ON PARTIALLY ORDERING OPERATOR ALGEBRAS

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1. Introduction. In this paper, we consider linear spaces and algebras with real scalars. It is well known that if X is a Banach space and $\mathfrak{B}$ is the set of all bounded linear operators which map X into itself, then $\mathfrak{B}$ is a Banach algebra. In this paper we shall show that $\mathfrak{B}$ can be partially ordered so that it becomes a partially ordered algebra in which norm convergence is equivalent to order convergence. This motivates a study of Banach algebras of operators in which one uses the order structure to obtain various results. In addition, it encourages a study of partially ordered algebras in general, since our result shows that among such algebras one finds all real Banach algebras of operators. Of course, there are many other real algebras which are naturally partially ordered and which have been studied from that point of view. In the final section we obtain various results to illustrate the value of using the order structure of a partially ordered algebra. For a general discussion of partially ordered linear spaces and algebras the reader may refer to **(1, 4–9)**.

2. Basic definitions. The definition of a partially ordered linear space (p.o.l.s.) which we use here is that given by Nakano; see **(7**, p. 23**)** and note that Nakano uses the term "semi-ordered linear space" instead of "p.o.l.s." We also use the definition of a partially ordered algebra (p.o.a.) given by Nakano; see **(7**, p. 112**)** and note that Nakano uses the term "semi-ordered ring" instead of "p.o.a." Real numbers will always be denoted by small Greek letters.

If X is a p.o.l.s., then the subset $X^+ = \{x: x \geqslant 0\}$ is called the positive cone in X. An element $u \in X^+$ is called an order unit if for every $x \in X$ there exists a real number α such that $-\alpha u \leqslant x \leqslant \alpha u$. A linear operator $A: X \to X$ is said to be positive if $A(X^+) \subset X^+$. A linear operator $B: X \to X$ is said to be regular if $B = A_1 - A_2$, where A_1 and A_2 are positive. Thus, if $\mathfrak{A}$ denotes the set of all regular linear operators which map X into itself, then $\mathfrak{A}$ is a p.o.a. We use I to denote the identity operator. It is clear that I is positive.

Definition 1. A sequence $\{x_n\}$ of elements from a p.o.l.s. X is said to be directed to 0 if $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$ and if $\inf\{x_n\} = 0$.

Definition 2. A sequence $\{y_n\}$ of elements from a p.o.l.s. X is said to order-converge (o-converge) to 0 if there exists a sequence $\{x_n\}$ which is directed

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to 0 such that $-x_n \leqslant y_n \leqslant x_n$ for all n. In this case we write o-lim $y_n = 0$; more generally, we write o-lim $y_n = y$ if and only if o-lim$(y_n - y) = 0$.

Definition 3. A p.o.l.s. X is said to be Dedekind σ-complete if for every sequence $\{x_n\}$, where $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$, $\inf\{x_n\}$ exists (**6**, pp. 9–11).

3. The main theorem. We begin with the following lemmas.

LEMMA 1. *Let X be a p.o.l.s. If $\{x_n\}$ and $\{y_n\}$ are sequences of elements from X and* o-lim $x_n = x$ *and* o-lim $y_n = y$, *then* o-lim$(x_n + y_n) = x + y$. *If α is any real number, then* o-lim $\alpha x_n = \alpha x$.

Proof. See (**7**, p. 36).

