is 0.

In proving the general case, we may assume that n of the S_j, say $S_1, S_2, \ldots, S_n$, are linearly independent and that S_{n+1} is linearly dependent on $S_{n-k+1}, S_{n-k+2}, \ldots, S_n$ but on no smaller number of S_j $(j = 1, 2, \ldots, n)$. Then the k-dimensional case of the first part of the proof shows that the analogue of (4) holds for the $k+1$ k-simplexes formed from the vertices $S_{n-k+1}, \ldots, S_{n+1}$. By Lemma 3 (applied $n-k$ times), we may prefix each of these simplexes with the vertices $S_1, S_2, \ldots, S_{n-k}$ and get our result.

We now introduce convenient bases for Σ and Σ^*.

THEOREM 1. *Let* $\nabla_0 = [S_0^1, S_0^2, \ldots, S_0^n]$ *be a fixed c.s. Let* $\mathfrak{B}$ *be the set of simplexes* Δ *such that* $(\Delta, \nabla_0) = (-1)^{[\frac{1}{2}n]}$. *For each* Δ_j *in* $\mathfrak{B}$, *let* ∇_j *be the face opposite to* ∇_0. *Let* $\mathfrak{B}^*$ *be the set of such faces. Then the sets* $\mathfrak{B}$ *and* $\mathfrak{B}^*$ *are dual bases of* Σ *and* Σ^*.

Proof. We first prove that $\mathfrak{B}$ spans Σ. Let $\mathfrak{B}_r$ $(r = 0, 1, 2, \ldots, n-1)$ be the set of simplexes $\Delta = [S_1, S_2, \ldots, S_n]$ such that

$$S_0^j = \{S_1, S_2, \ldots, S_j\} \qquad (j = 1, 2, \ldots, r).$$

Thus $\mathfrak{B}_0$ consists of all simplexes, and $\mathfrak{B}_{n-1} = \mathfrak{B}$. We now show that any member of $\mathfrak{B}_r$ is the signed sum of at most $n - r$ members of $\mathfrak{B}_{r+1}$. Let

$$\Delta = [S_1, S_2, \ldots, S_r, T_{r+1}, \ldots, T_n]$$

be in $\mathfrak{B}_r$. Let S_{r+1} be any 1-space in $S_0^{r+1} \cap \{T_{r+1}, \ldots, T_n\}$. Then, by Lemma 4, applied to the $(n - r)$-space $\{T_{r+1}, \ldots, T_n\}$, the $(n - r)$-simplex $[T_{r+1}, \ldots, T_n]$ is a signed sum of at most $n - r$ $(n - r)$-simplexes, each of which has S_{r+1} as a vertex. By Lemma 3, Δ is a signed sum of at most $n - r$ members of $\mathfrak{B}_{r+1}$.

To complete the proof, we invoke Lemma 1, which implies that, if Δ_j, ∇_k are in $\mathfrak{B}, \mathfrak{B}^*$, then $(\Delta_j, \nabla_k) = \delta_{jk}$. Thus $\mathfrak{B}$ is linearly independent, hence is a basis of Σ, and $\mathfrak{B}^*$ is the dual basis of Σ^*.

COROLLARY. *A simplex* Δ *is the signed sum of those members* Δ_j *of* $\mathfrak{B}$ *which have a face* ∇_j *in common with* Δ *and which have* ∇_0 *as the opposite face, the signature being positive or negative according as the common face* ∇_j *does or does not have the same orientations on* Δ *and* Δ_j. *The sum consists of at most* $n!$ *terms. If* Δ *is not a member of* $\mathfrak{B}$, *the sum of these signatures is* 0.

Proof. The first two statements follow from the equation $\Delta = \sum (\Delta, \nabla_j)\, \Delta_j$ which is valid since $\mathfrak{B}$ and $\mathfrak{B}^*$ are dual bases. The equations $(\Delta_j, \nabla_0) = (-1)^{[\frac{1}{2}n]}$ then imply the third statement.

