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SOME CONTRIBUTIONS TO THE THEORY OF NEAR-CRITICAL BISEXUAL BRANCHING PROCESSES

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Abstract

We investigate the probabilistic evolution of a near-critical bisexual branching process with mating depending on the number of couples in the population. We determine sufficient conditions which guarantee either the almost sure extinction of such a process or its survival with positive probability. We also establish some limiting results concerning the sequences of couples, females, and males, suitably normalized. In particular, gamma, normal, and degenerate distributions are proved to be limit laws. The results also hold for bisexual Bienaymé–Galton–Watson processes, and can be adapted to other classes of near-critical bisexual branching processes.

Keywords: Bisexual process; branching process; limiting behaviour

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1. Introduction

We consider the bisexual process with mating depending on the number of couples (introduced in [18]) as a two-type branching model $\{(F_n, M_n)\}_{n\geq 1}$ initiated with $Z_0 = N \geq 1$ couples (female–male mating units) and defined, for $n = 0, 1, \dots$, recursively by

$$(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{n,i}, m_{n,i}), \qquad Z_{n+1} = L_{Z_n}(F_{n+1}, M_{n+1}), \tag{1.1}$$

where the empty sum is taken as $(0, 0)$, $\{(f_{n,i}, m_{n,i})\}_{n\geq 0, i\geq 1}$ is a sequence of independent and identically distributed nonnegative, integer-valued random vectors, and $\{L_k\}_{k\geq 0}$ is a sequence of nonnegative real functions on $\mathbb{R}^+ \times \mathbb{R}^+$. Each L_k is assumed to be nondecreasing in each argument, integer-valued on the integers, and such that $L_k(x, 0) = L_k(0, y) = 0$, $x, y \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$, with $\mathbb{R}^+$ and $\mathbb{Z}^+$ denoting the nonnegative real numbers and nonnegative integer numbers respectively. From an intuitive viewpoint, $(f_{n,i}, m_{n,i})$ denotes the number of females and males descending from the ith couple of generation n. It follows that (F_{n+1}, M_{n+1}) represents the number of females and males in the $(n + 1)$th generation, which form Z_{n+1} couples according to the mating function L_{Z_n}. These couples reproduce independently through the same offspring distribution for each generation. It can be verified that $\{(Z_{n-1}, F_n, M_n)\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 0}$ are homogeneous Markov chains. The motivation behind this stochastic process

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is an interest in developing mathematical models to describe the probabilistic evolution of two-sex populations in which, because of environmental, social or other factors, matings between females and males could be influenced by the number of their progenitor couples.

When the mating functions do not depend on the number of couples, namely $L_k(x, y) = L(x, y)$, $x, y \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$, then the process is the classical bisexual Bienaymé–Galton–Watson process introduced in [5] which has received considerable attention in the literature; see, for example, [1]–[4], [6]–[10], and [17]. For a survey about this model we refer the reader to [11].

Note that if $Z_n = 0$ for some n, then the process (1.1) will become extinct. Let $q_N = \mathrm{P}(Z_n \to 0 \mid Z_0 = N)$ be the extinction probability when initially there are N couples in the population. Assuming the classical extinction–explosion duality in branching process theory

$$\mathrm{P}(Z_n \to 0 \mid Z_0 = N) + \mathrm{P}(Z_n \to \infty \mid Z_0 = N) = 1, \tag{1.2}$$

and imposing some requirements on the sequence of mating functions, necessary and sufficient conditions for its almost sure extinction and results concerning its limiting behaviour were established in [18] and [19]. In particular, it was proved that the asymptotic rate $r = \lim_{k \nearrow \infty} r_k$ exists and $r = \sup_{k \geq 1} r_k$ where, for $k = 1, 2, \ldots,$

$$r_k = \mathrm{E}[Z_n^{-1} Z_{n+1} \mid Z_n = k] = k^{-1}\, \mathrm{E}\bigg[L_k\bigg(\sum_{i=1}^{k} f_{n,i}, \sum_{i=1}^{k} m_{n,i}\bigg)\bigg].$$

Note that r_k represents the expected proportional change in the number of couples from one generation to the next if the current number of couples is k. Considering that the function $\mathcal{L}(k, x, y) = L_k(x, y)$, $k \in \mathbb{Z}^+$, $x, y \in \mathbb{R}^+$, is superadditive, it was also proved, assuming finite $\mu = (\mathrm{E}[f_{0,1}], \mathrm{E}[m_{0,1}])$, that if $\lim_{N \nearrow \infty} N^{-1} L_N(N\mu) < \infty$ then

$$q_N = 1, \quad N = 1, 2, \ldots, \qquad \text{if and only if} \qquad r \leq 1. \tag{1.3}$$

In analogy with asexual branching process theory, this result induces a classification for the bisexual processes given in (1.1) into supercritical ($r > 1$), critical ($r = 1$), and subcritical ($r < 1$) cases. We remark that in order to derive (1.3), since $r = \sup_{k \geq 1} r_k$, it is required that $r_k \leq r$, $k = 1, 2, \ldots$; in particular, for the critical case, it is necessary that $r_k \leq 1, k = 1, 2, \ldots$. Let us call the process near-critical if the sequence $\{r_k\}_{k \geq 1}$ approaches the asymptotic rate of a critical process as the number of couples goes to infinity, namely $\lim_{k \nearrow \infty} r_k = 1$, in such a way that $r_k > 1$ for some k. This situation has not been studied in bisexual process theory. We now present an example.

Example 1.1. It is well known that salmon live in the oceans of the northern hemisphere and enter the mouths of European and North American rivers at regular times. At the time of reproduction, the salmon return to the rivers where they hatched. The spawning process involves the mature salmon (male and female) swimming upstream overcoming strong river currents, waterfalls, and other obstacles to reach their home spawning ground. Then, the female releases her eggs and the male fertilizes them. After spawning, the adult salmon die. Taking into account this special conduct, in a first approximation it may be appropriate to describe the probabilistic evolution of the number of female and male salmon in a habitat in terms of a bisexual process (1.1).

Consider an offspring probability distribution such that $\mathrm{P}(f_{0,1} = 0)\,\mathrm{P}(m_{0,1} = 0) > 0$ and $\mathrm{E}[f_{0,1} + m_{0,1}] = 2$, and assume the sequence of mating functions $\{L_k\}_{k \geq 0}$, with

$$L_k(x, y) = \left\lfloor \tfrac{1}{2}(x + y) + b_k \right\rfloor \mathbf{1}_{\{x, y > 0\}}, \qquad x, y \in \mathbb{R}^+,$$

