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THE c-SUPPLEMENTED PROPERTY OF FINITE GROUPS

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Abstract The purpose of this paper is to study the influence of c-supplemented minimal subgroups on the p-nilpotency of finite groups. We obtain 'iff' and 'localized' versions of theorems of Itô and Buckley on nilpotence, p-nilpotence and supersolvability.

Keywords: c-supplemented; p-nilpotence; nilpotence; supersolvable; formation

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1. Introduction

All groups considered will be finite. For a formation $\mathcal{F}$ and a group G, we denote by $G^{\mathcal{F}}$ the $\mathcal{F}$-residual of G. Throughout this paper, $\mathcal{N}$, $\mathcal{N}_p$ and $\mathcal{U}$ will denote the class of nilpotent, p-nilpotent and supersolvable groups, respectively.

The ways in which minimal subgroups can be embedded in a nilpotent, p-nilpotent or supersolvable group have been investigated by a number of scholars. For example, a well-known theorem due to Itô [**9**, Satz III.5.3] asserts that a group G of odd order is nilpotent if every minimal subgroup of G lies in the centre of G. Another well-known theorem of Itô [**9**, Satz IV.5.5] states that a group G is p-nilpotent if every cyclic subgroup of G with order p or 4 (if $p = 2$) lies in the centre of G. Along the same lines, Buckley [**3**] showed in 1970 that a group G of odd order is supersolvable if every minimal subgroup of G is normal in G.

Since then, a series of papers have dealt with generalizations of the results of Itô and Buckley. Roughly speaking, one aim of these generalizations is to replace the normality condition by a weaker condition (see, for example, [**2**, **16**, **17**]). Another aim is to minimize the number of minimal subgroups or replace the global condition by a localized condition (see, for example, [**11**, **19**]). One further aim is to extend the results to a saturated formation by using the formation theory (examples are given in [**1**, **12**, **20**–**22**]).

So many related works inspire us to ask the following questions.

(i) To what degree does one weaken the sufficient conditions so that they become necessary?

(ii) To what degree can the directions mentioned be unified?

To answer these questions, we first note the following fact: a group G lies in a formation $\mathcal{F}$ if and only if (iff) the $\mathcal{F}$-residual $G^{\mathcal{F}}$ is an identity group. This observation indicates that if the sufficient conditions are restricted on $G^{\mathcal{F}}$, then they are naturally necessary and hence are sharp. Our second observation is focused on the concept of c-supplementation, which is weaker than both complementation and normality. The accepted concept of complementation was introduced by Hall and applied to characterizing the structure of solvable groups in his well-known series of papers [**7**,**8**]. The concept of c-supplementation was introduced in 2000 by Wang [**18**] and has since often been applied to determine the structure of some class of groups in recent years (see [**19**, **20**, **22**]). As a result of our observations, we give answers to the above questions by using the c-supplemented property of minimal subgroups of the residual subgroup. Our results actually record 'iff' and 'localized' versions of theorems by Itô and Buckley and some further related results.

A subgroup H of a group G is said to be *c-supplemented* in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leqslant \mathrm{core}_G(H)$ (the maximal normal subgroup of G that is contained in H). Hence, a complemented or normal subgroup must be a c-supplemented subgroup. But the converses do not hold in general (see [**18**]). A group G is called quasinilpotent if, given any chief factor A/B of G, every automorphism of A/B induced by an element of G is inner.

2. Preliminaries

We begin by giving some lemmas, which will be needed in §§ 3 and 4.

Lemma 2.1 (Wang [18, Lemma 2.1]). *Let G be a group. Then the following conditions apply.*

(i) *If H is c-supplemented in G, $H \leqslant M \leqslant G$, then H is c-supplemented in M.*

(ii) *Let $K \lhd G$ and $K \leqslant H$. Then H is c-supplemented in G if and only if H/K is c-supplemented in G/K.*

(iii) *Let π be a set of primes. Let N be a normal π'-subgroup of G and H a π-subgroup of G. If H is c-supplemented in G, then HN/N is c-supplemented in G/N. If furthermore N normalizes H, then the converse also holds.*

(iv) *Let L be a subgroup of G and $H \leqslant \Phi(L)$. If H is c-supplemented in G, then $H \lhd G$ and $H \leqslant \Phi(G)$.*

Lemma 2.2. *Let G be a group and let p be a prime number dividing $|G|$, with $(|G|, p-1) = 1$. Then*

(i) *if N is normal in G of order p, then N lies in $Z(G)$,*

(ii) *if G has cyclic Sylow p-subgroups, then G is p-nilpotent,*

(iii) *if M is a subgroup of G with index p, then M is normal in G.*

Proof. (i) Since $|\mathrm{Aut}(N)| = p-1$ and $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$, $|G/C_G(N)|$ must divide $(|G|, p-1) = 1$. It follows that $G = C_G(N)$ and $N \leqslant Z(G)$.

(ii) Let G_p be a Sylow p-subgroup of G with order p^n. Since G_p is cyclic, $|\mathrm{Aut}(G_p)| = p^{n-1}(p-1)$. Again, $N_G(G_p)/C_G(G_p)$ is isomorphic to a subgroup of $\mathrm{Aut}(G_p)$, so the order $|N_G(G_p)/C_G(G_p)|$ must divide $(|G|, p-1) = 1$. Thus, $N_G(G_p) = C_G(G_p)$, and statement (ii) follows by the well-known Burnside theorem.

(iii) We may assume that $\mathrm{core}_G(M) = 1$ by induction. As is widely known, the result is true in the case in which $p = 2$. So we assume that $p > 2$ and consequently G is of odd order as $(|G|, p-1) = 1$. Now we know that G is solvable by the odd-order theorem. Let N be a minimal normal subgroup of G. Then N is an elementary abelian q-group for some prime q. It is obvious that $G = MN$ and that $M \cap N$ is normal in G. Therefore, $M \cap N = 1$ and $|N| = |G : M| = p$. Now $N \leqslant Z(G)$ by statement (i) and, of course, M is normal in G, as desired. $\square$

We remark that the hypothesis $(|G|, p-1) = 1$ in Lemma 2.2 always holds when p is the smallest prime divisor of $|G|$. Hence, Lemma 2.2 (iii) extends a result of Frobenius (see [**13**]).

Lemma 2.3. *Let the p'-group H act on the p-group P. If H acts trivially on $\Omega_1(P)$ and P is quaternion-free if $p = 2$, then H acts trivially on P.*

Proof. Let P be a minimal counterexample. Then H acts trivially on every H-invariant proper subgroup of P. We may assume that $H = \langle x \rangle$ is of order q with $q \neq p$.

Set $G = [P]H$. Then G is not p-nilpotent. We shall show that every proper subgroup V of G is p-nilpotent: if $|V| = p^n$, then V is p-nilpotent, as desired. Assume q divides $|V|$. There then exists an element y in G such that $H \leqslant V^y$. Hence, $V^y = (V^y \cap P)H$. Since $V^y \cap P$ is an H-invariant proper subgroup of P, H acts trivially on $V^y \cap P$, so V^y is p-nilpotent and so is V.

Now G is a minimal non-p-nilpotent group (that is, every proper subgroup of a group is p-nilpotent but is not itself p-nilpotent). By the results of Itô and Schmidt [**9**, Sätze IV.5.4 and III.5.2], P is of exponent p if $p > 2$, and of exponent at most 4 if $p = 2$. If P is of exponent p, then H acts trivially on $\Omega_1(P) = P$: a contradiction. Hence, P is of exponent 4. On the other hand, $\Omega_1(P)H^g < G$ for any $g \in G$; therefore, $\Omega_1(P)H^g = \Omega_1(P) \times H^g$. It follows from $H^G = G$ that $\Omega_1(P) \leqslant Z(G)$. Pick $a, b \in P \setminus \Omega_1(P)$ such that $c = [a, b] \neq 1$. Then c is of order 2 as both c and b^2 are in $Z(G)$. Denote $\bar{R} = \langle a, b \rangle / \langle a^2 b^2, ca^2 \rangle$. We have $\bar{R} = \langle \bar{a},\ \bar{b} \mid \bar{a}^4 = 1,\ \bar{a}^2 = \bar{b}^2,\ \bar{a}\bar{b}\bar{a} = \bar{b} \rangle$. Hence, $\bar{R}$ is a quaternion and a section of P, which is contrary to the hypothesis. Thus, H must act trivially on P and the proof is complete. $\square$

Let G be a group. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. It is a natural generalization of the Fitting subgroup $F(G)$ of G. Its definition and properties can be found in [**10**, § X.13]. We now list the following basic facts which will be used.