4. The Sylow subgroup $\mathfrak{P}$.

We now turn to the group $\mathfrak{L}_n$ of unimodular projective transformations. Since we are concerned only with the permutations of simplexes effected by members of $\mathfrak{L}_n$, and since a scalar transformation leaves all simplexes fixed, we may work with $\mathfrak{L}_n$ via representative elements of the unimodular group. Similar considerations apply to the other groups dealt with in §6.

The order and existence of a useful p-Sylow subgroup of $\mathfrak{L}_n$ is given by the next two lemmas:

LEMMA 5. *The order of a* p-*Sylow subgroup of* $\mathfrak{G} = \mathfrak{L}_n$ *is* $d = q^{\frac{1}{2}n(n-1)}$.

LEMMA 6. *Let* ∇_0 *be a given c.s. Let* $\mathfrak{N}$ *be the set of elements of* $\mathfrak{G}$ *which leave* ∇_0 *fixed, and let* $\mathfrak{P}$ *be the subset of* $\mathfrak{N}$ *composed of elements whose orders are powers of* p. *Then* $\mathfrak{P}$ *is a* p-*Sylow subgroup of* $\mathfrak{G}$ *and* $\mathfrak{N}$ *is its normalizer in* $\mathfrak{G}$.

Proof. For Lemma 5, the order of $\mathfrak{G}$ is available **(2; 3)**. To prove Lemma 6, we let $\nabla_0 = [S_0^1, S_0^2, \ldots, S_0^n]$ and then choose an ordered basis $X = (x_1, x_2, \ldots, x_n)$ of V such that

$$(5) \qquad S_0^j = \{x_1, x_2, \ldots, x_j\} \qquad (j = 1, 2, \ldots, n).$$

Then, relative to the basis X, $\mathfrak{N}$ consists of all subdiagonal matrices, and $\mathfrak{P}$ of those which in addition have only 1's on the main diagonal. All conclusions now easily follow.

We proceed to set up a 1-1 correspondence between the elements of the p-Sylow subgroup $\mathfrak{P}$ defined in Lemma 6 and the members of the basis $\mathfrak{B}$ of Σ defined in Theorem 1. Again let X be a basis of V satisfying (5). Relative to X, each element P of $\mathfrak{P}$ is represented by a matrix whose rows $s_1, s_2, \ldots, s_n$ may be interpreted as vectors in V. It is easy to see that the simplex

$$\Delta = (-1)^{[\frac{1}{2}n]}[\{s_1\}, \{s_2\}, \ldots, \{s_n\}]$$

is a member of $\mathfrak{B}$ and that the correspondence θ defined by $\theta P = \Delta$ is 1-1 from $\mathfrak{P}$ onto $\mathfrak{B}$. From the fact that each row in the product of two matrices is the image of the corresponding row of the first matrix under the transformation corresponding to the second matrix, it follows that $(\theta P_1)P_2 = \theta(P_1P_2)$, where P_1 and P_2 are any two elements of $\mathfrak{P}$. Thus the right multiplication by P on the set $\mathfrak{P}$ is mapped by θ onto the application of P to the set $\mathfrak{B}$; and this mapping is an isomorphism since θ is 1-1. We may sum up the results of this paragraph in the following theorems:

THEOREM 2. *The dimension of Σ (or Σ^*) is equal to the order of a p-Sylow subgroup of $\mathfrak{G}$.*

THEOREM 3. *If θ is the mapping from $\mathfrak{P}$ onto $\mathfrak{B}$ defined in the preceding paragraph, then θ induces an isomorphism between the right regular representation of $\mathfrak{P}$ and the group $\mathfrak{P}$ considered as acting on the set $\mathfrak{B}$. The group $\mathfrak{P}$ is simply transitive on the members of $\mathfrak{B}$.*

5. The representation $\mathfrak{R}$. Two final geometric results are necessary for the proof of the main theorem.

LEMMA 7. *Let m be the index of the normalizer of a p-Sylow subgroup of $\mathfrak{G} = \mathfrak{L}_n$. Then*

(i) *m is the number of c.s.;*

(ii) $m = \left(\prod_{j=1}^{n} (q^j - 1)\right) \Big/ (q-1)^n.$

Proof. The first statement follows from Lemma 6 and the fact that the group G is transitive on the c.s. For the second statement, see **(13)**.