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where $\lfloor \cdot \rfloor$ denotes the integer-part function, $\mathbf{1}_{\{\cdot\}}$ is the indicator function, and $\{b_k\}_{k\geq 0}$ is a sequence of real numbers such that $b_k \geq 1$, $k = 1, 2, \ldots$, and $\lim_{k\nearrow\infty} k^{-1} b_k = 0$. Clearly,

$$r_k = k^{-1} \operatorname{E}[\lfloor T_{n,k} + b_k \rfloor \mathbf{1}_{\{\sum_{i=1}^k f_{n,i}>0, \sum_{i=1}^k m_{n,i}>0\}}], \qquad k = 1, 2, \ldots,$$

where $T_{n,k} = 1/2 \sum_{i=1}^k (f_{n,i} + m_{n,i})$. It can be verified that $r_k > 1$, $k \geq k_0$, for some $k_0 > 0$. We now prove that $\lim_{k\nearrow\infty} r_k = 1$. To this end, we introduce the modified rates $\widetilde{r}_k = k^{-1} \operatorname{E}[\lfloor T_{n,k} + b_k \rfloor]$. Since, for $n \in \mathbb{Z}^+$,

$$\operatorname{E}[T_{n,k}] + b_k - 1 \leq k\widetilde{r}_k \leq \operatorname{E}[T_{n,k}] + b_k + 1, \qquad k = 1, 2, \ldots,$$

using $\operatorname{E}[T_{n,k}] = k$, we can deduce that $\lim_{k\nearrow\infty} \widetilde{r}_k = 1$. Now, $\widetilde{r}_k - k^{-1} C_k \leq r_k \leq \widetilde{r}_k$, $k = 1, 2, \ldots$, where $C_k = \operatorname{E}[\lfloor T_{n,k} + b_k \rfloor (F_{n,k} + M_{n,k})]$ with $F_{n,k} = \prod_{i=1}^k \mathbf{1}_{\{f_{n,i}=0\}}$ and $M_{n,k} = \prod_{i=1}^k \mathbf{1}_{\{m_{n,i}=0\}}$. It is matter of some straightforward calculations to verify that $\operatorname{E}[(T_{n,k} + b_k)(F_{n,k} + M_{n,k})] = (k + b_k)(\operatorname{P}(f_{0,1} = 0)^k + \operatorname{P}(m_{0,1} = 0)^k)$, so it follows that $C_k = O(k \max\{\operatorname{P}(f_{0,1} = 0), \operatorname{P}(m_{0,1} = 0)\}^k)$, and we conclude that $\lim_{k\nearrow\infty} r_k = 1$.

In this paper we assume a process (1.1) such that (1.2) holds. The aim is to investigate, for the near-critical case, questions about its limiting evolution. The paper is organized as follows. In Section 2, using different probabilistic approaches based on martingale theory or stochastic difference equations, we provide some sufficient conditions which guarantee either the almost sure extinction of the process (Theorems 2.1 and 2.2) or its survival with positive probability (Theorem 2.3). Section 3 is devoted to investigating different kinds of limiting behaviour for $\{Z_n\}_{n\geq 0}$ (Theorem 3.1), $\{F_n\}_{n\geq 1}$, and $\{M_n\}_{n\geq 1}$ (Theorem 3.2), suitably normalized. In particular, gamma, normal, or degenerate distributions are derived as asymptotic laws. The results obtained in Sections 2 and 3 also hold for bisexual Bienaymé–Galton–Watson processes, and could be adapted to other classes of near-critical bisexual branching processes. Finally, in order to allow a more comprehensible reading, the proofs are relegated to Section 4.

2. Extinction probability

Our first result implies a slight improvement of the sufficient condition given in (1.3), because a finite number of r_k are allowed to be greater than 1.

Theorem 2.1. *Assume that $r_k \leq 1$, $k \geq k_0$, where k_0 is a positive integer. Then, $q_N = 1$, $N = 1, 2, \ldots$.*

Remark 2.1. Theorem 2.1 holds assuming that $\limsup_{k\nearrow\infty} r_k \leq 1$. Furthermore, along the lines of its proof, writing

$$r_k^\pi = \operatorname{E}[\pi(Z_n)^{-1} \pi(Z_{n+1}) \mid Z_n = k], \qquad k = 1, 2, \ldots,$$

where π is a nondecreasing and unbounded function on $\mathbb{R}^+$, we deduce that the existence of $k_0 > 0$ such that $r_k^\pi \leq 1$, $k \geq k_0$, implies $\operatorname{P}(\pi(Z_n) \to \infty \mid Z_0 = N) = 0$, or, equivalently, $\operatorname{P}(Z_n \to \infty \mid Z_0 = N) = 0$. Hence, by (1.2), $q_N = 1$, $N = 1, 2, \ldots$. Using this reasoning, sufficient conditions for almost sure extinction can be determined even if an infinite number of r_k are greater than 1. To this end, we will apply some probabilistic techniques considered in [13] for stochastic difference equations suitably adapted to the class of bisexual processes (1.1).

Note that $\{Z_n\}_{n\geq 0}$ satisfies, almost surely, the relation

$$Z_{n+1} = Z_n + Z_n \varepsilon_{Z_n} + \xi_{n+1}, \qquad n \in \mathbb{Z}^+, \tag{2.1}$$

where $\varepsilon_{Z_n} = r_{Z_n} - 1$ and $\xi_{n+1} = Z_{n+1} - \mathrm{E}[Z_{n+1} \mid Z_n]$.

Assuming $\mathrm{var}[Z_{n+1} \mid Z_n = k] < \infty$, $k = 1, 2, \dots$, it is easy to verify that $\{\xi_n\}_{n\geq 1}$ is a square-integrable martingale difference with respect to the sequence of σ-algebras $\{\mathcal{F}_n\}_{n\geq 0}$, where $\mathcal{F}_n = \sigma(Z_0, \dots, Z_n)$. Let us introduce, for $k = 1, 2, \dots$ and $\alpha > 0$, the *α-order absolute variation rates*:

$$R_{k,\alpha} = \mathrm{E}[|Z_n^{-1}(Z_{n+1} - \mathrm{E}[Z_{n+1} \mid Z_n])|^{\alpha} \mid Z_n = k] = k^{-\alpha}\,\mathrm{E}[|\xi_{n+1}|^{\alpha} \mid Z_n = k].$$

In particular, $\mathrm{var}[Z_{n+1} \mid Z_n = k] = k^2 R_{k,2}$, $k = 1, 2, \dots$.

Theorem 2.2. *Assume that*

(i) $\limsup_{k\nearrow\infty} 2\varepsilon_k R_{k,2}^{-1} < 1$,

(ii) $\lim_{k\nearrow\infty} \varepsilon_k^{-1} R_{k,2+\delta} = 0$, *for some* $0 < \delta \leq 1$.

Then $q_N = 1$, $N = 1, 2, \dots$.

We now state some sufficient conditions which guarantee a positive probability of survival.

Theorem 2.3. *Assume that there exists* $k_0 > 0$ *such that* $r_k > 1$, $k \geq k_0$, *and*

(i) $\liminf_{k\nearrow\infty} 2\varepsilon_k R_{k,2}^{-1} > 1$,

(ii) $\lim_{k\nearrow\infty} (\log k)^{1+\alpha} R_{k,2}^{-1} R_{k,2+\delta} = 0$, *for some* $0 < \delta \leq 1$ *and* $\alpha > 0$.

Then $q_N < 1$, $N \geq k_0$.

Remark 2.2. Taking into account that $\lim_{k\nearrow\infty} \varepsilon_k = 0$, if there exist $k_0 > 0$ and $M > 0$ such that

$$R_{k,2+\delta} \leq M R_{k,2} |(\log k)^{-(1+\alpha)} \varepsilon_k|, \qquad k \geq k_0,$$

then condition (ii) in Theorem 2.3 holds.