Lemma 2.4 (Wei *et al.* [22, Lemma 2.3]). *Let G be a group and N be a subgroup of G.*

(i) *If N is normal in G, then $F^*(N) \leqslant F^*(G)$.*

(ii) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{soc}(F(G)C_G(F(G))/F(G))$.*

(iii) *$F^*(F^*(G)) = F^*(G) \geqslant F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

(iv) *$C_G(F^*(G)) \leqslant F^*(G)$.*

(v) *If $N \leqslant Z(G)$, then $F^*(G/N) = F^*(G)/N$.*

(vi) *If $N \lhd G$ and $N \leqslant O_p(G)$ for some prime p, then $F^*(G/\Phi(N)) = F^*(G)/\Phi(N)$.*

Now recall that a class $\mathcal{F}$ of groups is called a formation provided that (i) if $G \in \mathcal{F}$ and $N \lhd G$, then $G/N \in \mathcal{F}$, and (ii) if $N_1, N_2 \lhd G$ such that $G/N_1, G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$.

Let $\mathcal{P}$ be the set of all prime numbers. Then, by a formation function f, we mean a function f defined on $\mathcal{P}$ such that $f(\mathcal{P})$, possibly empty, is a formation. A chief factor H/K of a group G is called f-central in G if $G/C_G(H/K) \in f(p)$ for all primes p dividing $|H/K|$. A formation $\mathcal{F}$ is said to be a local formation if there exists a formation function f such that $\mathcal{F}$ is the class of all groups G for which every chief factor of G is f-central in G. In this case, we write $\mathcal{F} = LF(f)$ and call f a local definition of $\mathcal{F}$. A formation $\mathcal{F}$ is called saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. It is well known that a formation $\mathcal{F}$ is saturated if and only if $\mathcal{F}$ is a local formation [**4**, § IV].

Let $\mathcal{F} = LF(f)$ with f integrated (i.e. $f(p) \subseteq \mathcal{F}$ for all $p \in \mathcal{P}$). A normal subgroup N of a group G is said to be $\mathcal{F}$-hypercentral in G provided that every G-chief factor below N is f-central. The product of all $\mathcal{F}$-hypercentral subgroups is again an $\mathcal{F}$-hypercentral subgroup of G. This unique maximal normal subgroup is called the $\mathcal{F}$-hypercentre of G and denoted by $Z_{\mathcal{F}}(G)$. We remark that $Z_{\mathcal{F}}(G)$ does not depend on the chosen integrated local definition and that $Z_\infty(G)$ becomes $Z_{\mathcal{N}}(G)$ in this notation (see [**4**, § IV.6.8]).

Lemma 2.5 (Wei *et al.* [22, Theorem 1.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

Lemma 2.6 (Wang and Li [19, Theorem 4.5]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every minimal subgroup of $F^*(H)$ is c-supplemented in G and $F^*(H)$ is quaternion-free, then $G \in \mathcal{F}$.*

Lemma 2.7 (Asaad *et al.* [1, Lemma 2]). *Let $\mathcal{F}$ be a saturated formation. Assume that G is a non-$\mathcal{F}$-group and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = F(G)M$, where $F(G)$ is the Fitting subgroup of G. Then*

(i) *$G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G,*

(ii) *$G^{\mathcal{F}}$ is a p-group for some prime p,*

(iii) *$G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$,*

(iv) *$G^{\mathcal{F}}$ is either an elementary abelian group or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

Lemma 2.8 (Ballester-Bolinches *et al.* [2, Theorem 4.3]). *Let G be a group. Suppose that every cyclic subgroup of $G^{\mathcal{N}}$ with order 4 is c-supplemented in G. Then G is nilpotent if and only if every minimal subgroup of $G^{\mathcal{N}}$ lies in $Z_{\mathcal{N}}(G)$.*

Lemma 2.9 (Dornhoff [5, Theorem 2.8]). *If a solvable group has a 2-Sylow subgroup P which is quaternion-free, then $P \cap Z(G) \cap G^{\mathcal{N}} = 1$.*

3. Main results

Theorem 3.1. *Let G_p be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every cyclic subgroup of $G_p \cap G^{\mathcal{N}_p}$ with order p or 4 (if $p = 2$) is c-supplemented in G.*

Proof. If G is p-nilpotent, then $G^{\mathcal{N}_p} = 1$, so the necessary condition holds.

Conversely, we shall prove that G is p-nilpotent under the sufficient condition. Let G be a minimal counterexample and let M be a maximal subgroup of G. Since $M/M \cap G^{\mathcal{N}_p} \cong MG^{\mathcal{N}_p}/G^{\mathcal{N}_p}$ is p-nilpotent, $M^{\mathcal{N}_p} \leqslant M \cap G^{\mathcal{N}_p}$. Now let M_p be a Sylow p-subgroup of M. Without losing generality we may assume that $M_p \leqslant G_p$. Then $M_p \cap M^{\mathcal{N}_p} \leqslant G_p \cap G^{\mathcal{N}_p}$. Hence, every cyclic subgroup of $M_p \cap M^{\mathcal{N}_p}$ with order p or 4 (if $p = 2$) is c-supplemented in M by Lemma 2.1 and M satisfies the hypotheses of the theorem. The minimality of G implies that M is p-nilpotent. Thus, G is a minimal non-p-nilpotent group. By results of Itô and Schmidt [**9**, Sätze IV.5.4 and III.5.2], G has a normal Sylow p-subgroup G_p and a cyclic Sylow q-subgroup G_q such that $G = [G_p]G_q$. Moreover, G_p is of exponent p if $p > 2$ and of exponent at most 4 if $p = 2$. On the other hand, the minimality of G implies that $G^{\mathcal{N}_p} = G_p$. Let P_0 be a minimal subgroup of G_p. Then, by the hypotheses, there exists a subgroup K_0 of G such that $G = P_0K_0$ and $P_0 \cap K_0 \leqslant \mathrm{core}_G(P_0)$. If P_0 is not normal in G, then K_0 is a maximal subgroup of G with index p. By applying Lemma 2.2 we see that K_0 is a normal subgroup of G. It follows from the fact that K_0 is nilpotent that G_q is normal in G: a contradiction. Therefore, every minimal subgroup of G_p is normal in G and, by Lemma 2.2, every minimal subgroup of G_p must be in the centre of G. If G_p has exponent p, then $G_p = \Omega_1(G_p)$ and $G = G_p \times G_q$: a contradiction. Thus, $p = 2$ and G_2 has exponent 4. Now let $P_1 = \langle x \rangle$ be a cyclic subgroup of G_2 with order 4. Then, by the hypotheses there is a subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1 \leqslant \mathrm{core}_G(P_1)$. If, furthermore,

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$|G:K_1|=4$, then $|G:K_1\langle x^2\rangle|=2$, and hence $K_1\langle x^2\rangle$ is normal in G and so is G_q: a contradiction. If $|G:K_1|=2$, then K_1 itself is normal in G. We still get a contradiction. Henceforth $K_1=G$ and P_1 is normal in G. Lemma 2.2 implies that $P_1G_q=P_1\times G_q$ and, consequently, $G_2=\Omega_2(G_2)$ centralizes G_q. This is a final contradiction. The proof is complete. $\square$

Some authors prefer the following equivalent form of Theorem 3.1. Similar equivalent forms of other results will be omitted.