LEMMA 8. *Let $\Delta_0 = [S_1, S_2, \ldots, S_n]$ be a simplex. Let ∇_σ be the face of Δ_0 corresponding to the permutation σ. For each σ, let $\{\Delta_{\sigma j}\}$ $(j = 1, 2, \ldots)$ be the set* *of simplexes which have ∇_σ as a positive face. Let m be the integer defined by Lemma 7. Then*

$$\sum_{\sigma, j} \epsilon(\sigma)\, \Delta_{\sigma j} = m\, \Delta_0. \tag{6}$$

Proof. An arbitrary simplex Δ makes one appearance on the left side of (6) for each face that Δ has in common with Δ_0, the signature being positive or negative according as this face does or does not have the same orientation on Δ and Δ_0. This face determines a unique opposite face ∇_0 of Δ. If we keep ∇_0 fixed and sum over those terms of (6) which give rise to ∇_0 in this way, then, by the corollary to Theorem 1, we get Δ_0. We then sum over ∇_0 to get the stated result.

In the case that $\mathfrak{G} = \mathfrak{L}_n$, we now state our main result:

THEOREM 4. *Let $\mathfrak{G} = \mathfrak{L}_n$ and let $\mathfrak{R}$ be the representation induced*[3] *in the space Σ by $\mathfrak{G}$. Further suppose that $\mathfrak{P}$ is a p-Sylow subgroup in $\mathfrak{G}$, that $\mathfrak{B}$ is the basis of Σ defined by Theorem* 1, *and that d and m are the order of $\mathfrak{P}$ and the index of the normalizer of $\mathfrak{P}$, respectively.* (These numbers are given by Lemma 5 and Lemma 7.) *Then*

(i) *in the sense of Theorem* 3, *$\mathfrak{R}$ restricted to $\mathfrak{P}$ is equivalent to the right regular representation of $\mathfrak{P}$; the degree of $\mathfrak{R}$ is thus d;*

(ii) *relative to $\mathfrak{B}$, $\mathfrak{R}$ is represented by a set of matrices each of which has only entries of* 0, 1 *or* -1; *in each row, at most $n!$ non-zero entries occur, and their sum is* 0, *if the row has more than one such entry;*

(iii) *if the base field F of the space Σ has characteristic prime to m, then $\mathfrak{R}$ is irreducible—in particular, this is so if the characteristic is* 0 *or p.*

Proof. Statements (i) and (ii) follow from Theorem 3 and the corollary to Theorem 1. To prove (iii), we show that the enveloping algebra of $\mathfrak{R}$ consists of all linear transformations from Σ to Σ. First, choose a basis X of V such that (5) holds, and set $S_j = \{x_j\}$ $(j = 1, 2, \ldots, n)$. We now note that, corresponding to each permutation σ of the numbers $1, 2, \ldots, n$, there exists an element Q_σ of $\mathfrak{L}_n$ such that $S_j Q_\sigma = S_{\sigma(j)}$. If σ is even, Q_σ may be defined by $x_j Q_\sigma = x_{\sigma(j)}$; if σ is odd, Q_σ may be defined by $x_1 Q_\sigma = -x_{\sigma(1)}$, $x_j Q_\sigma = x_{\sigma(j)}$, $j \neq 1$. If we now let Δ_0 be the simplex $[S_1, S_2, \ldots, S_n]$, it follows, by Theorem 3 and Lemma 8, that, for each Δ_i in $\mathfrak{B}$,

$$\Delta_i\, Q = m\, \Delta_0, \tag{7}$$

where $Q = \sum \epsilon(\sigma) P_j Q_\sigma$, the summation being over all permutations σ and all elements P_j of $\mathfrak{P}$. Now, let ∇' be the face of Δ_0 opposite to ∇_0 and let $\mathfrak{B}' = \{\Delta_i'\}$ (with $\Delta_0' = (-1)^{[\frac{1}{2}n]}\Delta_0$) be the corresponding basis of Σ, as given by Theorem 1. By Lemma 1, the only member of $\mathfrak{B}'$ which has ∇_0 as a face is Δ_0'. By the corollary to Theorem 1 and by (7),