Remark 2.3. Sufficient conditions for a positive probability of survival can be also established considering as a mathematical tool the following transition probability generating functions:

$$h_k(s) = \mathrm{E}[s^{Z_{n+1}} \mid Z_n = k], \qquad 0 \leq s \leq 1,\ k \in \mathbb{Z}^+,$$

For example, using some analytic techniques, it can be proved that if $\int_0^1 h_k(s)\,\mathrm{d}s \leq (1+k)^{-1}$ except for finitely many k, then $q_N < 1$, $N = 1, 2, \dots$.

3. Asymptotic behaviour

Assuming that $\mathrm{P}(Z_n \to \infty \mid Z_0 = N) > 0$, in this section we investigate the limiting evolution of the sequences $\{Z_n\}_{n\geq 0}$, $\{F_n\}_{n\geq 1}$, and $\{M_n\}_{n\geq 1}$, all suitably normalized. We will prove their convergence in distribution to gamma, Gaussian, or degenerate laws.

One of the hypotheses that we will require is $\lim_{k\to\infty} k^{1-\alpha}\varepsilon_k = c > 0$, for some $\alpha < 1$. Therefore, the function $g(k) = k\varepsilon_k$, $k = 1, 2, \dots$, is such that $g(k) = ck^{\alpha} + o(k^{\alpha})$. For technical reasons, we will extend g to a twice continuously differentiable function on $\mathbb{R}$ as follows:

$$g(x) = \begin{cases} cx^\alpha + o(x^\alpha) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let us introduce the sequence of real numbers $\{a_n\}_{n\geq 0}$ in the recursive form

$$a_0 = 1, \qquad a_{n+1} = a_n + g(a_n), \quad n \in \mathbb{Z}^+.$$

The next result summarizes the probabilistic limiting evolution of $\{Z_n\}_{n\geq 0}$.

Theorem 3.1. *Assume that the following conditions hold:*

(i) $\lim_{k\to\infty} k^{1-\alpha}\varepsilon_k = c > 0$, *for some* $0 < \alpha < 1$, $\varepsilon_k > 0$,

(ii) $\lim_{k\to\infty} k^{2-\beta} R_{k,2} = d > 0$, *for some* $\beta \leq 1 + \alpha$,

(iii) *there exist constants* k_0 *and* $M > 0$ *such that* $R_{k,2+\delta} \leq M R_{k,2}^{\delta}$, $k \geq k_0$, $\delta > 0$, $M > 0$.

Then we have the following results.

(a) *If* $\beta = 1 + \alpha$ *and* $d < 2c$, *we have*

$$\lim_{n\nearrow\infty} \mathrm{P}(n^{-1} Z_n^{1-\alpha} \leq x \mid Z_k \to \infty) = \Gamma_{a,b}(x), \qquad x \in \mathbb{R},$$

where $\Gamma_{a,b}$ *denotes the gamma distribution function with parameters*

$$a = \frac{2c - d\alpha}{(1-\alpha)d} \quad \text{and} \quad b = \frac{d(1-\alpha)^2}{2}.$$

(b) *If* $0 < \alpha < 1$ *and* $\beta < 1 + \alpha$ *then, on* $\{Z_k \to \infty\}$, *the following results hold.*

(i) *For* $\beta < 3\alpha - 1$, *the sequence* $\{a_n^{-1} Z_n\}_{n\geq 0}$ *converges almost surely to* 1 *and* $\{(Z_n - a_n)/g(a_n)\}_{n\geq 0}$ *is almost surely convergent.*

(ii) *For* $\beta \geq 3\alpha - 1$, *the sequence* $\{a_n^{-1} Z_n\}_{n\geq 0}$ *converges in probability to* 1 *and*

$$\lim_{n\nearrow\infty} \mathrm{P}(\Delta_n^{-1/2} g(a_n)^{-1}(Z_n - a_n) \leq x \mid Z_k \to \infty) = \Phi(x), \qquad x \in \mathbb{R},$$

where Φ *is the standard normal distribution function and*

$$\Delta_n = \begin{cases} dc^{-3}(1-\alpha)^{-1}\log n & \text{if } \beta = 3\alpha - 1, \\ d(\beta - 3\alpha + 1)^{-1} c^{(\beta-2)/(1-\alpha)} & \\ \qquad \times ((1-\alpha)n)^{(\beta-3\alpha+1)/(1-\alpha)} & \text{if } \beta > 3\alpha - 1. \end{cases}$$

Remark 3.1. Theorem 3.1 makes sense because, under its hypotheses, it can be verified that $\mathrm{P}(Z_n \to \infty \mid Z_0 = N) > 0$. In fact, from Theorem 3.1(i) we have

$$\liminf_{k\nearrow\infty} 2\varepsilon_k R_{k,2}^{-1} = \liminf_{k\nearrow\infty} \frac{2ck^{1+\alpha-\beta}(1 + o(1))}{d + o(1)} = \begin{cases} 2cd^{-1} & \text{if } \beta = 1 + \alpha, \\ \infty & \text{if } \beta < 1 + \alpha, \end{cases}$$

so condition (i) of Theorem 2.3 holds. Furthermore, considering Theorem 3.1(ii), (iii), and Remark 2.2, condition (ii) of Theorem 2.3 is also satisfied, so we can deduce that $q_N < 1$ irrespective of whether $\beta = 1 + \alpha$, $d < 2c$, or $\beta < 1 + \alpha$, $0 < \alpha < 1$.

Since $Z_k \to \infty$ cannot be easily checked, the following consequence of Theorem 3.1 is interesting from a practical viewpoint.

Corollary 3.1. *Under the hypotheses considered in Theorem 3.1, the following results hold.*

(a) *If* $\beta = 1+\alpha$ *and* $d < 2c$ *then, for* $x \in \mathbb{R}$,

 (i) $\lim_{n\nearrow\infty} \mathrm{P}(n^{-1}Z_n^{1-\alpha} \le x) = q_N\, \mathbf{1}_{\{x\ge 0\}} + (1-q_N)\Gamma_{a,b}(x)$,

 (ii) $\lim_{n\nearrow\infty} \mathrm{P}(n^{-1}Z_n^{1-\alpha} \le x \mid Z_n > 0) = \Gamma_{a,b}(x)$.

(b) *If* $\beta \ge 3\alpha - 1$ *then, for* $x \in \mathbb{R}$,

$$\lim_{n\nearrow\infty} \mathrm{P}(\Delta_n^{-1/2} g(a_n)^{-1}(Z_n - a_n) \le x \mid Z_n > 0) = \Phi(x).$$

Before investigating the limiting behaviour of $\{F_n\}_{n\ge 1}$ and $\{M_n\}_{n\ge 1}$, we establish the following proposition.

Proposition 3.1. *Under the hypotheses considered in Theorem 3.1, we have*

$$\limsup_{n\nearrow\infty} \mathrm{E}[n^{-1}Z_n^{1-\alpha}] < \infty.$$

Theorem 3.2. *Under the hypotheses considered in Theorem 3.1, if* $\beta = 1+\alpha$ *and* $d < 2c$, *then*

$$\lim_{n\to\infty} \mathrm{P}(n^{-1}F_n^{1-\alpha} \le x \mid Z_k \to \infty) = \Gamma_{a,\mu_1^{1-\alpha}b}(x), \qquad x \in \mathbb{R},$$

where $\mu_1 = \mathrm{E}[f_{0,1}]$.