LEMMA 2. *Let X be a Dedekind σ-complete p.o.l.s. If $\mathfrak{A}$ is the p.o.a. of all regular linear operators which map X into itself, then $\mathfrak{A}$ is a Dedekind σ-complete p.o.a.*

Proof. If $A_1, A_2 \in \mathfrak{A}$, then $A_1 \geqslant A_2$ is defined to mean that $A_1 - A_2$ is a positive linear operator. Now let us take a sequence $\{A_n\}$ of operators from $\mathfrak{A}$ such that $A_1 \geqslant A_2 \geqslant \ldots \geqslant 0$, the latter symbol 0 denoting the zero operator. For each $x \in X^+$ we have $A_1(x) \geqslant A_2(x) \geqslant \ldots \geqslant 0 \in X$; hence, $F(x) = \inf\{A_n(x)\}$ exists for each $x \in X^+$. From Lemma 1 we have $F(\alpha x) = \alpha F(x)$ and $F(x + y) = F(x) + F(y)$ for all $x, y \in X^+$ and all $\alpha \geqslant 0$. Since $X = X^+ - X^+$, we may define a positive linear operator A such that $A(x) = F(x)$ for all $x \in X^+$. It is easy to verify that $A = \inf\{A_n\}$. Therefore, $\mathfrak{A}$ is Dedekind σ-complete.

From this result it follows that if $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ and $\inf\{\alpha_n\} = 0$ and if $A \in \mathfrak{A}$ and $A \geqslant 0$, then $\inf\{\alpha_n A\} = 0 \in \mathfrak{A}$.

THEOREM. *Let X be a Banach space and let $\mathfrak{B}$ denote the Banach algebra of all norm-bounded linear operators which map X into itself. It is possible to partially order $\mathfrak{B}$ so that it becomes a Dedekind σ-complete p.o.a. in which norm convergence and o-convergence are equivalent.*

Proof. We begin by partially ordering X as in (**2**, Theorem **1**). This is done by selecting a fixed element $u \in X$ such that $||u|| = 4$. Define

$$K = \{\lambda(u + x) : \lambda \geqslant 0 \quad \text{and} \quad ||x|| \leqslant 1\}$$

and then define $x \leqslant y$ to mean that $y - x \in K$. Clearly $K = X^+$ and u is an order unit. As shown in (**2**, Theorem 1), norm convergence and o-convergence in X are equivalent. Furthermore, X is Dedekind σ-complete.

Now let $\mathfrak{A}$ denote the p.o.a. of all regular linear operators which map X into itself. If $A \in \mathfrak{A}$ and $A \geqslant 0$, then for all $y \in X$ such that $||y|| \leqslant 1$ we have $u + y \geqslant 0$ and $u - y \geqslant 0$ so that

$$\begin{aligned} 2\,||A(u)|| = ||A(u + y) + A(u - y)|| &\geqslant ||A(u) - A(y)|| \\ &\geqslant ||A(y)|| - ||A(u)||. \end{aligned}$$

The first inequality in the latter expression comes from the fact that the norm is monotone on X^+. Therefore, $3||A(u)|| \geqslant ||A(y)||$ for all $y \in X$ such that $||y|| \leqslant 1$; hence, $A \in \mathfrak{B}$ and $||A|| \leqslant 3||A(u)||$. Since every operator in $\mathfrak{A}$ is the difference of two positive operators, we see that $\mathfrak{A} \subset \mathfrak{B}$.

We shall now show that $\mathfrak{B} \subset \mathfrak{A}$. By the Hahn–Banach theorem there exists a linear functional f defined on X such that $f(u) = 20/3$ and $||f|| = 5/3$. Now if $z \in X^+$, then $z = \lambda(u + x)$, where $\lambda \geqslant 0$ and $||x|| \leqslant 1$. Hence,

$$f(z) = \lambda[f(u) + f(x)] \geqslant 5\lambda \geqslant ||z|| \qquad \text{for all } z \in X^+.$$

Now define $U(x) = f(x)u$ for all $x \in X$ It is clear that $U \in \mathfrak{A}$ and that $U \geqslant 0$. Now take $A \in \mathfrak{B}$ and $||A|| \leqslant 1$ and $z \in X^+$ ($z \neq 0$); hence

$$U(z) \pm A(z) = f(z)[u \pm f(z)^{-1} A(z)]$$

and, since $||\pm f(z)^{-1}A(z)|| \leqslant ||z||^{-1} ||A(z)|| \leqslant 1$, we have that $U(z) \pm A(z) \in X^+$. Hence, $U \pm A \geqslant 0$ and since $2A = (U + A) - (U - A)$, we see that $A \in \mathfrak{A}$. Therefore, $\mathfrak{B} \subset \mathfrak{A}$.