[3]Since the elements of $\mathfrak{G}$ leave (Δ, ∇) invariant, they induce well-defined linear transformations in Σ and Σ^*. A similar remark applies in the case of Theorem 4′.

$$\Delta_0' Q = m\,\Delta_0',\ \Delta_i' Q = 0, \qquad i \neq 0. \tag{8}$$

By Theorem 3, applied to the basis $\mathfrak{B}'$, there exists, for each i, an element P_i' of $\mathfrak{G}$ such that

$$\Delta_0' P_i' = \Delta_i'. \tag{9}$$

Now, for each pair i, j, we set

$$T_{ij} = \frac{1}{m} P_i'^{-1}\, Q P_j'.$$

By (8) and (9), it follows that $\Delta_i' T_{ij} = \Delta_j'$; $\Delta_k' T_{ij} = 0$, $k \neq i$. Since the T_{ij} form a basis for the linear transformations from Σ to Σ, the proof of irreducibility is complete.

6. The symplectic, unitary and orthogonal groups. In this section, we consider the modifications necessary in the preceding development if the group $\mathfrak{L}_n$ is replaced by the other classical linear groups.

In the case of the unitary group, V denotes a vector space over a field of q^2 elements and of characteristic p; in the other cases, the field is to have q elements.

The symplectic group, $\mathfrak{S}_{2n}$, has an invariant, skew, bilinear form of pairs of vectors, $x = (\alpha_i)$, $y = (\beta_i)$, which may be taken as

$$(x, y) = \sum_{j=1}^{n} (\alpha_j \beta_{n+j} - \alpha_{n+j}\beta_j). \tag{*}$$

For the unitary group, $\mathfrak{U}_{2n}$, this is to be replaced by

$$(x, y) = \sum_{j=1}^{n} (\alpha_j \bar{\beta}_{n+j} + \alpha_{n+j}\bar{\beta}_j), \tag{*}$$

with $\bar{\beta} = \beta^q$; for $\mathfrak{U}_{2n+1}$, a term $\alpha_{2n+1}\bar{\beta}_{2n+1}$ is added.

For the first orthogonal group, $\mathfrak{O}_{2n}$, we choose the quadratic form

$$Q(x) = \sum_{j=1}^{n} \alpha_j \alpha_{n+j}; \tag{*}$$

a term α^2_{2n+1} is to be added in the case of $\mathfrak{O}_{2n+1}$; an irreducible quadratic form in α_{2n+1} and α_{2n+2} is to be added for $\mathfrak{O}'_{2n+2}$, the second orthogonal group. In these three cases, we introduce the inner product

$$(x, y) = Q(x + y) - Q(x) - Q(y).$$

Thus, in all cases, the concept of orthogonality of pairs of vectors exists.

Unless the contrary is stated, it is assumed in what follows that $\mathfrak{G}$ is any one of the groups S_{2n}, $\mathfrak{U}_{2n}$, $\mathfrak{U}_{2n+1}$, $\mathfrak{O}_{2n}$, $\mathfrak{O}_{2n+1}$ or $\mathfrak{O}'_{2n+2}$ with $n \geqslant 1$, the group $\mathfrak{G} = \mathfrak{O}_{2n+1}$ being excluded if q is even, since then $\mathfrak{G}$ is isomorphic to $\mathfrak{S}_{2n}$.

The symbol $c(S^r)$ denotes the subspace orthogonal to S^r.