Corollary 3.2. *Under the hypotheses considered in Theorem 3.1, if* $\beta < 3\alpha - 1$ *then, on* $\{Z_k \to \infty\}$, *we obtain that* $\{a_n^{-1}F_n\}_{n\ge 1}$ *converges in probability to* μ_1.

Similar results to Theorem 3.2 and Corollary 3.2 can be derived for $\{M_n\}_{n\ge 1}$.

4. Proofs

Proof of Theorem 2.1. Assume that $r_k \le 1$, $k \ge k_0$, where k_0 is a positive integer. Let us introduce, for $n_0 > 0$ fixed, the stopping time

$$T_{n_0}(k_0) = \begin{cases} \inf\{n \ge n_0 \colon Z_n < k_0\} & \text{if } \min_{n\ge n_0} Z_n < k_0, \\ \infty & \text{otherwise,} \end{cases}$$

and consider $\{Y_n\}_{n\ge 0}$, with $Y_n = Z_{n_0+n}\,\mathbf{1}_{\{T_{n_0}(k_0)\ge n_0+n\}} + Z_{T_{n_0}(k_0)}\,\mathbf{1}_{\{T_{n_0}(k_0)<n_0+n\}}$. Clearly, Y_n is $\mathcal{F}_{n_0+n}$-measurable, $n \ge 0$, where we recall that $\mathcal{F}_n = \sigma(Z_0, \dots, Z_n)$.

If $Z_{n_0} \ge k_0, \dots, Z_{n_0+n} \ge k_0$, $n \ge 0$, then $T_{n_0}(k_0) \ge n_0 + n + 1$; hence, using the fact that $\mathrm{E}[Z_{n+1} \mid Z_n] = Z_n r_{Z_n} \le Z_n$ almost surely on $\{Z_n \ge k_0\}$, it follows on $\{Z_{n_0} \ge k_0, \dots, Z_{n_0+n} \ge k_0\}$ that

$$\mathrm{E}[Y_{n+1} \mid \mathcal{F}_{n_0+n}] = \mathrm{E}[Z_{n_0+n+1} \mid \mathcal{F}_{n_0+n}] \le Z_{n_0+n} = Y_n \quad \text{almost surely (a.s.).}$$

If for some $k \in \{1, \dots, n\}$, $n \ge 1$, $Z_{n_0} \ge k_0, \dots, Z_{n_0+k-1} \ge k_0$ and $Z_{n_0+k} < k_0$, then $T_{n_0}(k_0) = n_0 + k < n_0 + n + 1$, and on $\{Z_{n_0} \ge k_0, \dots, Z_{n_0+k-1} \ge k_0, Z_{n_0+k} < k_0\}$ we obtain

$$\mathrm{E}[Y_{n+1} \mid \mathcal{F}_{n_0+n}] = \mathrm{E}[Z_{n_0+k} \mid \mathcal{F}_{n_0+n}] = Y_n \quad \text{a.s.}$$

Finally, if $Z_{n_0} < k_0$ then $T_{n_0}(k_0) = n_0$. Hence, on $\{Z_{n_0} < k_0\}$, we have

$$\mathrm{E}[Y_{n+1} \mid \mathcal{F}_{n_0+n}] = \mathrm{E}[Z_{n_0} \mid \mathcal{F}_{n_0+n}] = Y_n \quad \text{a.s.}$$

Thus, $\{Y_n\}_{n\geq 0}$ is a nonnegative supermartingale with respect to $\{\mathcal{F}_{n_0+n}\}_{n\geq 0}$ and, by the martingale convergence theorem (see [20]), we derive the almost sure convergence of $\{Y_n\}_{n\geq 0}$ to the nonnegative and finite limit

$$Y_\infty = \lim_{k\nearrow\infty} Z_k \mathbf{1}_{\{\inf_{n\geq n_0} Z_n \geq k_0\}} + Z_{T_{n_0}(k_0)} \mathbf{1}_{\{\inf_{n\geq n_0} Z_n < k_0\}}.$$

Thus, for $N = 1, 2, \dots,$

$$\mathrm{P}(Z_n \to \infty \mid Z_0 = N) = \mathrm{P}\biggl(\bigcup_{n_0=1}^{\infty}\Bigl\{\inf_{n\geq n_0} Z_n \geq k_0\Bigr\} \cap \{Z_n \to \infty\} \biggm| Z_0 = N\biggr) = 0.$$

Taking into account (1.2), the proof is complete.

Proof of Theorem 2.2. Suppose that $x \neq 0$, $x+h > 0$, and $0 < \delta \leq 1$, such that condition (ii) holds. We can derive

$$\log(x+h) \leq \log x + \frac{h}{x} - \frac{1}{2}\Bigl(\frac{h}{x}\Bigr)^2 + \frac{1}{2}\Bigl|\frac{h}{x}\Bigr|^{2+\delta}. \tag{4.1}$$

Taking $x = Z_n + 1$ and $h = Z_n \varepsilon_{Z_n} + \xi_{n+1}$, from (2.1) we deduce that $x + h = Z_{n+1} + 1$. Hence, applying (4.1) and taking expectations, we obtain

$$\begin{aligned}\mathrm{E}[\log(Z_{n+1}+1) \mid Z_n = k] \leq \log(k+1) &+ \mathrm{E}\biggl[\frac{Z_n\varepsilon_{Z_n} + \xi_{n+1}}{Z_n+1} \biggm| Z_n = k\biggr] \\ &- \frac{1}{2}\mathrm{E}\biggl[\biggl(\frac{Z_n\varepsilon_{Z_n} + \xi_{n+1}}{Z_n+1}\biggr)^2 \biggm| Z_n = k\biggr] \\ &+ \frac{1}{2}\mathrm{E}\biggl[\biggl|\frac{Z_n\varepsilon_{Z_n} + \xi_{n+1}}{Z_n+1}\biggr|^{2+\delta} \biggm| Z_n = k\biggr].\end{aligned}$$

Using the properties of $\{\xi_n\}_{n\geq 1}$ and that $|a+b|^r \leq C_r(|a|^r + |b|^r)$, $r > 0$, for some $C_r > 0$ (see [16, p. 157]), there exists $C > 0$ such that

$$\begin{aligned}\mathrm{E}[\log(Z_{n+1}+1) \mid Z_n = k] &\leq \log(k+1) + \frac{2\varepsilon_k - \varepsilon_k^2 + R_{k,2}}{2} + C(\varepsilon_k^{2+\delta} + R_{k,2+\delta}) \\ &\leq \log(k+1) + \varepsilon_k(1+o(1)) - \tfrac{1}{2}R_{k,2}(1+o(1)).\end{aligned}$$

Hence, from (i) and (ii) for k large enough, we derive

$$\mathrm{E}[\log(Z_{n+1}+1) \mid Z_n = k] \leq \log(k+1)$$

or, equivalently,

$$\mathrm{E}[(\log(Z_n+1))^{-1}\log(Z_{n+1}+1) \mid Z_n = k] \leq 1.$$

Using Remark 2.1 with $\pi(x) = \log(x+1)$ and Theorem 2.1, the proof is complete.

For the proof of Theorem 2.3, we need the following auxiliary result.