Corollary 3.2. *Let H be a normal subgroup of a group G such that G/H is p-nilpotent, where p is a prime divisor of $|G|$ with $(|G|,p-1)=1$. Then G is p-nilpotent if and only if every cyclic subgroup of H with order p or 4 (if $p=2$) is c-supplemented in G.*

An identical argument using Lemma 2.3 yields the following result.

Theorem 3.3. *Let G_p be a Sylow p-subgroup of a group G and assume that G_p is quaternion-free, where p is a prime divisor of $|G|$ with $(|G|,p-1)=1$. Then G is p-nilpotent if and only if every minimal subgroup of $G_p\cap G^{\mathcal{N}_p}$ is c-supplemented in G.*

If p is an arbitrary prime number, the corresponding result is as follows, which is an 'iff' version of Itô's theorem on p-nilpotence.

Theorem 3.4. *Let G_p be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if every minimal subgroup of $G_p\cap G^{\mathcal{N}_p}$ lies in $Z_{\mathcal{N}_p}(G)$ and every cyclic subgroup of $G_p\cap G^{\mathcal{N}_p}$ with order 4 (if $p=2$) is c-supplemented in G.*

Proof. It will suffice to prove the 'if' part.

Let G be a minimal counterexample and let M be a maximal subgroup of G. We shall show that M satisfies the hypotheses of the theorem. First we see easily that $\mathcal{N}_p=LF(f)$, where $f(q)=\{1\}$ if $q=p$ and $f(q)=\{\text{all groups}\}$ if $q\neq p$. By [**4**, § IV.3.8.(b)], $\mathcal{N}_p=LF(F)$ with F integrated and full (i.e. $S_qF(q)=F(q)$ for all $q\in\mathcal{P}$), where $F(q)=S_qf(q)\cap\mathcal{N}_p$. Now let H/K be a G-chief factor below $Z_{\mathcal{N}_p}(G)$. Since H/K is F-central in G, $G/C_G(H/K)\in F(q)$, where q is an arbitrary prime divisor of $|H/K|$. Noting that $M\cap H/M\cap K\cong MK\cap H/K\leqslant H/K$, we have

$$M/C_M(M\cap H/M\cap K)\cong M/C_M(MK\cap H/K).$$

On the other hand, $C_G(H/K)\leqslant C_G(MK\cap H/K)$, so

$$G/C_G(H/K)\sim G/C_G(MK\cap H/K)\in F(q).$$

It follows that

$$\begin{aligned} M/C_M(M\cap H/M\cap K) &\cong M/C_M(MK\cap H/K)\\ &\cong MC_G(MK\cap H/K)/C_G(MK\cap H/K)\\ &\leqslant G/C_G(MK\cap H/K)\in F(q). \end{aligned}$$

Hence, $M/C_M(M \cap H/M \cap K) \in F(q)$ for any prime divisor q of $|H/K|$. This shows that $M \cap Z_{\mathcal{N}_p}(G) \leqslant Z_{\mathcal{N}_p}(M)$. Now let M_p be a Sylow p-subgroup of M. Without loss of generality, we may assume that $M_p \leqslant G_p$. For any minimal subgroup P_1 of $M_p \cap M^{\mathcal{N}_p}$, by the hypotheses, we have $P_1 \leqslant M \cap Z_{\mathcal{N}_p}(G) \leqslant Z_{\mathcal{N}_p}(M)$. Moreover, every cyclic subgroup of $M_p \cap M^{\mathcal{N}_p}$ with order 4 (if $p = 2$) is c-supplemented in M. Hence, M satisfies the hypotheses of the theorem. The minimality of G implies that M is p-nilpotent. Now, G is minimal non-p-nilpotent. It follows that G has a normal Sylow p-subgroup G_p and a cyclic Sylow q-subgroup G_q such that $G = [G_p]G_q$. Moreover, G_p is of exponent p if $p > 2$ and of exponent at most 4 if $p = 2$. Given that $Z_{\mathcal{N}_p}(G) < G$, $Z_{\mathcal{N}_p}(G)$ is nilpotent. Let P and Q be the Sylow p-subgroup and the Sylow q-subgroup of $Z_{\mathcal{N}_p}(G)$, respectively. Then both P and Q are normal in G and, since $Q < G_q$ and G_q is cyclic, $Q \leqslant \Phi(G_q) \leqslant Z(G)$. If H/K is a G-chief factor below P, then $H \leqslant Z_{\mathcal{N}}(G)$ as $G = C_G(H/K)$. Thus, we have $P \leqslant Z_{\mathcal{N}}(G)$ and therefore $Z_{\mathcal{N}_p}(G) = Z_{\mathcal{N}}(G) = Z(G)$. It follows from $G^{\mathcal{N}_p} = G_p$ and the hypotheses that $\Omega_1(G_p) \leqslant Z(G)$. If G_p is of exponent p, then $G_p = \Omega_1(G_p) \leqslant Z(G)$, and $G = G_p \times G_q$: a contradiction. Therefore, $p = 2$ and G_2 is of exponent 4. By applying Theorem 3.1 we conclude that G is 2-nilpotent, which is contrary to the hypothesis on G. This completes the proof. □

By minimizing the number of minimal subgroups, we can give an extension of Lemma 2.8. It is also an 'iff' version of Itô's theorem on nilpotence.

Theorem 3.5. *A group G is nilpotent if and only if every minimal subgroup of $F^*(G^{\mathcal{N}})$ lies in $Z_{\mathcal{N}}(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{N}})$ with order 4 is c-supplemented in G.*

Proof. Only the 'if' part needs to be verified.

Let G be a counterexample of minimal order. Then we have the following hypotheses.

(i) Every proper normal subgroup of G is nilpotent.

Suppose that M is a maximal normal subgroup of G. Since $M^{\mathcal{N}} \lhd M \cap G^{\mathcal{N}}$ and $M \cap G^{\mathcal{N}} \lhd G^{\mathcal{N}}$, by Lemma 2.4 (i), $F^*(M^{\mathcal{N}}) \leqslant F^*(M \cap G^{\mathcal{N}}) \leqslant F^*(G^{\mathcal{N}})$. Moreover, $M \cap Z_{\mathcal{N}}(G) \leqslant Z_{\mathcal{N}}(M)$. Now we see easily that M satisfies the hypotheses of the theorem. The minimal choice of G implies that M is nilpotent.

(ii) $F(G)$ is the unique maximal normal subgroup of G.

In fact, since the class of nilpotent groups is a Fitting class, G has a unique maximal normal subgroup M, say. The nilpotency of M implies that $M = F(G)$.

(iii) $G^{\mathcal{N}} = G$, $G' = G$ and $F^*(G) = F(G) < G$.

If $G^{\mathcal{N}} < G$, then $G^{\mathcal{N}}$ is nilpotent by (i). Thus, $F^*(G^{\mathcal{N}}) = G^{\mathcal{N}}$ by Lemma 2.4 (iii). Now Lemma 2.8 implies immediately that G is nilpotent: a contradiction. Hence, we must have $G^{\mathcal{N}} = G$. From (ii) we know that $G/F(G)$ is simple. If $G/F(G)$ is cyclic of prime order, then $G^{\mathcal{N}} \leqslant F(G)$ and consequently $G = F(G)$ is nilpotent, which is a contradiction. Thus, $G/F(G)$ is a simple non-abelian group. Now $G' \nleqslant F(G)$ implies that $G' = G$. If $F(G) < F^*(G)$, then $F^*(G) = G$ by (ii). Again by Lemma 2.8, G is nilpotent, which is a contradiction.

(iv) The final contradiction.