Since $\mathfrak{A} = \mathfrak{B}$, we may regard $\mathfrak{B}$ as the p.o.a. of regular operators which map X into itself. By Lemma 2, $\mathfrak{B}$ is Dedekind σ-complete. Furthermore, U is an order unit in $\mathfrak{B}$. The calculations given above show that if $A \in \mathfrak{B}$, then $-||A||U \leqslant A \leqslant ||A||U$. From this and Lemma 2 it follows that in $\mathfrak{B}$ norm convergence implies o-convergence.

Let us now take $A, B \in \mathfrak{B}$ such that $-A \leqslant B \leqslant A$. Since $2A \geqslant A + B \geqslant 0$ and $2A \geqslant A - B \geqslant 0$, we have from above that

$$||A + B|| \leqslant 3||A(u) + B(u)|| \leqslant 6||A(u)||$$

and

$$||A - B|| \leqslant 3||A(u) - B(u)|| \leqslant 6||A(u)||,$$

where the second inequality in each expression comes from the fact that the norm is monotone on X^+. Hence,

$$2||B|| = ||A + B + B - A|| \leqslant ||A + B|| + ||A - B|| \leqslant 12||A(u)||$$

or

$$||B|| \leqslant 6||A(u)||.$$

Now let $\{B_n\}$ be a sequence of elements from $\mathfrak{B}$ which o-converges to $0 \in \mathfrak{B}$. There must be a sequence $\{A_n\}$ of elements from $\mathfrak{B}$ which is directed to 0 such that $-A_n \leqslant B_n \leqslant A_n$. Hence, $||B_n|| \leqslant 6||A_n(u)||$. Since

$$\inf\{A_n\} = 0 \in \mathfrak{B},$$

we must have $\inf\{A_n(u)\} = 0 \in X$. But in X o-convergence is equivalent to norm convergence; hence, $\lim ||A_n(u)|| = 0$ and, therefore, $\lim ||B_n|| = 0$. Thus, in $\mathfrak{B}$ o-convergence implies norm convergence.

We have shown that in $\mathfrak{B}$ norm convergence and o-convergence are equivalent. This completes the proof of the theorem.

Remark. As a consequence of the fact that in $\mathfrak{B}$ norm convergence and o-convergence are equivalent, we can assert that if $\{A_n\}$ is a sequence of elements from $\mathfrak{B}$ which is directed to $0 \in \mathfrak{B}$ and $B \in \mathfrak{B}$, $B \geqslant 0$, then both of the sequences $\{A_n B\}$ and $\{BA_n\}$ are directed to 0. However, this might not be true if we choose a different way to partially order $\mathfrak{B}$, as the following example shows.

Let X be the Banach space of all bounded sequences of real numbers with supremum norm. Assume that X is partially ordered componentwise. Thus, X^+ is the set of all sequences with non-negative components and $u = (1, 1, \ldots)$ is an order unit. Furthermore, X is Dedekind σ-complete. It is well known that $\mathfrak{B}$ is the same as the p.o.a. of regular linear operators.

There is a positive linear functional f defined on X such that if $x = (\alpha_1, \alpha_2, \ldots)$ and $\lim \alpha_n = \alpha$, then $f(x) = \alpha$. Define $B(x) = f(x)u$. Define

$$A_n(x) = (0, \ldots, 0, \alpha_{n+1}, \alpha_{n+2}, \ldots), \quad \text{where} \quad x = (\alpha_1, \alpha_2, \ldots).$$

It is clear that $A_1 \geqslant A_2 \geqslant \ldots \geqslant 0$ and $\inf\{A_n\} = 0$. However, it is easily seen that $BA_n = B$ for all n; hence, $\inf\{BA_n\} = B \neq 0$. Nevertheless, it is true that $\inf\{A_n B\} = 0$; in fact, the following is true.