Lemma 4.1. *Let $\{X_n\}_{n\geq 0}$ be a sequence of nonnegative random variables and let $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of σ-algebras such that X_n is $\mathcal{F}_n$-measurable for each n. Suppose that for any constant C^* there exists a positive integer n such that $\mathrm{P}(X_n > C^*) > 0$ and, moreover, $\mathrm{P}(X_n \to 0) + \mathrm{P}(X_n \to \infty) = 1$. If f is a positive and decreasing function on $\mathbb{R}^+$ and for some constant $A > 0$, we have*

$$\mathrm{E}[f(X_{n+1}) \mid \mathcal{F}_n] \leq f(X_n) \quad \textit{a.s. on } \{X_n > A\}, \qquad n \in \mathbb{Z}^+,$$

then $\mathrm{P}(X_n \to \infty) > 0$.

Proof. Let $Y_n^* = \min\{f(X_n), f(A)\}$, $n \in \mathbb{Z}^+$. It can be verified that

$$\mathrm{E}[Y_{n+1}^* \mid \mathcal{F}_n] \leq \min\{\mathrm{E}[f(X_{n+1}) \mid \mathcal{F}_n], f(A)\} \leq Y_n^* \quad \text{a.s.}, \qquad n \in \mathbb{Z}^+.$$

Thus, $\{Y_n^*\}_{n\geq 0}$ is a nonnegative supermartingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ and, by the martingale convergence theorem, it is almost surely convergent to a finite and nonnegative random variable Y^*. Because $\{Y_n^*\}_{n\geq 0}$ is bounded, we also derive its L^1-convergence. Suppose that $\mathrm{P}(X_n \to \infty) = 0$. Then $\mathrm{P}(X_n \to 0) = 1$, and it follows that $\mathrm{E}[Y^*] = f(A)$. Since $\{\mathrm{E}[Y_n^*]\}_{n\geq 0}$ is decreasing, $\mathrm{E}[Y_n^*] \geq f(A)$, $n = 0, 1, \dots$. Now, $Y_n^* \leq f(A)$, $n = 0, 1, \dots$. Hence, we deduce that, for every n, $Y_n^* = f(A)$ almost surely and, consequently, since f is decreasing, we deduce that $X_n \leq A$ almost surely, $n = 0, 1, \dots$, contradicting the first assumption.

Proof of Theorem 2.3. Let $x \geq 3$, $x + h > 3$, $0 < \delta \leq 1$, and $\alpha > 0$. Write $f(x) = (\log x)^{-\alpha}$. It was proved in [13, Theorem 2] that, for some $C_1 > 0$,

$$f(x+h) \leq f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + C_1\frac{|h|^{2+\delta}}{x^{2+\delta}(\log x)^{\alpha+1}} + \mathbf{1}_{(-\infty,-2h]}(x). \tag{4.2}$$

Let $x = Z_n + 3$ and $h = Z_n\varepsilon_{Z_n} + \xi_{n+1}$. From (2.1) we deduce that $x + h = Z_{n+1} + 3$. Hence, applying (4.2) and taking expectations, for $k > 0$ we obtain

$$\begin{aligned}\mathrm{E}[f(Z_{n+1}+3) \mid Z_n = k] \leq{}& f(k+3) + f'(k+3)\,\mathrm{E}[Z_n\varepsilon_{Z_n} + \xi_{n+1} \mid Z_n = k] \\ &+ \tfrac{1}{2}f''(k+3)\,\mathrm{E}[(Z_n\varepsilon_{Z_n} + \xi_{n+1})^2 \mid Z_n = k] \\ &+ C_1\,\mathrm{E}\bigg[\frac{|Z_n\varepsilon_{Z_n} + \xi_{n+1}|^{2+\delta}}{(Z_n+3)^{2+\delta}[\log(Z_n+3)]^{\alpha+1}} \;\bigg|\; Z_n = k\bigg] \\ &+ \mathrm{P}(Z_n + 3 \leq -2(Z_n\varepsilon_{Z_n} + \xi_{n+1}) \mid Z_n = k).\end{aligned}$$

Using the properties of $\{\xi_n\}_{n\geq 1}$, we have

$$\begin{aligned}\mathrm{E}[f(Z_{n+1}+3) \mid Z_n = k] \leq{}& f(k+3) + f'(k+3)k\varepsilon_k + \frac{f''(k+3)k^2(\varepsilon_k^2 + R_{k,2})}{2} \\ &+ C_1\frac{k^{2+\delta}(\varepsilon_k^{2+\delta} + R_{k,2+\delta})}{(k+3)^{2+\delta}[\log(k+3)]^{\alpha+1}} \\ &+ \mathrm{P}(Z_n + 3 \leq -2(Z_n\varepsilon_{Z_n} + \xi_{n+1}) \mid Z_n = k).\end{aligned}$$

Since $\{\varepsilon_k\}_{k\geq 1}$ is bounded (by the Markov inequality), for k large enough we obtain

$$\mathrm{P}(2(Z_n\varepsilon_{Z_n} + \xi_{n+1}) \leq -Z_n - 3 \mid Z_n = k) \leq \mathrm{P}\bigg(\frac{|\xi_{n+1}|}{Z_n} \geq \widetilde{C}_2 \;\bigg|\; Z_n = k\bigg) \leq C_2 R_{k,2+\delta},$$

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where $\widetilde{C}_2$ and C_2 are positive constants independent of k. In this situation, given δ and α such that condition (ii) of Theorem 2.3 holds, since $f'(x) = -\alpha x^{-1}(\log x)^{-\alpha-1}$ and $f''(x) = \alpha x^{-2}(\log x)^{-\alpha-1}(1+o(1))$, we have

$$\mathrm{E}[f(Z_{n+1}+3) \mid Z_n = k] \le f(k+3) - \frac{\alpha\varepsilon_k}{[\log(k+3)]^{\alpha+1}}(1+o(1)) + \frac{\alpha R_{k,2}}{2[\log(k+3)]^{\alpha+1}}(1+o(1)).$$

Now, from Theorem 2.3(i), we have

$$-\frac{\varepsilon_k(1+o(1))}{[\log(k+3)]^{\alpha+1}} + \frac{R_{k,2}(1+o(1))}{2[\log(k+3)]^{\alpha+1}} \le 0.$$

Thus, for Z_n large enough,

$$\mathrm{E}[f(Z_{n+1}+3) \mid Z_n] \le f(Z_n+3) \quad \text{a.s.}$$

Now, $r_k > 1$, $k \ge N$, implies that $\mathrm{E}[Z_{n+1} \mid Z_n = k] > k$, $k \ge N$. Thus, there exists $N^* > N$ such that $\mathrm{P}(Z_{n+1} = N^* \mid Z_n = N) > 0$. As a consequence, for $C > 0$ there exists $n_0 \ge 1$ such that $\mathrm{P}(Z_{n+n_0} > C \mid Z_n = N) > 0$. By Lemma 4.1, it follows that $\mathrm{P}(Z_n \to \infty \mid Z_0 = N) > 0$.