Since $F^*(G) = F(G)$ is not an identity group, we may choose the smallest prime divisor p of $|F(G)|$ such that $O_p(G) \neq 1$. For any Sylow q-subgroup G_q of G, where $q \neq p$, consider the subgroup $G_0 = O_p(G)G_q$. It is clear that $G_0^{\mathcal{N}} \leqslant O_p(G)$ and $G_0 \cap Z_{\mathcal{N}}(G) \leqslant Z_{\mathcal{N}}(G_0)$. Hence, every minimal subgroup of $G_0^{\mathcal{N}}$ lies in $Z_{\mathcal{N}}(G_0)$ and every cyclic subgroup of $G_0^{\mathcal{N}}$ with order 4 is c-supplemented in G_0. By Lemma 2.8, G_0 is nilpotent. Hence, $G_0 = O_p(G) \times G_q$ and $O^p(G) \leqslant C_G(O_p(G))$. Consequently, $G/C_G(O_p(G))$ is a p-group and, by (i) and (iii), $C_G(O_p(G)) = G$, namely $O_p(G) \leqslant Z(G)$. Now we consider the factor group $\bar{G} = G/O_p(G)$. First we have $F^*(\bar{G}) = F^*(G)/O_p(G)$ by Lemma 2.4 (v). For any element $\bar{x}$ of odd prime order in $F^*(\bar{G})$, since $O_p(G)$ is the Sylow p-subgroup of $F^*(G)$, $\bar{x}$ can be viewed as the image of an element x of odd prime order in $F^*(G)$. It follows that x lies in $Z_{\mathcal{N}}(G)$ and $\bar{y}$ lies in $Z_{\mathcal{N}}(\bar{G})$, for $Z_{\mathcal{N}}(G/O_p(G)) = Z_{\mathcal{N}}(G)/O_p(G)$. This shows that $\bar{G}$ satisfies the hypotheses of the theorem. By the choice of G, we conclude that $\bar{G}$ is nilpotent and so G is nilpotent. This is the final contradiction and we are done. □

Now we turn our attention to the topic of localization. First we can localize the conditions in Theorem 3.1 as follows.

Theorem 3.6. *Let G be a group such that G is S_4-free and let G_p be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every cyclic subgroup of $G_p \cap G^{\mathcal{N}_p}$ with order p or 4 (if $p = 2$) is c-supplemented in $N_G(G_p)$.*

Proof. We only need to prove the 'if' part.

Suppose that G is a minimal counterexample. Then we have the following claims.

(i) M is p-nilpotent whenever $G_p \leqslant M < G$. In particular, $N_G(G_p)$ is p-nilpotent.

It is clear that M satisfies the hypotheses of the theorem. The minimality of G implies that M is p-nilpotent. If $N_G(G_p) = G$, then, by Theorem 3.1, G is p-nilpotent: a contradiction. Hence, $N_G(G_p) < G$ and $N_G(G_p)$ is p-nilpotent.

(ii) $O_{p'}(G) = 1$.

If not, write $N = O_{p'}(G)$ and consider G/N. Then G_pN/N is a Sylow p-subgroup of G/N. Since $(G/N)^{\mathcal{N}_p} = G^{\mathcal{N}_p}N/N$, $(G_pN/N) \cap (G/N)^{\mathcal{N}_p} = (G_p \cap G^{\mathcal{N}_p})N/N$. Now let P_1N/N be a cyclic subgroup of $(G_pN/N) \cap (G/N)^{\mathcal{N}_p}$ with order p or 4. We may assume that $P_1 = \langle x \rangle$ is a cyclic subgroup of $G_p \cap G^{\mathcal{N}_p}$ such that x is of order p or 4 (if $p = 2$). Since P_1 is c-supplemented in $M = N_G(G_p)$, there exists a subgroup K_1 of M such that $M = P_1K_1$ and $P_1 \cap K_1 \leqslant \mathrm{core}_M(P_1)$. It follows that $MN/N = (P_1N/N)(K_1N/N)$, where $(P_1N/N) \cap (K_1N/N) = (P_1 \cap K_1N)N/N$. If $P_1 \cap K_1N = P_1$, then $P_1 \leqslant K_1N$ and $|K_1|_p = |K_1N|_p = |MN|_p = |M|_p$. Furthermore, $K_1 = M$ and P_1 is normal in M. Consequently, P_1N/N is normal in MN/N. Now assume that $P_1 \cap K_1N < P_1$. Then $P_1 \cap K_1N \leqslant \langle x^p \rangle$ and either $P_1 \cap K_1N = 1$ or $P_1 \cap K_1N = \langle x^p \rangle$. If the former holds, then P_1N/N is complemented in MN/N. If the latter holds, then $P_1 \cap K_1N$ lies in $\Phi(M)$ as $x^p \in \Phi(G_p) \leqslant \Phi(M)$. We can infer from the fact that $\langle x^p \rangle$ is c-supplemented in M that $P_1 \cap K_1N$ is normal in M. In this case P_1N/N is c-supplemented in MN/N. To summarize, every cyclic subgroup of $(G_pN/N) \cap (G/N)^{\mathcal{N}_p}$ with order p or 4 is c-supplemented in $MN/N = N_{G/N}(G_pN/N)$. The choice of G implies that G/N is p-nilpotent and hence G is p-nilpotent: a contradiction.

(iii) $G/O_p(G)$ is p-nilpotent and $C_G(O_p(G)) \leqslant O_p(G)$.

Suppose that $G/O_p(G)$ is not p-nilpotent. Then, by Frobenius's theorem (see [**14**, Theorem 10.3.2]), there exists a subgroup of G_p properly containing $O_p(G)$ such that its G-normalizer is not p-nilpotent. By applying (i), we may choose a subgroup P_1 of G_p properly containing $O_p(G)$ such that $N_G(P_1)$ is not p-nilpotent but $N_G(P_2)$ is p-nilpotent whenever $P_1 < P_2 \leqslant G_p$. It is clear that $P_1 < P_0 \leqslant G_p$ for some Sylow p-subgroup P_0 of $N_G(P_1)$. Since $P_0 \cap (N_G(P_1))^{\mathcal{N}_p} \leqslant G_p \cap G^{\mathcal{N}_p}$, by Lemma 2.1 (i), every cyclic subgroup of $P_0 \cap (N_G(P_1))^{\mathcal{N}_p}$ with order p or 4 is c-supplemented in P_0. The choice of P_1 implies that $N_G(P_0)$ is p-nilpotent, and hence $N_{N_G(P_1)}(P_0)$ is also p-nilpotent. It follows that every cyclic subgroup of $P_0 \cap (N_G(P_1))^{\mathcal{N}_p}$ with order p or 4 is c-supplemented in $N_{N_G(P_1)}(P_0)$. This shows that $N_G(P_1)$ satisfies the hypotheses of the theorem. Therefore, $N_G(P_1)$ is p-nilpotent by the minimality of G, which is contrary to our choice. Now, $G/O_p(G)$ is p-nilpotent, so G is p-solvable with $O_{p'}(G) = 1$. Consequently, we obtain $C_G(O_p(G)) \leqslant O_p(G)$ (by [**6**, Theorem 6.3.2]).

(iv) $G = G_pG_q$, where G_q is an elementary abelian Sylow q-subgroup of G for a prime $q \neq p$. Moreover, G_p is maximal in G and $G_qO_p(G)/O_p(G)$ is a minimal normal in $G/O_p(G)$.