Definition 4. If V underlies an orthogonal group, and if q is even, a subspace is *isotropic* if each of its vectors annuls the quadratic form Q. In all other cases, a subspace is isotropic if every two of its vectors are orthogonal.

Definition 5. A special $2r$-simplex is an ordered set of $2r$ isotropic 1-spaces $[S_1, S_2, \ldots, S_{2r}]$ for which there exist vectors s_j in S_j such that

$$(s_j, s_k) = 0, \quad (s_{r+j}, s_{r+k}) = 0, \quad (s_j, s_{r+k}) = \delta_{jk} \qquad (j, k = 1, 2, \ldots, r).$$

It is clear that the vertices of such a simplex are linearly independent. The vertices S_j and S_{r+j} are termed *opposite*. We shorten "special $2n$-simplex" to "simplex".

The existence of isotropic n-spaces and of simplexes follows at once from the equations (*). In each case, the first n basis vectors span an isotropic n-space and the $2n$ 1-spaces generated by the first $2n$ basis vectors are the vertices of a simplex.

Definition 6. A *special composition sequence* (s.c.s.) is a sequence of n isotropic subspaces $[S^1, S^2, \ldots, S^n]$ such that $S^j \subset S^{j+1}$ $(j = 1, 2, \ldots, n-1)$.

Definition 7. An *admissible permutation* (a.p.) of the numbers $1, 2, \ldots, 2n$ is a permutation σ such that $\sigma(n+j) \equiv n + \sigma(j) \pmod{2n}$ $(j = 1, 2, \ldots, n)$.

It is to be noted that an a.p. is determined by its effect on $1, 2, \ldots, n$. The a.p. form a group of order $2^n\, n!$ isomorphic to the hyper-octahedral group **(15)**. Each a.p. σ induces a permutation $\bar{\sigma}$ of the n pairs $(j, n+j)$. We set $\epsilon'(\sigma) = \epsilon(\sigma)\,\epsilon(\bar{\sigma})$.

Definition 8. Let $\Delta = [S_1, S_2, \ldots, S_{2n}]$ be a simplex and $\nabla = [S^1, S^2, \ldots, S^n]$ an s.c.s. Suppose that there exists an a.p. σ such that

$$S^j = \{S_{\sigma(1)}, S_{\sigma(2)}, \ldots, S_{\sigma(j)}\} \qquad (j = 1, 2, \ldots, n).$$

Then ∇ is termed a face of Δ: a positively or negatively oriented face according as $\epsilon'(\sigma)$ is 1 or -1. The face ∇_1 of Δ which is opposite to ∇ is defined by

$$\nabla_1 = [S_1^1, S_1^2, \ldots, S_1^n],\ S_1^j = \{S_{\sigma(n+1)}, S_{\sigma(n+2)}, \ldots, S_{\sigma(n+j)}\} \quad (j = 1, 2, \ldots, n).$$

The spaces Σ and Σ^* are defined as in §3.

Lemmas 1, 2, 3 and 4 have analogues which are:

LEMMA 1′. *If $\nabla = [S^1, S^2, \ldots, S^n]$ and $\nabla_1 = [S_1^1, S_1^2, \ldots, S_1^n]$ are two faces of a simplex $\Delta = [S_1, S_2, \ldots, S_{2n}]$, then a necessary and sufficient condition for ∇ and ∇_1 to be opposite is that*

$$(3') \qquad S^j \cap c(S_1^j) = 0, \qquad j = 1, 2, \ldots, n.$$

If ∇ and ∇_1 are two s.c.s. for which (3′) holds, there exists a simplex Δ, uniquely determined to within an ordering of its vertices, which has ∇ and ∇_1 as (opposite) faces.