Proof of Theorem 3.1. Let us consider the function

$$G(x) = \int_1^x \frac{1}{g(y)}\,\mathrm{d}y, \qquad x \ge 1.$$

By Theorem 3.1(i) and l'Hôpital's rule, we deduce that $G(x) \sim (c(1-\alpha))^{-1}x^{1-\alpha}$, $x \nearrow \infty$. From (2.1) and the hypotheses of the theorem, it is easy to check that

$$Z_{n+1} = Z_n + g(Z_n) + \xi_{n+1} \quad \text{a.s.}, \qquad n \in \mathbb{Z}^+.$$

Note that

$$\mathrm{E}[\xi_{n+1}^2 \mid \mathcal{F}_n] = Z_n^2 R_{Z_n,2} \quad \text{a.s.}$$

and

$$\lim_{x\nearrow\infty} g'(x)G(x) = \alpha(1-\alpha)^{-1}.$$

Let us prove part (a). Since $\beta = 1+\alpha$, we have

$$\lim_{k\nearrow\infty} \frac{\mathrm{E}[\xi_{n+1}^2 \mid Z_n = k]}{g^2(k)G(k)} = c^{-1}d(1-\alpha).$$

Let $\lambda = (1-\alpha)^{-1}$ and $\gamma = c^{-1}d(1-\alpha)$. We therefore have $\lambda\gamma = dc^{-1} < 2$. Hence, by [14, Theorem 1],

$$\lim_{n\nearrow\infty} \mathrm{P}(n^{-1}G(Z_n) \le x \mid Z_k \to \infty) = \Gamma_{2\gamma^{-1}-\lambda+1,\gamma/2}(x), \qquad x \in \mathbb{R}.$$

Also, since $G(x) \sim (c(1-\alpha))^{-1}x^{1-\alpha}$, $x \nearrow \infty$, we obtain

$$\lim_{n\nearrow\infty} \lambda c^{-1} Z_n^{1-\alpha} G(Z_n)^{-1} = 1 \quad \text{a.s.}, \qquad \text{on } \{Z_k \to \infty\}.$$

Now we will prove part (b). Considering (2.1) and the hypotheses of the theorem, we have

$$Z_{n+1} = Z_n + Z_n \varepsilon_{Z_n}(1 + \eta_{n+1}) \quad \text{a.s.,} \qquad n \in \mathbb{Z}^+, \tag{4.3}$$

where

$$\eta_{n+1} = \xi_{n+1}(Z_n \varepsilon_{Z_n})^{-1} \quad \text{on } \{Z_n \varepsilon_{Z_n} \neq 0\}.$$

It is clear that $\mathrm{E}[\eta_{n+1} \mid Z_n = k] = 0$ and $\mathrm{E}[\eta_{n+1}^2 \mid Z_n = k] = R_{k,2}\varepsilon_k^{-2}$. For simplicity, let us write $h(k) = \mathrm{E}[\xi_{n+1}^2 \mid Z_n = k]$ and $\varphi(k) = \mathrm{E}[\eta_{n+1}^2 \mid Z_n = k]$. Also, let us introduce the positive real function $\widehat{\varphi}^2(x) = g^{-2}(x)h(x)$, for $x > 0$ and $g(x) \neq 0$. Obviously, $\widehat{\varphi}^2(k) = \varphi^2(k)$ for all integers $k \neq 0$, and $\widehat{\varphi}^2(x) \sim dc^{-2}x^{\beta-2\alpha}$, $x \nearrow \infty$. Considering

$$\psi(x) = \int_1^x \frac{\widehat{\varphi}^2(y)}{g(y)}\,\mathrm{d}y, \qquad x \geq 1,$$

by conditions (i) and (ii), if $\beta < 3\alpha - 1$ then $\psi(\infty) < \infty$. Otherwise,

$$\psi(x) \sim \begin{cases} \dfrac{d}{c^3}\dfrac{1}{\beta - 3\alpha + 1}x^{\beta-3\alpha+1} & \text{if } \beta > 3\alpha - 1, \qquad x \nearrow \infty, \\ \dfrac{d}{c^3}\log x & \text{if } \beta = 3\alpha - 1, \qquad x \nearrow \infty, \end{cases} \tag{4.4}$$

and therefore $\psi(\infty) = \infty$ if $\beta \geq 3\alpha - 1$.

To conclude the proof, it is sufficient to apply [12, Theorem 3]. To this end, it will be necessary to check that its requirements are satisfied. In fact, from (4.3) and [12, Conditions (A1)–(A3)], [12, Theorem 3] holds. Also, [12, Conditions (A4)–(A7)] are satisfied. From the conditions of [12], we have the following.

(A4) Since the process is critical, $\lim_{x\nearrow\infty} x^{-1}g(x) = 0$. Moreover, the function g is ultimately concave and, since $0 < \alpha < 1$, g' is also ultimately convex.

(A5) Let G^{-1} be the inverse function of G. Then we can verify, as $x \nearrow \infty$, that $G^{-1}(x) \sim (c(1-\alpha)x)^{1/1-\alpha}$. Hence, $(\widehat{\varphi}^2 \circ G^{-1})(x) \sim (d/c^2)(c(1-\alpha)x)^{(\beta-2\alpha)/(1-\alpha)}$. Moreover, since $\beta < 1 + \alpha$, it is ultimately convex. We conclude that

$$\lim_{t\nearrow\infty} \int_1^t x^{-2}(\widehat{\varphi}^2 \circ G^{-1})(x)\,\mathrm{d}x = \lim_{t\nearrow\infty} dc^{-2}(c(1-\alpha))^{(\beta-2\alpha)/(1-\alpha)} \int_1^t x^{(\beta-2)/(1-\alpha)}\,\mathrm{d}x < \infty.$$

(A6) If $\psi(\infty) = \infty$ then $\beta \geq 3\alpha - 1$. Moreover, $|\widehat{\varphi}^{-2}(x)g''(x)g(x)| \sim Ax^{4\alpha-\beta-2}$ as $x \nearrow \infty$, for some constant $A > 0$, and therefore is ultimately decreasing since $4\alpha - \beta - 2 < 0$. Also, if $\psi(\infty) < \infty$ then $|g''(x)g(x)| \sim Bx^{2(\alpha-1)}$ as $x \nearrow \infty$, for some constant $B > 0$, and therefore is ultimately decreasing since $\alpha < 1$.

(A7) Using the fact that $g'(x) \sim c\alpha x^{\alpha-1}$ as $x \nearrow \infty$ and (4.4), we deduce that

$$\lim_{x\nearrow\infty} g'(x)\psi^{1/2}(x) = 0.$$

If $\widehat{\psi}(x) = (\psi \circ G^{-1})(x)$ then, for $\beta < 3\alpha - 1$, $\widehat{\psi}(\infty) < \infty$ is satisfied. Hence, by [12, Theorem 3], we obtain part (b)(i).

On the other hand, if $\beta \geq 3\alpha - 1$ then

$$\widehat{\psi}(x) = (\psi \circ G^{-1})(x) \sim \begin{cases} \dfrac{d}{\beta - 3\alpha + 1} c^{(\beta-2)/(1-\alpha)}((1-\alpha)x)^{(\beta-3\alpha+1)/(1-\alpha)} & \text{if } \beta > 3\alpha - 1, \\ \dfrac{d}{c^3(1-\alpha)} \log x & \text{if } \beta = 3\alpha - 1, \end{cases} \tag{4.5}$$

and $\widehat{\psi}(\infty) = \infty$. Again, applying [12, Theorem 3], it follows that

$$\lim_{n \nearrow \infty} \mathrm{P}(\widehat{\psi}^{-1/2}(n) g(a_n)^{-1}(Z_n - a_n) \leq x \mid Z_k \to \infty) = \Phi(x).$$

Finally, part (b)(ii) is obtained from (4.5) and Slutsky's theorem.