For any prime divisor q of $|G|$ not equal to p, since G is p-solvable, there is a Sylow q-subgroup G_q of G such that $G_0 = G_pG_q$ is a subgroup of G (see [**6**, Theorem 6.3.5]). If $G_0 < G$, then G_0 is p-nilpotent by (i). This leads to $G_q \leqslant C_G(O_p(G)) \leqslant O_p(G)$: a contradiction. Thus, $G = G_pG_q$ is solvable. Now let $T/O_p(G)$ be a minimal normal subgroup of $G/O_p(G)$ contained in $O_{pp'}(G)/O_p(G)$. Then $T = O_p(G)(T \cap G_q)$. If $T \cap G_q < G_q$, then $G_pT < G$ and therefore G_pT is p-nilpotent. As a result, $1 < T \cap G_q \leqslant C_G(O_p(G)) \leqslant O_p(G)$: a contradiction. Thus, $T = O_{pp'}(G)$ and hence $G_qO_p(G)/O_p(G)$ is an elementary abelian q-group complementing $G_p/O_p(G)$. This yields that G_p is maximal in G.

(v) $|G_p : O_p(G)| = p$.

Let P_0 be a maximal subgroup of G_p containing $O_p(G)$ and let $G_0 = P_0O_{pp'}(G)$. Then P_0 is a Sylow p-subgroup of G_0. The maximality of G_p in G implies that either $N_G(P_0) = G$ or $N_G(P_0) = G_p$. If the latter holds, then $N_{G_0}(P_0) = P_0$. It is easy to see that G_0 satisfies the hypotheses of the theorem. Therefore, G_0 is p-nilpotent and $G_q \leqslant C_G(O_p(G)) \leqslant O_p(G)$: a contradiction. Hence, $N_G(P_0) = G$ and $P_0 = O_p(G)$.

(vi) $G = G^{\mathcal{N}_p}L$, where $L = \langle a\rangle[G_q]$ is a non-abelian split extension of a normal Sylow q-subgroup G_q by a cyclic p-subgroup $\langle a\rangle$, $a^p \in Z(L)$ and the action of a (by conjugate) on G_q is irreducible.

From (iii) we see that $G^{\mathcal{N}_p} \leqslant O_p(G)$. Clearly, $T = G^{\mathcal{N}_p}G_q$ is normal in G. Let P_0 be a maximal subgroup of G_p containing $G^{\mathcal{N}_p}$. Then, by the maximality of G_p, either $N_G(P_0) = G_p$ or $N_G(P_0) = G$. If $N_G(P_0) = G_p$, then $N_M(P_0) = P_0$, where $M = P_0T = P_0G_q$. Evidently, $P_0 \cap M^{\mathcal{N}_p} \leqslant G_p \cap G^{\mathcal{N}_p}$, and hence M satisfies the hypotheses of the theorem. By the minimality of G, M is p-nilpotent. It follows that $T = G^{\mathcal{N}_p}G_q = G^{\mathcal{N}_p} \times G_q$ and so $G_q \lhd G$: a contradiction. Therefore, $P_0 \lhd G$ and $P_0 \leqslant O_p(G)$. We may thus infer from (v) that $O_p(G) = P_0$ and hence that $G_p/G^{\mathcal{N}_p}$ is a cyclic group. Now, applying the Frattini argument, we have $G = G^{\mathcal{N}_p}N_G(G_q)$. Therefore, we may assume that $G = G^{\mathcal{N}_p}L$, where $L = \langle a\rangle[G_q]$ is a non-abelian split extension of a normal Sylow q-subgroup G_q by a cyclic p-group $\langle a\rangle$. Now that $|G_p : O_p(G)| = p$ and $O_p(G) \cap N_G(G_q) \lhd N_G(G_q)$, we have $a^p \in Z(L)$. Also since G_p is maximal in G, $G^{\mathcal{N}_p}G_q/G^{\mathcal{N}_p}$ is minimal normal in $G/G^{\mathcal{N}_p}$ and consequently a acts irreducibly on G_q.

(vii) $G^{\mathcal{N}_p}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

By Lemma 2.7 it will suffice to show that there exists a p-nilpotent maximal subgroup M of G such that $G = G^{\mathcal{N}_p}M$. In fact, let M be a maximal subgroup of G containing L. Then $M = L(M \cap G^{\mathcal{N}_p})$ and $G = G^{\mathcal{N}_p}M$. It is clear that $M \cap G^{\mathcal{N}_p} \lhd G$, and hence $M = (\langle a\rangle(M \cap G^{\mathcal{N}_p}))G_q$. Write $P_0 = \langle a\rangle(M \cap G^{\mathcal{N}_p})$ and let M_0 be a maximal subgroup of M containing P_0. Then $M_0 = P_0(M_0 \cap G_q)$ and $G^{\mathcal{N}_p}M_0 < G$. By applying (i) we see that $G^{\mathcal{N}_p}M_0$ is p-nilpotent. Therefore, $M_0 \cap G_q \leqslant C_G(O_p(G)) \leqslant O_p(G)$. It follows that $M_0 \cap G_q = 1$ and so P_0 is maximal in M. In this case, if $P_0 \lhd M$, then $\langle a\rangle = P_0 \cap L \lhd L$, which is contrary to (vi). Hence, $N_M(P_0) = P_0$ and M satisfies the hypotheses of the theorem. The choice of G implies that M is p-nilpotent, as desired.

(viii) The exponent of $G^{\mathcal{N}_p}$ is not p.

Assume that $G^{\mathcal{N}_p}$ has exponent p. If $G^{\mathcal{N}_p} \leqslant Z(G_p)$, then, by the Frattini argument,

$$G = N_G(G^{\mathcal{N}_p}) = C_G(G^{\mathcal{N}_p})N_G(G_p).$$

Since $N_G(G_p) = G_p$ and $G_p \leqslant C_G(G^{\mathcal{N}_p})$, $G = N_G(G^{\mathcal{N}_p}) = C_G(G^{\mathcal{N}_p})$. Consequently, $G^{\mathcal{N}_p}G_q = G^{\mathcal{N}_p} \times G_q$ and $G_q \lhd G$: a contradiction. Now we assume that $G^{\mathcal{N}_p} \nleqslant Z(G_p)$.

Let P_1 be a minimal subgroup of $G^{\mathcal{N}_p}$ not contained in $Z(G_p)$. Then, by the hypotheses, there is a subgroup K_1 of G_p such that $G_p = P_1K_1$ and $P_1 \cap K_1 = 1$. In general, we may find minimal subgroups $P_1, P_2, \dots, P_m$ of $G^{\mathcal{N}_p}$ and also subgroups $K_1, K_2, \dots, K_m$ of G_p such that $G_p = P_iK_i$, $P_i \cap K_i = 1$, for each i and

$$1 \neq G^{\mathcal{N}_p} \cap K_1 \cap \cdots \cap K_m \leqslant Z(G_p).$$

Furthermore, we may assume that $P_i \leqslant K_1 \cap \cdots \cap K_{i-1}$, $i = 2, 3, \dots, m$. Henceforth,

$$K_1 \cap \cdots \cap K_{i-1} = P_i(K_1 \cap \cdots \cap K_i)$$

for $i = 2, 3, \dots, m$. Since K_i is maximal in G_p, $G^{\mathcal{N}_p} \cap K_i$ is normal in G_p. We may assume that $G_p = G^{\mathcal{N}_p}\langle a\rangle$. It is easy to see that $(G^{\mathcal{N}_p} \cap K_i)\langle a\rangle$ is a complement of P_i in G_p, so we may replace K_i by $(G^{\mathcal{N}_p} \cap K_i)\langle a\rangle$ and so we may further assume that $\langle a\rangle \leqslant K_i$ for each i. As $G_p = G^{\mathcal{N}_p}\langle a\rangle$, $K_1 \cap \dots \cap K_m = (G^{\mathcal{N}_p} \cap K_1 \cap \dots \cap K_m)\langle a\rangle$. It follows from $G^{\mathcal{N}_p} \cap K_1 \cap \dots \cap K_m \leqslant Z(G_p)$ that $K_1 \cap \dots \cap K_m$ is abelian.

Now we claim that p is even. If it is not the case, then, by [**6**, Theorem 6.5.2], $K_1 \cap \dots \cap K_m \leqslant O_p(G)$. Consequently, $G_p = G^{\mathcal{N}_p}(K_1 \cap \dots \cap K_m) \leqslant O_p(G)$: a contradiction. We proceed to consider the following two cases.

