REPRESENTATION OF m AS $\sum_{k=-n}^{n} \varepsilon_k k$

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J.H. van Lint has recently shown [1] that if $A(n, m)$ denotes the number of representations of m in the form $\sum_{k=-n}^{n} \varepsilon_k k$, where $\varepsilon_k = 0$ or 1 for $-n \le k \le n$ then

$$(1) \qquad A(n, 0) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2}.$$

Using this result, the fact that $A(n, m)$ is a non-increasing function of $|m|$, and a simple recurrence relation for $A(n, m)$ we derive the following extension of (1):

$$(2) \qquad A(n, [0(n)]) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2}$$

where $[0(n)]$ is any integral valued function $m(n) = 0(n)$.

We note that $A(0, 0) = 2$ and define $A(0, m) = 0$ for all $m \neq 0$. Now if $n \ge 1$ the number of representations $m = \sum_{k=-n}^{n} \varepsilon_k k$ is $2A(n-1, m)$ when $\varepsilon_{-n} = \varepsilon_n$; $A(n-1, m+n)$ when $\varepsilon_{-n} = 1$, $\varepsilon_n = 0$; and $A(n-1, m-n)$ when $\varepsilon_{-n} = 0$, $\varepsilon_n = 1$. Thus

$$(3) \qquad A(n, m) = 2A(n-1, m) + A(n-1, m+n) + A(n-1, m-n) \quad \text{for } n \ge 1 \text{ and all } m.$$

It is a trivial consequence of the definition that $A(n, -m) = A(n, m)$ for $n \ge 0$ and all m so that in the proof of the following lemma we need consider only non-negative m.

LEMMA. $A(n, m)$ <u>is a non-increasing function of</u> $|m|$.

<u>Proof</u>. Mr. Gary Bunce has verified the assertion by computer for $n \le 44$ (a table of values of $A(n, 0)$, $n \le 44$ is appended). If we assume the assertion holds for $0, \ldots, n-1$ then since $A(n, m+1) - A(n, m) = 2\{A(n-1, m+1) - A(n-1, m)\} + \{A(n-1, m+1+n) - A(n-1, m+n)\} + \{A(n-1, m+1-n) - A(n-1, m-n)\} \le 0$ for $m \ge n$ it suffices to show $A(n, m)$ is a non-increasing function of m for $0 \le m \le n$, $n \ge 44$.

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As in [1] we note that $A(n,m)$ is the coefficient of x^m in $\prod_{k=-n}^{n}(1+x^k)$ and hence if C is the unit circle

$$A(n,m) = \frac{1}{2\pi i}\int_C \prod_{k=-n}^{n}(1+z^k)\frac{dz}{z^{m+1}} = \frac{2^{2n+2}}{\pi}\int_0^{\pi/2}\cos 2mx \prod_{k=1}^{n}\cos^2 kx\, dx.$$

Throughout the remainder of the proof we extend $A(n,m)$ to all real values of m by means of this equation and thus have

$$\frac{dA(n,m)}{dm} = -\frac{2^{2n+3}}{\pi}\left\{\int_0^{\pi/2n} x\sin 2mx \prod_{k=1}^{n}\cos^2 kx\, dx + \int_{\pi/2n}^{\pi/2} x\sin 2mx \prod_{k=1}^{n}\cos^2 kx\, dx\right\}$$

$$= -\frac{2^{2n+3}}{\pi}(I_1 + I_2).$$

Since $\frac{dA(n,m)}{dm} \le 0$ for $0 \le m \le 1$ it suffices to show $I_1 \ge |I_2|$ for $1 \le m \le n$, $n \ge 44$.

It is easily shown that for $0 \le x \le 1/3$ we have $\tan x \le \frac{12}{11}x$ and hence, after inspecting the derivative of the following function, that $e^{(12/11)x^2}\cos^2 x \ge 1$. Thus

$$I_1 \ge \int_0^{1/3n} x\sin 2mx \exp\left(-\sum_{k=1}^{n}\frac{12}{11}k^2x^2\right)dx$$

$$\ge m e^{-\frac{2}{99}(2n+3+\frac{1}{n})}\int_0^{1/3n}\left(2x^2 - \frac{4m^2x^4}{3}\right)dx$$

$$\ge \frac{1}{46n^3}e^{-(4n/99)}$$

for $1 \le m \le n$, $n \ge 44$.