Proof of Proposition 3.1. On $\{Z_n > 0\}$ we define $\widetilde{Z}_{n+1} = Z_n^{-1}(Z_{n+1} - Z_n)$, $n \in \mathbb{Z}^+$. We have

$$Z_{n+1}^{1-\alpha} = Z_n^{1-\alpha}(1 + \widetilde{Z}_{n+1})^{1-\alpha} = Z_n^{1-\alpha}(1 + (1-\alpha)\widetilde{Z}_{n+1} + T_2(\widetilde{Z}_{n+1})),$$

where $T_2(x)$ denotes the remainder of the first-order Taylor expansion of the function $(1+x)^{1-\alpha}$ around 0.

Using the fact that $\varepsilon_k = r_k - 1$, $k = 1, 2, \ldots$, we obtain

$$Z_{n+1}^{1-\alpha} = Z_n^{1-\alpha}(1 + (1-\alpha)(\varepsilon_{Z_n} + Z_n^{-1}(Z_{n+1} - \mathrm{E}[Z_{n+1} \mid Z_n])) + T_2(\widetilde{Z}_{n+1})) \quad \text{a.s.}$$

and, therefore,

$$\mathrm{E}[Z_{n+1}^{1-\alpha} \mid Z_n] = Z_n^{1-\alpha}(1 + (1-\alpha)\varepsilon_{Z_n} + \mathrm{E}[T_2(\widetilde{Z}_{n+1}) \mid Z_n]) \quad \text{a.s.}$$

Taking into account [15, p. 182], we have $|T_2(x)| \leq C|x|^2$, $x > 0$, for some $C > 0$. Hence, we deduce that

$$|\,\mathrm{E}[T_2(\widetilde{Z}_{n+1}) \mid Z_n]| \leq C Z_n^{-2}\,\mathrm{E}[(Z_{n+1} - Z_n)^2 \mid Z_n] = C(R_{Z_n,2} + \varepsilon_{Z_n}^2) \quad \text{a.s.}$$

Since we are assuming that conditions (i) and (ii) of Theorem 3.1 hold, $k^{1-\alpha}\varepsilon_k = c + \eta_k$ and $k^{2-\beta}R_{k,2} = d + \eta_k^*$, for $c, d > 0$, $\alpha < 1$, and $\beta \leq 1 + \alpha$, with $\lim_{k \nearrow \infty} \eta_k = \lim_{k \nearrow \infty} \eta_k^* = 0$.

If $\beta = 1 + \alpha$ then

$$|Z_n^{1-\alpha}\,\mathrm{E}[T_2(\widetilde{Z}_{n+1}) \mid Z_n]| \leq C(d + \eta_{Z_n}^* + Z_n^{\alpha-1}(c + \eta_{Z_n})^2) \quad \text{a.s.}$$

and, consequently,

$$\begin{aligned} \mathrm{E}[Z_{n+1}^{1-\alpha} \mid Z_n] &\leq Z_n^{1-\alpha} + (1-\alpha)(c + \eta_{Z_n}) + C(d + \eta_{Z_n}^* + Z_n^{\alpha-1}(c + \eta_{Z_n})^2) \\ &= Z_n^{1-\alpha} + (1-\alpha)c + Cd + \widetilde{\eta}_{Z_n} \quad \text{a.s.,} \end{aligned}$$

where $\widetilde{\eta}_k = (1-\alpha)\eta_k + C\eta_k^* + Ck^{\alpha-1}(c + \eta_k)^2$. Clearly, $\lim_{k \to \infty} \widetilde{\eta}_k = 0$. Thus,

$$\mathrm{E}[Z_{n+1}^{1-\alpha}] \leq \mathrm{E}[Z_n^{1-\alpha}] + ((1-\alpha)c + Cd)\,\mathrm{P}(Z_n > 0) + \mathrm{E}[\widetilde{\eta}_{Z_n}\,\mathbf{1}_{\{Z_n > 0\}}]$$

and, by induction on n, we have

$$\mathrm{E}[Z_{n+1}^{1-\alpha}] \leq ((1-\alpha)c + Cd) \sum_{k=1}^{n} \mathrm{P}(Z_k > 0) + \sum_{k=1}^{n} \mathrm{E}[\widetilde{\eta}_{Z_k}\,\mathbf{1}_{\{Z_k > 0\}}] + N.$$

Thus, we derive

$$\mathrm{E}\left[\frac{Z_{n+1}^{1-\alpha}}{n+1}\right] \le \frac{1}{n}\left(((1-\alpha)c + Cd)\sum_{k=1}^{n}\mathrm{P}(Z_k > 0) + \sum_{k=1}^{n}\mathrm{E}[\widetilde{\eta}_{Z_k}\mathbf{1}_{\{Z_k>0\}}] + N\right).$$

By Cesaro's lemma, $\lim_{n\nearrow\infty} n^{-1}\sum_{k=1}^{n}\mathrm{P}(Z_k > 0) = \lim_{n\nearrow\infty}\mathrm{P}(Z_n > 0) = 1 - q_N$. Since $\widetilde{\eta}_k$ is bounded, by the dominated convergence theorem we have

$$\lim_{n\nearrow\infty} n^{-1}\sum_{k=1}^{n}\mathrm{E}[\widetilde{\eta}_{Z_k}\mathbf{1}_{\{Z_k>0\}}] = \lim_{n\nearrow\infty}\mathrm{E}[\widetilde{\eta}_{Z_n}\mathbf{1}_{\{Z_n>0\}}] = \mathrm{E}[\lim_{n\nearrow\infty}\widetilde{\eta}_{Z_n}\mathbf{1}_{\{Z_n>0\}}] = 0$$

because, on $\{Z_k \to 0\}$, $\lim_{n\nearrow\infty}\widetilde{\eta}_{Z_n}\mathbf{1}_{\{Z_n>0\}} = 0$ and, on $\{Z_k \to \infty\}$,

$$\lim_{n\nearrow\infty}\widetilde{\eta}_{Z_n}\mathbf{1}_{\{Z_n>0\}} = (\lim_{n\nearrow\infty}\widetilde{\eta}_{Z_n})\mathbf{1}_{\{Z_k\to\infty\}} = 0.$$

Therefore, we conclude that

$$\limsup_{n\nearrow\infty}\mathrm{E}[(n+1)^{-1}Z_{n+1}^{1-\alpha}] \le ((1-\alpha)c + Cd)(1-q_N) < \infty.$$

On the other hand, if $\beta < 1+\alpha$ then

$$|Z_n^{1-\alpha}\mathrm{E}[T_2(\widetilde{Z}_{n+1}) \mid Z_n]| \le C(Z_n^{-1-\alpha+\beta}(d + \eta_{Z_n}^*) + Z_n^{\alpha-1}(c + \eta_{Z_n})^2) \quad \text{a.s.};$$

hence,

$$\mathrm{E}[Z_{n+1}^{1-\alpha} \mid Z_n] \le Z_n^{1-\alpha} + (1-\alpha)c + \overline{\eta}_{Z_n} \quad \text{a.s.},$$

where $\overline{\eta}_k = C(k^{-1-\alpha+\beta})(d + \eta_k^*) + k^{\alpha-1}(c+\eta_k)^2)$. Clearly, $\lim_{k\nearrow\infty}\overline{\eta}_k = 0$. Finally, using a similar reasoning to the previous case, we obtain

$$\limsup_{n\nearrow\infty}\mathrm{E}[(n+1)^{-1}Z_{n+1}^{1-\alpha}] = (1-\alpha)c(1-q_N) < \infty.$$