Case 1 ($|\langle a\rangle| = 2^n$, $n > 1$). Since $K_1 \cap \dots \cap K_m$ is an abelian normal subgroup of G_2 and $a \in K_1 \cap \dots \cap K_m$, $\Phi(K_1 \cap \dots \cap K_m) = \langle a^2\rangle \lhd G_2$ and so $\Omega_1(\langle a^2\rangle) = \langle c\rangle \leqslant Z(G_2)$, where $c = a^{2^{n-1}}$. Again, $c \in Z(L)$ by (vi), so $c \in Z(G)$. Now we consider the factor group $G/\langle c\rangle$. For any element x of $G^{\mathcal{N}_2}$, since $G^{\mathcal{N}_2}$ has exponent 2, by the hypotheses, $G_2 = \langle x\rangle K$ with $\langle x\rangle \cap K \leqslant \mathrm{core}_{G_2}(\langle x\rangle)$. In view of $c \in \Phi(G_2)$ we see that $G_2/\langle c\rangle = (\langle x\rangle\langle c\rangle/\langle c\rangle)(K/\langle c\rangle)$ and $(\langle x\rangle\langle c\rangle/\langle c\rangle) \cap (K/\langle c\rangle) \leqslant \mathrm{core}_{G_2/\langle c\rangle}(\langle x\rangle\langle c\rangle/\langle c\rangle)$. This shows that $G/\langle c\rangle$ satisfies the hypotheses of the theorem. The choice of G implies that $G/\langle c\rangle$ is 2-nilpotent and therefore G is also 2-nilpotent: a contradiction.

Case 2 ($|\langle a\rangle| = 2$). As $G_2 = G^{\mathcal{N}_2}\langle a\rangle$ is not normal in G, we have $\langle a\rangle \cap G^{\mathcal{N}_2} = 1$. On the other hand, since a acts irreducibly on G_q, a is an involutive automorphism of G_q; consequently, G_q is cyclic of order q and $b^a = b^{-1}$, where $G_q = \langle b\rangle$. In this case, $G^{\mathcal{N}_2}$ is minimal normal in G. In fact, let N be a minimal normal subgroup of G contained in $G^{\mathcal{N}_2}$ and let $H = NL$. Since $N\langle a\rangle$ is maximal but not normal in H, we see that $N_H(N\langle a\rangle) = N\langle a\rangle$. Noting that $N\langle a\rangle \cap H^{\mathcal{N}_2} \leqslant N$, every minimal subgroup of $N\langle a\rangle \cap H^{\mathcal{N}_2}$ is c-supplemented in $N_H(N\langle a\rangle) = N\langle a\rangle$ by Lemma 2.1 (i). If further $H < G$, then the choice of G implies that H is 2-nilpotent. Consequently, $NG_q = N \times G_q$ and so $1 \neq N \cap Z(G_2) \leqslant Z(G)$. The choice of N implies that N is of order 2. In this case, if $N \nleqslant \Phi(G_2)$, then N has a complement to G_2. By applying the theorem of Gaschütz [**9**, Satz I.17.4], N also has a complement to G, say M. It follows that M is a normal subgroup of G. Furthermore, G/M is cyclic of order 2 and so $N \leqslant G^{\mathcal{N}_2} \leqslant M$: a contradiction. Hence, $N \leqslant \Phi(G_2)$. Now we consider the factor group G/N. For any minimal subgroup $\langle x\rangle N/N$ of $(G/N)^{\mathcal{N}_2} = G^{\mathcal{N}_2}/N$, by the hypotheses, $G_2 = \langle x\rangle K$ with $\langle x\rangle \cap K \leqslant \mathrm{core}_{G_2}\langle x\rangle$, where $x \in G^{\mathcal{N}_2}$. Now we see that $G_2/N = (\langle x\rangle N/N)(K/N)$ with $(\langle x\rangle N/N) \cap (K/N) \leqslant \mathrm{core}_{G_2/N}(\langle x\rangle N/N)$, so $\langle x\rangle N/N$ is c-supplemented in G_2/N. This yields at once that G/N is 2-nilpotent. Consequently, $G^{\mathcal{N}_2} \leqslant N$ and so $G^{\mathcal{N}_2} = N$, which is contrary to $H < G$. Hence, $G^{\mathcal{N}_2}$ must be a minimal normal subgroup of G and hence it is an elementary abelian 2-group. Since $G^{\mathcal{N}_2} \cap N_G(G_q) \lhd N_G(G_q)$, we know that $G^{\mathcal{N}_2} \cap N_G(G_q) = 1$ and so b acts fixed-point-freely on $G^{\mathcal{N}_2}$. We may assume that $N_1 = \{1, c_1, c_2, \dots, c_q\}$ is a subgroup of $G^{\mathcal{N}_2}$ with $c_1 \in Z(G_2)$ and that $b = (c_1, c_2, \dots, c_q)$ is a permutation of the set $\{c_1, c_2, \dots, c_q\}$. Noting that $b^a = b^{-1}$ and $(c_1)^{a^{-1}ba} = (c_1)^{b^{-1}}$, $(c_2)^a = c_q$. By using $(b^i)^a = b^{-i}$ and $(c_1)^{a^{-1}b^ia} = (c_1)^{b^{-i}}$, we see that $(c_{i+1})^a = c_{q-i+1}$ for $i = 1, 2, \dots, (q+1)/2$. Hence, N_1 is normalized by both $G^{\mathcal{N}_2}$ and L, and so N_1 is normal in G. The minimal normality of $G^{\mathcal{N}_2}$ implies that $G^{\mathcal{N}_2} = N_1$. Thus, we have $Z(G_2) = \{1, c_1\}$. Since $G^{\mathcal{N}_2}$, $K_1, \dots, K_m$ are maximal subgroups of G_2 and $1 \neq G^{\mathcal{N}_2} \cap K_1 \cap \dots \cap K_m \leqslant Z(G_2)$, we get $\Phi(G_2) \leqslant Z(G_2)$. In view of the fact that G_2

is not abelian, $\Phi(G_2) = G_2' = Z(G_2)$, namely G_2 is an extra-special 2-group. By applying [**14**, Theorem 5.3.8], there exists some positive integer h such that $|G_2| = 2^{2h+1}$. Hence, $|G^{\mathcal{N}_2}| = 2^{2h}$. However, $2^{2h} - 1 = (2^h + 1)(2^h - 1)$ and $q = 2^{2h} - 1$, and hence $h = 1$, $q = 3$ and $|G_2| = 2^3$. Now we see that $L \cong S_3$ and $G^{\mathcal{N}_2} G_3 \cong A_4$. Therefore, $G \cong S_4$, which is contrary to the hypothesis on G.

(ix) The final contradiction.

By (vii) and (viii) we see that $p = 2$ and $G^{\mathcal{N}_2}$ has exponent 4. Now, using Lemma 2.7, we see that $\Phi(G^{\mathcal{N}_2})$ is an elementary abelian 2-group. For any minimal subgroup P_1 of $\Phi(G^{\mathcal{N}_2})$, since P_1 is c-supplemented in G_2, we have $P_1 \lhd G_2$ and therefore $\Phi(G^{\mathcal{N}_2}) \leqslant Z(G_2)$. By the Frattini argument we further obtain

$$G = N_G(\Phi(G^{\mathcal{N}_2})) = C_G(\Phi(G^{\mathcal{N}_2})) N_G(G_2).$$

As $N_G(G_2) = G_2$ and $G_2 \leqslant C_G(\Phi(G^{\mathcal{N}_2}))$, we get $\Phi(G^{\mathcal{N}_2}) \leqslant Z(G)$.