PROPOSITION 1. *Let X be a p.o.l.s. and let $\mathfrak{A}$ be the p.o.a. of regular linear operators which map X into itself. If $\{A_n\}$ is a sequence of elements from $\mathfrak{A}$ which is directed to* 0 *and $B \in \mathfrak{A}$, $B \geqslant 0$, then the sequence $\{A_n B\}$ is directed to* 0.

Proof. We must have $\inf\{A_n(y)\} = 0 \in X$ for all $y \in X^+$. For each $x \in X^+$ we have $B(x) \in X^+$; hence, $\inf\{A_n B(x)\} = 0 \in X$. This means that $\inf\{A_n B\} = 0 \in \mathfrak{A}$.

COROLLARY. *If $\{E_n\}$ is a sequence of elements from $\mathfrak{A}$ such that* o-lim $E_n = E$ *and if $F \in \mathfrak{A}$, then* o-lim $E_n F = EF$.

4. Additional results. The preceding corollary states that in some p.o. algebras multiplication is "left-continuous" with respect to o-convergence. Of course, in other cases multiplication may not be continuous at all with respect to o-convergence. This lack of continuity may make certain desirable results untrue or impossible to prove. For example, suppose that $\mathfrak{A}$ is a p.o.a. and $A \in \mathfrak{A}$ is such that

$$\text{o-lim} \sum_{k=1}^{n} A^k = E.$$

Is it true that $(I + E)(I - A) = I$, where I denotes the multiplicative identity? I do not know how to answer this question without using the fact that multiplication is left-continuous with respect to o-convergence. Nevertheless, it is possible to obtain some results on the existence of inverses, etc., without using the fact that multiplication is continuous in any sense. This we shall now do.

Assumption. In what follows $\mathfrak{A}$ will denote a p.o.a. which is Dedekind σ-complete. We use I to denote the multiplicative identity and assume that $I \geqslant 0$.

PROPOSITION 2. *If $A \in \mathfrak{A}$, $A \geqslant 0$, and the sequence*

$$\left\{\sum_{k=1}^{n} A^k\right\}$$

is bounded above, then $I - A$ has an inverse and $(I - A)^{-1} \geqslant I$.

Proof. Define

$$B_n = I + \sum_{k=1}^{n} A^k \quad \text{and} \quad B = \sup\{B_n\}.$$

We note that $\inf\{B - B_n\} = 0$. Now

$$\begin{aligned} B(I - A) &= (B - B_n)(I - A) + B_n(I - A) \\ &\leqslant B - B_n + I - A^{n+1} \leqslant I + B - B_n. \end{aligned}$$

Therefore, $B(I - A) \leqslant I$ or $B \leqslant I + BA$. Since $A \geqslant 0$, we can show by induction that $B \leqslant B_n + BA^{n+1}$. If we put $H_n = B_n + BA^{n+1}$, then it is easily verified that $B \leqslant H_1 \leqslant H_2 \leqslant \ldots \leqslant B + B^2$. Hence, we can define $H = sup\{H_n\}$. Now let us define

$$D_n = \sum_{k=1}^{n} BA^k \leqslant B^2,$$

so that $D = sup\{D_n\}$ can be defined. Note that $\inf\{D - D_n\} = 0$ and that $D - D_n \geqslant BA^{n+1} \geqslant 0$. Therefore,

$$H_1 \leqslant H_2 \leqslant \ldots \leqslant B + D - D_2 \leqslant B + D - D_1,$$

which means that $H_n \leqslant B + D - D_m$ for all m, n. Therefore,

$$H = \sup\{H_n\} \leqslant \inf\{B + D - D_m\} = B.$$

From above we have $B \leqslant H$; hence, $B = H$. Also from above we have

$$B \leqslant I + BA \leqslant H = B.$$

Therefore, $B = I + BA$ or $B(I - A) = I$.