Proof of Theorem 3.2. On $\{Z_n > 0\}$, we define $\widetilde{F}_{n+1} = (\mu_1 Z_n)^{-1}(F_{n+1} - \mu_1 Z_n)$, $n \in \mathbb{Z}^+$. We have

$$\begin{aligned} F_{n+1}^{1-\alpha} &= (\mu_1 Z_n)^{1-\alpha}(1 + \widetilde{F}_{n+1})^{1-\alpha} \\ &= (\mu_1 Z_n)^{1-\alpha}(1 + (1-\alpha)\widetilde{F}_{n+1} + T_2(\widetilde{F}_{n+1})) \\ &= (\mu_1 Z_n)^{1-\alpha} + (1-\alpha)(\mu_1 Z_n)^{-\alpha}(F_{n+1} - \mu_1 Z_n) + (\mu_1 Z_n)^{1-\alpha}T_2(\widetilde{F}_{n+1}), \end{aligned} \tag{4.6}$$

where $T_2(x)$ is as defined in the proof of Proposition 3.1.

Let $W_{n+1} = (\mu_1 Z_n)^{-\alpha}(F_{n+1} - \mu_1 Z_n)\mathbf{1}_{\{Z_n>0\}}$, $n \in \mathbb{Z}^+$. We have that $\mathrm{E}[W_{n+1} \mid Z_n] = 0$ almost surely. Hence, $\mathrm{E}[W_{n+1}] = 0$. Also,

$$\mathrm{E}[W_{n+1}^2 \mid Z_n] = \mu_1^{-2\alpha}Z_n^{1-2\alpha}\rho_1^2\mathbf{1}_{\{Z_n>0\}} \quad \text{a.s.},$$

where $\rho_1^2 = \mathrm{var}[f_{0,1}]$, so $\mathrm{E}[W_{n+1}^2] = \mu_1^{-2\alpha}\rho_1^2\mathrm{E}[Z_n^{1-2\alpha}\mathbf{1}_{\{Z_n>0\}}]$.

If $2\alpha \ge 1$ then $Z_n^{1-2\alpha}\mathbf{1}_{\{Z_n>0\}} \le \mathbf{1}_{\{Z_n>0\}}$. Thus, $\mathrm{E}[W_{n+1}^2] \le \mu_1^{-2\alpha}\rho_1^2\mathrm{P}(Z_n > 0)$, which converges to $\mu_1^{-2\alpha}\rho_1^2(1-q_N) < \infty$ as $n \nearrow \infty$. Consequently,

$$\lim_{n\nearrow\infty}\mathrm{E}[(n+1)^{-2}W_{n+1}^2] = \lim_{n\nearrow\infty}(n+1)^{-2}\mathrm{E}[W_{n+1}^2] = 0.$$

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Hence, $\{(n+1)^{-1}W_{n+1}\}_{n\geq 0}$ converges in L^2 to 0, and therefore converges in probability to 0. In particular, $\{(n+1)^{-1}(\mu_1 Z_n)^{-\alpha}(F_{n+1}-\mu_1 Z_n)\}_{n\geq 0}$, conditioned on $\{Z_k \to \infty\}$, converges in probability to 0.

On the other hand, if $0 < 2\alpha < 1$ then $\mathrm{E}[Z_n^{1-2\alpha}] \leq \mathrm{E}[Z_n^{1-\alpha}]$, and, therefore, taking into account Proposition 3.1, we have

$$\limsup_{n\nearrow\infty} \mathrm{E}[n^{-1}Z_n^{1-2\alpha}] \leq \limsup_{n\nearrow\infty} \mathrm{E}[n^{-1}Z_n^{1-\alpha}] < \infty.$$

Thus, $\{\mathrm{E}[n^{-1}Z_n^{1-2\alpha}]\}_{n\geq 0}$ is bounded, so there exists a positive constant K such that

$$\lim_{n\nearrow\infty} \mathrm{E}[(n+1)^{-2}W_{n+1}^2] \leq K \lim_{n\nearrow\infty} n^{-1}\mu_1^{-2\alpha}\rho_1^2 = 0,$$

and it follows that $\{(n+1)^{-1}W_{n+1}\}_{n\geq 0}$ converges in L^2 to 0, and, therefore, conditioned on $\{Z_k \to \infty\}$, also converges in probability to 0.

Now, by [15, p. 182], on $\{Z_n > 0\}$ we have for some $C > 0$ that

$$(\mu_1 Z_n)^{1-\alpha} T_2(\widetilde{F}_{n+1}) \leq C(\mu_1 Z_n)^{-1-\alpha}(F_{n+1}-\mu_1 Z_n)^2.$$

Let us consider $Y_{n+1} = (\mu_1 Z_n)^{-1-\alpha}(F_{n+1}-\mu_1 Z_n)^2\,\mathbf{1}_{\{Z_n>0\}},\ \ n \in \mathbb{Z}^+$. It can be verified that

$$\mathrm{E}[|Y_{n+1}|] = \mu_1^{-1-\alpha}\rho_1^2\,\mathrm{E}[Z_n^{-\alpha}\,\mathbf{1}_{\{Z_n>0\}}] \leq \mu_1^{-1-\alpha}\rho_1^2\,\mathrm{P}(Z_n > 0).$$

The last term converges to $\mu_1^{-1-\alpha}\rho_1^2(1-q_N) < \infty$ as $n \nearrow \infty$, and therefore

$$\{\mathrm{E}[|(n+1)^{-1}Y_{n+1}|]\}_{n\geq 0}$$

converges to 0. Hence, $\{(n+1)^{-1}Y_{n+1}\}_{n\geq 0}$ converges in probability to 0. As a consequence, conditioned on $\{Z_k \to \infty\}$, it is easy to verify that the sequence

$$\{(n+1)^{-1}(\mu_1 Z_n)^{1-\alpha}T_2(\widetilde{F}_{n+1})\}_{n\geq 0}$$

converges in probability to 0.

Finally, from (4.6) we have

$$\frac{F_{n+1}^{1-\alpha}}{n+1} = \frac{n}{n+1}\frac{(\mu_1 Z_n)^{1-\alpha}}{n} + \frac{(1-\alpha)(\mu_1 Z_n)^{-\alpha}(F_{n+1}-\mu_1 Z_n)}{n+1} + \frac{(\mu_1 Z_n)^{1-\alpha}T_2(\widetilde{F}_{n+1})}{n+1}.$$

We have proved that, conditioned on $\{Z_k \to \infty\}$, the two last terms of the above sum converge in probability to 0. By Theorem 3.1, conditioned on $\{Z_k \to \infty\}$, we have that $\{n^{-1}Z_n^{1-\alpha}\}_{n\geq 1}$ converges in distribution to a gamma law with parameters $a = (2c - d\alpha)/(1-\alpha)d$ and $b = d(1-\alpha)^2/2$. We conclude the proof using Slutsky's theorem.

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