Next we consider the factor group $G/\Phi(G^{\mathcal{N}_2})$. Let x be an element of $G^{\mathcal{N}_2}$. Then x is of order at most 4 and, by the hypotheses, $G_2 = \langle x \rangle K$ with $\langle x \rangle \cap K \leqslant \mathrm{core}_{G_2}\langle x \rangle$. But $x^2 \in \Phi(G^{\mathcal{N}_2})$, so $\langle x^2 \rangle K$ is a group. Now that $|G_2 : \langle x^2 \rangle K| \leqslant 2$, $\Phi(G^{\mathcal{N}_2}) \leqslant \langle x^2 \rangle K$. It follows that

$$G_2/\Phi(G^{\mathcal{N}_2}) = (\langle x \rangle \Phi(G^{\mathcal{N}_2})/\Phi(G^{\mathcal{N}_2}))(\langle x^2 \rangle K/\Phi(G^{\mathcal{N}_2}))$$

with

$$\begin{aligned}(\langle x \rangle \Phi(G^{\mathcal{N}_2})/\Phi(G^{\mathcal{N}_2})) \cap (\langle x^2 \rangle K/\Phi(G^{\mathcal{N}_2})) &= (\langle x \rangle \cap K)\Phi(G^{\mathcal{N}_2})/\Phi(G^{\mathcal{N}_2}) \\ &\leqslant \mathrm{core}_{G_2/\Phi(G^{\mathcal{N}_2})}(\langle x \rangle \Phi(G^{\mathcal{N}_2})/\Phi(G^{\mathcal{N}_2})).\end{aligned}$$

This means that $G/\Phi(G^{\mathcal{N}_2})$ satisfies the hypotheses of the theorem. By the choice of G, $G/\Phi(G^{\mathcal{N}_2})$ is 2-nilpotent. Of course, G is 2-nilpotent, which is the final contradiction. Our proof is now complete. □

Similarly, by using Lemma 2.9 in the first paragraph of claim (ix), we can also localize the conditions in Theorem 3.3.

Theorem 3.7. *Let G be a group such that G is S_4-free. Also let G_p be a Sylow p-subgroup of G and assume that G_p is quaternion-free, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every minimal subgroup of $G_p \cap G^{\mathcal{N}_p}$ is c-supplemented in $N_G(G_p)$.*

4. Applications

Thompson once proved that a group G is solvable if G has a nilpotent maximal subgroup M of odd order [**14**, Theorem 10.4.2]. Later on, Deskins and Janko showed that the result still holds if the Sylow 2-subgroup of M is allowed to have class at most 2 [**9**, Satz IV.7.4]. As an application of our main results, we can give another generalization of Thompson's result.

Theorem 4.1. *Let M be a nilpotent maximal subgroup of a group G and let M_2 be a Sylow 2-subgroup of M. If one of the following conditions holds, then G is solvable.*

(i) *Every cyclic subgroup of $M_2 \cap G^{\mathcal{N}_2}$ with order 2 or 4 is c-supplemented in G,*

(ii) *Every minimal subgroup of $M_2 \cap G^{\mathcal{N}_2}$ is c-supplemented in G and M_2 is quaternion-free.*

(iii) *G is S_4-free and every cyclic subgroup of $M_2 \cap G^{\mathcal{N}_2}$ with order 2 or 4 is c-supplemented in M_2.*

(iv) *G is S_4-free and quaternion-free and very minimal subgroup of $M_2 \cap G^{\mathcal{N}_2}$ is c-supplemented in M_2.*

Proof. Let G be a minimal counterexample and let $M_{2'}$ be the normal 2-complement of M. Then $M_{2'}$ is normal in G by [**15**, Theorem 1]. It is clear that $G/M_{2'}$ satisfies the hypotheses of the theorem. The minimality of G implies that $G/M_{2'}$ is solvable and therefore G is solvable as M is nilpotent: a contradiction. Hence, $M_{2'} = 1$ and M_2 is maximal in G. If M_2 is normal in G, then G/M_2 is cyclic and, of course, G is solvable: a contradiction. Therefore, $M_2 = N_G(M_2)$ and M_2 is a Sylow 2-subgroup of G. By applying our main results we find that G is 2-nilpotent. Of course G is still solvable. This contradiction shows that Theorem 4.1 holds. □

In the following, we shall extend Theorem 3.5 to formations. The result is an 'iff' version of Itô's theorem on nilpotence in the formation universe.

Theorem 4.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{N}$. Then a group $G \in \mathcal{F}$ if and only if every minimal subgroup of $F^*(G^{\mathcal{F}})$ lies in $Z_{\mathcal{F}}(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is c-supplemented in G.*

Proof. Of course it is only the sufficiency of the condition that is in question.

Suppose that G is a minimal counterexample. Then $G^{\mathcal{F}} \neq 1$, $Z_{\mathcal{F}}(G) < G$ and $G/Z_{\mathcal{F}}(G) \notin \mathcal{F}$. It follows that there is a maximal subgroup M of G containing $Z_{\mathcal{F}}(G)$ such that $G/\operatorname{core}_G(M) \notin \mathcal{F}$, otherwise we would have $(G/Z_{\mathcal{F}}(G))/(\Phi(G/Z_{\mathcal{F}}(G))) \in \mathcal{F}$ and then $G/Z_{\mathcal{F}}(G) \in \mathcal{F}$ as $\mathcal{F}$ is saturated: a contradiction. Furthermore, $G^{\mathcal{F}} \nleqslant M$ and $G = G^{\mathcal{F}}M$. Therefore, $M/M \cap G^{\mathcal{F}} \cong G/G^{\mathcal{F}} \in \mathcal{F}$ and $M^{\mathcal{F}} \leqslant M \cap G^{\mathcal{F}}$. In view of [**4**, § IV.6.10] we see that $[G^{\mathcal{F}}, Z_{\mathcal{F}}(G)] = 1$, and hence every minimal subgroup of $F^*(G^{\mathcal{F}})$ lies in $Z(G^{\mathcal{F}})$. Moreover, if $|F^*(G^{\mathcal{F}})|$ is even, every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is c-supplemented in $G^{\mathcal{F}}$. By Theorem 3.5, $G^{\mathcal{F}}$ is nilpotent, and hence $F^*(G^{\mathcal{F}}) = G^{\mathcal{F}} \leqslant F(G)$ and $G = F(G)M$. Now, every G-chief factor H/K below $Z_{\mathcal{F}}(G)$ is actually an M-chief factor and $\operatorname{Aut}_M(H/K)$ is isomorphic to $\operatorname{Aut}_G(H/K)$ as $F(G)$ centralizes H/K. Hence, $Z_{\mathcal{F}}(G) \leqslant Z_{\mathcal{F}}(M)$ and every minimal subgroup of $F^*(G^{\mathcal{F}} \cap M) = G^{\mathcal{F}} \cap M$ lies in $Z_{\mathcal{F}}(M)$. Moreover, every cyclic subgroup of $F^*(M^{\mathcal{F}}) = M^{\mathcal{F}}$ with order 4 is c-supplemented in M. By the minimality of G, $M \in \mathcal{F}$; consequently, $G^{\mathcal{F}}$ is a p-group for some prime p by Lemma 2.7 (ii). If $G^{\mathcal{F}}$ has exponent p, then $G^{\mathcal{F}} = \Omega_1(G^{\mathcal{F}}) \leqslant Z_{\mathcal{F}}(G)$ and so $G/Z_{\mathcal{F}}(G) \in \mathcal{F}$: a contradiction. Hence, $p = 2$ and $G^{\mathcal{F}}$ has exponent 4. At the same time, $\Omega_1(G^{\mathcal{F}}) \leqslant M$, otherwise $G = Z_{\mathcal{F}}(G)M$

and $G/Z_{\mathcal{F}}(G) \cong M/M \cap Z_{\mathcal{F}}(G) \in \mathcal{F}$, which is also a contradiction. Hence, there is an element x of $G^{\mathcal{F}}$ with order 4 such that $x \notin M$. By the hypotheses, there is a subgroup L of G such that $G = \langle x \rangle L$ and $\langle x \rangle \cap L \leqslant \mathrm{core}_G(\langle x \rangle)$. Now we claim that $\langle x \rangle \lhd G$. If this claim is false, then $L < G$. Let S be a maximal subgroup of G containing L. Since $x^2 \in \Phi(G^{\mathcal{F}}) \leqslant \Phi(G)$, $x^2 \in S$ and $|G : S| = 2$. Thus, $S \lhd G$, and G/S is a 2-group. Furthermore, $x \in G^{\mathcal{F}} \leqslant S$: a contradiction. Now we see that $G = \langle x \rangle M$ and $\langle x^2 \rangle \leqslant M$, so $|G : M| = 2$ and $M \lhd G$. We can infer from the fact that G/M is a 2-group that $x \in G^{\mathcal{F}} \leqslant M$, which is a final contradiction. This completes the proof. □