Similar arguments show that $(I - A)B = I$. Thus, $B = (I - A)^{-1}$. Clearly, $B \geqslant I \geqslant 0$.

PROPOSITION 3. *If $E, F \in \mathfrak{A}$, $E \leqslant I$, $F \geqslant 0$, and $EF \geqslant I$ (or $FE \geqslant I$), then E has an inverse and $E^{-1} \geqslant I$.*

Proof. Define $A = I - E \geqslant 0$. Thus, $(I - A)F \geqslant I$ or $F \geqslant I + AF$. By induction it can be shown that

$$F \geqslant I + \sum_{k=1}^{n} A^k.$$

Hence, by Proposition 2 we see that $(I - A)^{-1}$ exists and $(I - A)^{-1} \geqslant I$. Since $E = I - A$, the proposition is proved.

COROLLARY. *If* $0 < \alpha \leqslant \beta$ *and* $\alpha I \leqslant H \leqslant \beta I$, *then* H *has an inverse and* $H^{-1} \geqslant \beta^{-1} I$.

Proof. Use Proposition 3 with $E = \beta^{-1} H$ and $F = \beta\alpha^{-1} I$.

PROPOSITION 4. *If* $A, E \in \mathfrak{A}$, *where* $-A \leqslant E \leqslant A$, *and if the sequence*

$$\left\{\sum_{k=1}^{n} A^k\right\}$$

is bounded above, then $I - E$ *has an inverse.*

Proof. It is easy to show that $-A^k \leqslant E^k \leqslant A^k$ for all k. Define

$$B_n = I + \sum_{k=1}^{n} A^k \quad \text{and} \quad B = \sup\{B_n\}.$$

Define

$$F_n = I + \sum_{k=1}^{n} E^k.$$

Clearly $-B_n \leqslant F_n \leqslant B_n$. Now put $H_n = B_n + F_n$; it is easily verified that $0 \leqslant H_1 \leqslant H_2 \leqslant \ldots \leqslant 2B$. We may therefore define $H = \sup\{H_n\}$ and then put $F = H - B$. Now

$$\begin{aligned} H - H_n &= \sup\{H_m - H_n : m \geqslant n + 1\} \\ &= \sup\left\{\sum_{k=n+1}^{m} (A^k + E^k) : m \geqslant n + 1\right\} \\ &\leqslant \sup\left\{\sum_{k=n+1}^{m} 2A^k : m \geqslant n + 1\right\} \\ &= \sup\{2(B_m - B_n) : m \geqslant n + 1\} \\ &= 2(B - B_n). \end{aligned}$$

Therefore by writing

$$F - F_n = (H - B) - (H_n - B_n) = (H - H_n) - (B - B_n),$$

we see that

$$-2(B - B_n) \leqslant F - F_n \leqslant 2(B - B_n).$$

We now note that

$$\begin{aligned} F(I - E) &= (F - F_n)(I - E) + F_n(I - E) \\ &= F - F_n - (F - F_n)E + I - E^{n+1} \\ &\leqslant 2(B - B_n) + 2(B - B_n)A + I + A^{n+1} \\ &\leqslant I + 3(B - B_n) + 2(B - B_{n+1}) \leqslant I + 5(B - B_n). \end{aligned}$$

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In the next to last inequality we use the fact that $A^{n+1} \leqslant B - B_n$,

$$I + B_n A = B_{n+1},$$

and $BA = B - I$. Since $\inf\{5(B - B_n)\} = 0$, we have $F(I - E) \leqslant I$. We may follow through inequalities similar to those given above to show that $F(I - E) \geqslant I - 5(B - B_n)$; hence, $F(I - E) \geqslant I$. Therefore,

$$F(I - E) = I.$$

Finally, we may repeat the arguments of the preceding paragraph to show that $(I - E)F = I$. Hence, $F = (I - E)^{-1}$.