LEMMA 2′. *If $[S_1, S_2, \ldots, S_{2n}]$ is a simplex and σ is an a.p., then*

$$[S_{\sigma(1)}, S_{\sigma(2)}, \ldots, S_{\sigma(2n)}] = \epsilon'(\sigma)[S_1, S_2, \ldots, S_{2n}].$$

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LEMMA 3′. *Let* $[S_1, S_2]$ *be a special* 2-*simplex contained in a* 2-*space* S^2. *Let* $\{\Delta\}$ *be a collection of special* $(2n-2)$-*simplexes contained in* $c(S^2)$. *For each* Δ, *let* Δ' *be the special* $2n$-*simplex which has* S_1 *and* S_2 *as its first and* $(n+1)$*st vertices and the vertices of* Δ, *taken in positive order, as its remaining vertices. Then* $\sum \Delta = 0$ *implies* $\sum \Delta' = 0$.

LEMMA 4′. *Let* Δ *be a simplex and* S *an isotropic* 1-*space. Then* Δ *can be expressed as a sum of simplexes each of which has* S *as a vertex.*

Proof. The proofs of Lemmas 1′, 2′ and 3′ are virtually the same as those of Lemmas 1, 2 and 3, and so may be omitted. As a first step in the proof of Lemma 4′, we check two special cases. If $n = 1$, $\Delta = [S_1, S_2]$, $S \neq S_1$, $S \neq S_2$, then it is easy to verify that $[S_1, S]$ and $[S, S_2]$ are simplexes (see Definition 5), and that

$$[S_1, S_2] = [S_1, S] + [S, S_2].$$

Next suppose that $n = 2$, $\Delta = [S_1, S_2, S_3, S_4]$, and that S is orthogonal to exactly one vertex of Δ, say to S_1. Then, if

$$T = c(S) \cap \{S_2, S_3\}, \quad U = c(S) \cap \{S_3, S_4\},$$

the required conclusion may be drawn from the equation

$$[S_1, S_2, S_3, S_4] = [S, T, S_3, U] + [S_1, S_2, T, S] + [S_1, S, U, S_4].$$

The rest of the proof consists in showing that any other case can be reduced to one of these two cases. We may suppose that $n \geqslant 2$ and that S is not orthogonal to a pair of opposite vertices, say S_1 and S_{n+1}, since, then, the restriction to $c(\{S_1, S_{n+1}\})$ and an application of Lemma 3′ effectively replaces n by $n-1$. Thus we may suppose that the two vertices S_{n+1} and S_{n+2} are not orthogonal to S. Now set

$$T = c(S) \cap \{S_{n+1}, S_{n+2}\}, \quad U = c(T) \cap \{S_1, S_2\}.$$

Then the following is a relation among special 4-simplexes, all in one 4-space:

$$[S_1, S_2, S_{n+1}, S_{n+2}] = [S_1, U, T, S_{n+2}] + [U, S_2, S_{n+1}, T]. \tag{10}$$

By Lemma 3′, if the vertices $S_3, \ldots, S_n, S_{n+3}, \ldots, S_{2n}$ are adjoined to these 4-simplexes, an expression is obtained for Δ as a sum of two simplexes each of which has at least one vertex orthogonal to S. If $n \geqslant 3$, this construction can be repeated, with the indices 1 and 2 replaced by 2 and 3, to yield a second vertex orthogonal to S. Finally, if $n \geqslant 2$ and S is orthogonal to two vertices of Δ (which may be taken as non-opposite), say to S_1 and S_2, and not orthogonal to S_{n+1} and S_{n+2}, then the same construction yields, on the right side of (10), two simplexes, each of which has a pair of opposite vertices orthogonal to S; and this case has already been considered.

In the statement of Theorem 1, the number $(-1)^{[\frac{1}{2}n]}$ is to be replaced by $(-1)^n$, in the present case; in the corollary to Theorem 1, $n!$ by $2^n\, n!$. No changes are required in the proof.