Using Theorems 3.6 and 3.7, we can localize Lemmas 2.5 and 2.6 as follows. They are 'iff' and 'localized' versions of Buckley's theorem in the formation universe.

Theorem 4.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group such that G is S_4-free. Then $G \in \mathcal{F}$ if and only if, for every prime divisor p of $|G_0|$ and some Sylow p-subgroup P of G_0, every cyclic subgroup of $P \cap (G_0)^{\mathcal{N}_p}$ with order p or 4 (if $p = 2$) is c-supplemented in $N_G(P)$, where $G_0 = F^*(G^{\mathcal{F}})$.*

Proof. Only the 'if' part needs to be verified.

Let p be the smallest prime divisor of $|G_0|$. Since every cyclic subgroup of $P \cap (G_0)^{\mathcal{N}_p}$ with order p or 4 (if $p = 2$) is c-supplemented in $N_{G_0}(P)$, G_0 is p-nilpotent. In particular, G_0 is solvable by the odd-order theorem. Now, applying Lemma 2.4, we see that $F^*(G^{\mathcal{F}}) = F(G^{\mathcal{F}})$. Therefore, $N_G(P) = G$ for any prime divisor p of $|G_0|$ and the result follows by Lemma 2.5. This completes the proof. □

Similarly we can prove the following.

Theorem 4.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group such that G is S_4-free and quaternion-free. Then $G \in \mathcal{F}$ if and only if, for every prime divisor p of $|G_0|$ and some Sylow p-subgroup P of G_0, every minimal subgroup of $P \cap (G_0)^{\mathcal{N}_p}$ is c-supplemented in $N_G(P)$, where $G_0 = F^*(G^{\mathcal{F}})$.*

Remark 4.5. The hypothesis that $(|G|, p-1) = 1$ is always satisfied when p is the smallest prime divisor of $|G|$. Hence, in this case the related theorems are always true. However, this hypothesis is necessary. For example, if we let $G = S_3$ be the symmetric group of degree 3 and $p = 3$, then G is not 3-nilpotent.

Remark 4.6. The hypothesis that G is S_4-free in the localized results is necessary. For instance, if we let $G = S_4$, and let G_2 be a Sylow 2-subgroup of G, then G_2 is a dihedral group of order 8 and $N_G(G_2) = G_2$. It is easy to see that every cyclic subgroup of $G_2 \cap G^{\mathcal{N}_2}$ with order 2 or 4 is c-supplemented in $N_G(G_2)$ and G_2 is quaternion-free, but G is not 2-nilpotent.

Remark 4.7. The theorems in § 4 are not true for non-saturated formations. For example, let $\mathcal{F}$ be the formation composed of all groups G such that $G^{\mathcal{U}}$ is an elementary abelian. Clearly, $\mathcal{U} \subseteq \mathcal{F}$, but $\mathcal{F}$ is not saturated. Set $G = SL(2,3)$, $H = Z(G)$. Then G/H is isomorphic to A_4 and so $G/H \in \mathcal{F}$. The other hypotheses in the theorems are satisfied since H is of order 2, but G does not belong to $\mathcal{F}$.

Remark 4.8. The quaternion-free hypothesis in the related results is necessary. For instance, if we take $G = GL(2,3)$, then we see that the elements

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate $GL(2,3)$, and the following relations hold:

$$a^8 = b^2 = c^3 = 1, \qquad b^{-1}ab = a^3, \qquad c^{-1}a^2c = ab, \qquad c^{-1}abc = aba^2, \qquad b^{-1}cb = c^2.$$

Also, we see that $G_2 = \langle a, b\rangle$ is a Sylow 2-subgroup of $GL(2,3)$ and a semi-dihedral group of order 16. Furthermore, $G^{\mathcal{N}_2} = G'' = \langle a^2, ab\rangle$ is a quaternion group of order 8. It is easy to see that every minimal subgroup of $G_2 \cap G^{\mathcal{N}_2}$ is contained in $Z(G) = \langle a^4\rangle$, but G itself is not 2-nilpotent.

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References

1. M. Asaad, A. Ballester-Bolinches and M. C. Pedraza-Aguilera, A note on minimal subgroups of finite groups, *Commun. Alg.* **24** (1996), 2771–2776.
2. A. Ballester-Bolinches, Y. Wang and X. Guo, c-supplemented subgroups of finite groups, *Glasgow Math. J.* **42** (2000), 383–389.
3. J. Buckley, Finite groups whose minimal subgroups are normal, *Math. Z.* **16** (1970), 15–17.
4. H. Doerk and T. Hawkes, *Finite solvable groups* (Walter de Gruyter, Berlin, 1992).
5. L. Dornhoff, M-groups and 2-groups, *Math. Z.* **100** (1967), 226–256.
6. D. Gorenstein, *Finite groups* (Chelsea, New York, 1980).
7. P. Hall, A characteristic property of soluble groups, *J. Lond. Math. Soc.* **12** (1937), 198–200.
8. P. Hall, Complemented groups, *J. Lond. Math. Soc.* **12** (1937), 201–204.
9. B. Huppert, *Endliche Gruppen, I* (Springer, 1967).
10. B. Huppert and N. Blackburn, *Finite groups, III* (Springer, 1982).
11. S. Li, On minimal subgroups of finite groups, III, *Commun. Alg.* **26** (1998), 2453–2461.
12. Y. Li, Y. Wang and H. Wei, The influence of π-quasinormality of some subgroups of a finite group, *Arch. Math.* **81** (2003), 245–252.
13. O. Ore, Contributions to the theory of groups of finite order, *Duke Math. J.* **5** (1939), 431–460.
14. D. J. S. Robinson, *A course in the theory of groups* (Springer, 1980).
15. J. S. Rose, On finite group insoluble groups with nilpotent maximal subgroups, *J. Alg.* **48** (1977), 182–196.
16. Y. Wang, c-normality of groups and its properties, *J. Alg.* **180** (1996), 954–965.
17. Y. Wang, The influence of minimal subgroups on the structure of finite groups, *Acta Math. Sinica* **16** (2000), 63–70.
18. Y. Wang, Finite groups with some subgroups of Sylow subgroups c-supplemented, *J. Alg.* **224** (2000), 467–478.

19. Y. Wang and Y. Li, On some sufficient conditions of supersolvability of finite groups, *Publ. Math. Debrecen* **64** (2004), 269–284.
20. Y. Wang, H. Wei and Y. Li, A generalization of Kramer's theorem and its applications, *Bull. Austral. Math. Soc.* **65** (2002), 467–475.
21. H. Wei, Y. Wang and Y. Li, On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups, II, *Commun. Alg.* **31** (2003), 4807–4816.
22. H. Wei, Y. Wang and Y. Li, On c-supplemented maximal and minimal subgroups of Sylow subgroups of finite groups, *Proc. Am. Math. Soc.* **132** (2004), 2197–2204.