From inequality (4) of [1] we have

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$$|I_2| \le \int_{\pi/2n}^{\pi/n} \frac{\pi}{n} e^{-\frac{n}{2}} dx + \int_{\pi/n}^{3\pi/2n} \frac{3\pi}{2n} e^{-\frac{n}{2} + \frac{1}{4 \sin x}} dx$$

$$+ \int_{3\pi/2n}^{\pi/2} x|\sin 2mx| e^{-\frac{n}{2} + \frac{1}{2 \sin x}} dx ,$$

and since

$$0 \le \int_{3\pi/2n}^{\pi/2} x|\sin 2mx| dx \le \int_{0}^{\pi/2} x|\sin 2mx| dx$$

$$= \sum_{k=0}^{[\frac{m-1}{2}]} \int_{2k\pi/2m}^{(2k+1)\pi/2m} x \sin 2mx\, dx - \sum_{k=0}^{[\frac{m-2}{2}]} \int_{(2k+1)\pi/2m}^{(2k+2)\pi/2m} x \sin 2mx\, dx$$

$$= \sum_{k=0}^{[\frac{m-1}{2}]} \frac{(4k+1)\pi}{4m^2} + \sum_{k=0}^{[\frac{m-2}{2}]} \frac{(4k+3)\pi}{4m^2} = \frac{\pi}{4}$$

we have

$$|I_2| \le \frac{\pi^2}{2n^2} \exp(-\frac{n}{2}) + \frac{3\pi^2}{4n^2} \exp(-\frac{n}{2} + \frac{1}{4(\frac{\pi}{n} - \frac{\pi^3}{6n^3})}) + \frac{\pi}{4} \exp(-\frac{n}{2} + \frac{1}{2(\frac{3\pi}{2n} - \frac{9\pi^3}{16n^3})})$$

$$\le \frac{\pi^2}{2n^2} e^{-\frac{n}{2}} + \frac{3\pi^2}{4n^2} e^{-\frac{21n}{50}} + \frac{\pi}{4} e^{-\frac{7n}{18}} \le 0.8 e^{-\frac{7n}{18}}$$

for $1 \le m \le n$, $n \ge 44$. Hence $\log \frac{I_1}{|I_2|} \ge \frac{23}{66} n - 3 \log n - 3.61$ which is positive increasing for $n \ge 44$.

In view of the lemma, (2) will be established if we prove the following

THEOREM. For fixed non-negative integral r we have

$$A(n, rn) \sim (\frac{3}{\pi})^{1/2} 2^{2n+1} n^{-3/2} .$$

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Proof. The case $r = 0$ is van Lint's result. For $r = 1$, from (3) and the lemma we have

$$\left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} \sim A(n, 0) \geq A(n, n+1) = \frac{1}{2} A(n+1, 0) - A(n, 0) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} .$$

Assume the theorem holds for $0, 1, \ldots, r-1$, $r > 1$. Then from (3) and the lemma we have

$$\left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} \sim A(n, 0) \geq A(n, rn) =$$

$$A(n+1, rn-n-1) - 2A(n, rn-n-1) - A(n, rn-2n-2) \geq$$

$$A(n+1, (r-1)(n+1)) - 3A(n, 0) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} .$$

This completes the proof of the theorem

n	A(n, 0)	n	A(n, 0)
0	2	23	119 59017 50512
1	4	24	449 54482 17544
2	8	25	1694 04112 01280
3	20	26	6398 32332 68592
4	52	27	24217 35046 98128
5	152	28	91841 71814 43568
6	472	29	3 48937 57977 33080
7	1520	30	13 27997 12712 51072
8	5044	31	50 62214 33283 74912
9	17112	32	193 25677 61386 20652
10	59008	33	738 82308 42287 89704
11	2 06260	34	2828 27657 54086 98552
12	7 29096	35	10840 42279 93495 01944
13	26 01640	36	41599 05262 08542 82392
14	93 58944	37	1 59810 72165 23633 28040
15	339 04324	38	6 14593 82190 54464 43632
16	1235 80884	39	23 65956 60978 26858 03440
17	4529 02072	40	91 16747 52821 27845 78024
18	16678 37680	41	351 61507 93945 39408 08880
19	61685 10256	42	1357 28405 65572 40093 97408
20	2 29032 60088	43	5243 63286 58618 90105 62588
21	8 53384 50344	44	20273 83210 00799 83213 73176
22	31 89952 97200		

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1. J.H. van Lint, Representation of 0 as $\sum_{k=-N}^{N} \varepsilon_k k$. Proc. Amer. Math. Soc. 18 (1967) 182-184.

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