The analogue of Lemma 5 is:

LEMMA 5′. *The order of a p-Sylow subgroup of* $\mathfrak{G}$ *is*

$$d = q^{n^2},\quad q^{n(2n-1)},\quad q^{n(2n+1)},\quad q^{n(n-1)},\quad q^{n^2}\quad \textit{or}\quad q^{n(n+1)}$$

according as

$$\mathfrak{G} = \mathfrak{S}_{2n},\quad \mathfrak{U}_{2n},\quad \mathfrak{U}_{2n+1},\quad \mathfrak{O}_{2n},\quad \mathfrak{O}_{2n+1}\quad \textit{or}\quad \mathfrak{O}'_{2n+2}.$$

Proof. See **(2; 3)** for the order of $\mathfrak{G}$.

The statement of Lemma 6 goes over intact and the proof is similar; so both may be omitted. The same remark applies to Theorems 2 and 3.

We now note an exception that occurs (only) in the case that $\mathfrak{G} = \mathfrak{O}_{2n}$. Then the isotropic n-spaces form two families such that two members of the same family (of opposite families) intersect in a space of dimension $n - r$ with r even (odd), and such that the elements of $\mathfrak{O}_{2n}$ permute these n-spaces within their separate families **(3**, p. 48**)**. The first property implies that at most one-half of the 2^n isotropic n-spaces spanned by sets of n vertices of a given simplex can fail to intersect a given isotropic n-space; thus, in the corollary to Theorem 1, the number $n!$ may be replaced by $2^{n-1}\, n!$, in this case. The second property implies that the group $\mathfrak{O}_{2n}$ is not transitive on all of the s.c.s., only on one-half of them. Thus the analogue of Lemma 7 takes the following form:

LEMMA 7′. *Let m be the index of the normalizer of a p-Sylow subgroup of* $\mathfrak{G}$. *Then*

(i) *if* $\mathfrak{G} = \mathfrak{O}_{2n}$, *m is one-half the number of s.c.s.; if* $\mathfrak{G} = \mathfrak{S}_{2n}$, $\mathfrak{U}_{2n}$, $\mathfrak{U}_{2n+1}$, $\mathfrak{O}_{2n+1}$ *or* $\mathfrak{O}'_{2n+2}$, *m is the number of s.c.s.;*

(ii) *if* $\mathfrak{G} = \mathfrak{S}_{2n}$, $\mathfrak{U}_{2n}$, $\mathfrak{U}_{2n+1}$, $\mathfrak{O}_{2n}$, $\mathfrak{O}_{2n+1}$ *or* $\mathfrak{O}'_{2n+2}$, *then*

$$m = \left(\prod_{j=1}^{n} (q^{2j} - 1)\right) \Big/ (q-1)^n,\ \left(\prod_{j=1}^{2n} (q^j - (-1)^j)\right) \Big/ (q^2 - 1)^n,$$

$$\left(\prod_{j=1}^{2n+1} (q^j - (-1)^j)\right) \Big/ (q^2 - 1)^n,$$

$$(q^n - 1)\left(\prod_{j=1}^{n-1} (q^{2j} - 1)\right) \Big/ (q-1)^n,$$

$$\left(\prod_{j=1}^{n} (q^{2j} - 1)\right) \Big/ (q-1)^n \quad \textit{or} \quad (q^{n+1} + 1)\left(\prod_{j=2}^{n} (q^{2j} - 1)\right) \Big/ (q-1)^{n-1}.$$

Proof. Part (ii) is easily established by counting the number of s.c.s. using induction on n. If S is an isotropic 1-space, one may invoke the induction hypothesis on the quotient space $c(S)/S$ with the induced definition of isotropy We omit the details.

In the modified statement of Lemma 8, only admissible permutations are to be considered; if $\mathfrak{G} = \mathfrak{O}_{2n}$, a further restriction is to be made to even permutations. No essential change occurs in the proof. The analogue of Theorem 4 may be stated as follows:

THEOREM 4′. *Let $\mathfrak{G}$ be one of the groups $\mathfrak{S}_{2n}$, $\mathfrak{U}_{2n}$, $\mathfrak{U}_{2n+1}$, $\mathfrak{O}_{2n}$, $\mathfrak{O}_{2n+1}$ or $\mathfrak{O}'_{2n+2}$. Let $\mathfrak{R}$ be the representation induced*[3] *in the space Σ by $\mathfrak{G}$. Further suppose that $\mathfrak{P}$ is a p-Sylow subgroup in $\mathfrak{G}$, that $\mathfrak{B}$ is the basis of Σ defined by Theorem* 1′, *and that d and m are the order of $\mathfrak{P}$ and the index of the normalizer of $\mathfrak{P}$, respectively.* (These numbers are given by Lemma 5′ and Lemma 7′). *Then all conclusions of Theorem* 4 *are valid if, in* (ii), *the number $n!$ is replaced by $2^{n-1}\, n!$ if $\mathfrak{G} = \mathfrak{O}_{2n}$, and by $2^n\, n!$ in all other cases.*

The proof of Theorem 4 carries over without essential change.

Theorems 4 and 4′ imply the theorem stated in the introduction.

7. Concluding remarks. Our first remarks take the form of two conjectures which, if true, provide converses to the theorem of the introduction:

CONJECTURE 1. *The group $\mathfrak{G}$ does not have an irreducible representation of degree d over a field whose characteristic divides m.*

We are able to prove the following weaker result:

THEOREM 5. *Using the notations of Theorems* 4 *and* 4′, *if the characteristic of F divides m, then the representation $\mathfrak{R}$ is reducible.*

Proof. It is convenient to introduce "boundary" operators b and b^* on Σ and Σ^*: for each simplex Δ, let $b\Delta$ be the signed sum of the faces of Δ, the sign of a face being that of its orientation on Δ; for each c.s. ∇, let $b^*\nabla$ be the sum of those simplexes which have ∇ as a positive face; then extend b and b^* to all of Σ and Σ^* by linearity. Lemmas 8 and 8′ may now be rewritten as: $b^*b\Delta_0 = m\Delta_0$. Thus, if Δ is a simplex and ∇ a c.s., it follows that $(b^*b\,\Delta, \nabla) = m(\Delta, \nabla)$, and this is easily seen to be equivalent to $(b^*\nabla, b\Delta) = m(\Delta, \nabla)$. The assumption $m = 0$ then implies that $b^*\,\Sigma^*$ and $b\Sigma$ are orthogonal. It is easy to see that neither of them is 0. Hence $b^*\,\Sigma^*$ is a proper non-zero subspace of Σ. This subspace is invariant under $\mathfrak{R}$: if G is an element of $\mathfrak{G}$ and ∇ is a c.s., then $(b^*\nabla)G = b^*(\nabla G)$. Thus $\mathfrak{R}$ is reducible.

CONJECTURE 2. *The notation being that of Theorems* 4 *and* 4′, *any proper subgroup $\mathfrak{H}$ of $\mathfrak{G}$ does not have an irreducible representation of degree d. In particular, the restriction of $\mathfrak{R}$ to $\mathfrak{H}$ is reducible.*

If $\mathfrak{G} = \mathfrak{L}_2$, $\mathfrak{L}_3$, $\mathfrak{U}_3$ or $\mathfrak{S}_4$, this statement follows from results of Moore **(11)**, Wiman **(14)**, Hartley **(6)** and Mitchell **(9; 10)**, who have shown that, in these cases, every proper subgroup $\mathfrak{H}$ of $\mathfrak{G}$ has order less than d^2.

In **(13)**, an alternative method is used to derive the character of $\mathfrak{R}$ in the case that $\mathfrak{G} = \mathfrak{L}_n$. There, use is made of a correspondence between $\mathfrak{L}_n$ and the symmetric group of degree n. If $\mathfrak{G}$ is one of the other groups considered in this paper, a similar correspondence exists between $\mathfrak{G}$ and the hyper-octahedral group of the appropriate degree, and yields the character of $\mathfrak{R}$. However, this method leans heavily on a previous determination of the characters of the symmetric and hyper-octahedral groups and does not deal with the representation itself.

Our final observation is that the special case $n = 3$ of the corollary to Theorem 1 also follows from a theorem on graphs (**8**, p. 126).

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