PROPOSITION 5. *If A, $U \in \mathfrak{A}$ and $-U \leqslant A^n \leqslant U$ for all n and $|\alpha| < 1$, then $I - \alpha A$ has an inverse.*

Proof. The proof here is similar to that given for Proposition 4 and will therefore be omitted.

Remark. By looking at the main theorem of this paper, the reader will note that Proposition 5 generalizes the well-known fact that if B is a bounded linear operator mapping a Banach space into itself and $||B|| < 1$, then $I - B$ has an inverse.

Finally, we obtain a result on the existence of "square roots."

PROPOSITION 6. *If $E \in \mathfrak{A}$ and $0 \leqslant E \leqslant I$, then there exists a unique element $A \in \mathfrak{A}$ such that $0 \leqslant A \leqslant I$ and $A^2 = E$.*

Proof. Define $A_1 = I$ and then $A_{n+1} = A_n - 1/2(A_n{}^2 - E)$ for all n. We note that $A_n E = EA_n$ for all n. Assume that for some k we have

$$E \leqslant A_k{}^2 \leqslant A_k \leqslant I;$$

this is at least true for $k = 1$. Thus $A_{k+1} \geqslant A_k - A_k{}^2 + E \geqslant E$. It can be shown that

$$A_{k+1}{}^2 = E + 1/4(A_k{}^2 - E)[(2I - A_k)^2 - E] \geqslant E.$$

Since $A_{k+1} \leqslant A_k \leqslant I$, we see that $E \leqslant A_{k+1}{}^2 \leqslant A_{k+1} \leqslant I$. Thus, we have shown by induction that $E \leqslant A_n{}^2 \leqslant A_n \leqslant I$ for all n and, hence,

$$E \leqslant \ldots \leqslant A_2 \leqslant A_1.$$

Therefore, we can define $A = \inf\{A_n\}$. Now

$$A^2 - E = A(A - A_n) + (A - A_n)A_n + A_n{}^2 - E.$$

Therefore, since $2(A_n - A) \geqslant A_n{}^2 - E \geqslant 0$, we have

$$-2(A_n - A) \leqslant A^2 - E \leqslant 2(A_n - A).$$

Since $\inf\{A_n - A\} = 0$, we see that $A^2 = E$.

To show that A is unique we proceed as follows. Assume that $B \in \mathfrak{A}$ and $0 \leqslant B \leqslant I$ and $B^2 = E$. Therefore, if $B \leqslant A_k$, then

$$2A_{k+1} = 2A_k + B^2 - A_k^2 = 2A_k + B(B - A_k) + (B - A_k)A_k$$
$$\geqslant A2_k + 2(B - A_k) = 2B;$$

hence, $A_{k+1} \geqslant B$. Since $A_1 = I \geqslant B$, we have $B \leqslant A_n$ for all n. Hence, $B \leqslant A$.

Now

$$0 = A^2 - B^2 = A(A - B) + (A - B)B \geqslant A(A - B) \geqslant 0,$$

so that $A(A - B) = 0$. Similarly, we can show that $(A - B)A = 0$ and $(A - B)^2 = 0$. If we put $F = A + (A - B)$, then $F^2 = A^2 = E$ and

$$0 \leqslant F = 2A - B \leqslant 2A - A^2 = I - (I - A)^2 \leqslant I.$$

By the arguments of the preceding paragraph, we must have $F \leqslant A$. Hence, $A - B \leqslant 0$. Therefore, we have $A = B$.

Remark. The reader should note that not every positive element of $\mathfrak{A}$ has a square root. Also note that even I may have more than one positive square root. Examples may be obtained by taking $\mathfrak{A}$ to be the p.o.a. of 2×2 matrices with real entries. The positive elements of $\mathfrak{A}$ are those matrices with only non-negative real entries.

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