AN IMPROVED ERROR BOUND FOR THE COMPOUND POISSON APPROXIMATION OF A NEARLY HOMOGENEOUS PORTFOLIO

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ABSTRACT

For the case of a portfolio with identical claim amount distributions, Gerber's error bound for the compound Poisson approximation is improved (in the case $\lambda \geqslant 1$). The result can also be applied to more general portfolios by partitioning them into homogeneous subportfolios.

KEYWORDS

Compound Poisson distribution; homogeneous portfolio.

Consider n distributions of the form

$$P_i = (1 - p_i)\delta_0 + p_i Q_i, \qquad i = 1, \ldots, n, \tag{1}$$

where δ_0 is the probability measure concentrated in 0, $p_i \in (0, 1)$, and where Q_i is a distribution on $(0, \infty)$. The interpretation of this representation is the following. For each probability measure P on $[0, \infty)$ (i.e. for the distribution P of a risk X) with $P\{0\} < 1$ (this means that we are dealing with a "risk") and an arbitrary event A, one can write

$$\begin{aligned} P(A) &= P(A \cap \{0\}) + P(A \cap (0, \infty)) \\ &= P\{0\}\delta_0(A) + (1 - P\{0\})P(A \mid (0, \infty)) \\ &= (1 - p)\delta_0(A) + pQ(A), \end{aligned}$$

with $p = 1 - P\{0\}$ and $Q(A) = P(A \mid (0, \infty))$ the conditional distribution of A, given the event $(0, \infty)$ (i.e. given a positive claim). In the special case where P admits the representation

$$P = \sum_{k=0}^{\infty} q_k D^{*k} \tag{2}$$

(with a distribution D on $(0, \infty)$, $D^{*0} = \delta_0$, and $q_k \geqslant 0$, $k = 0, 1, 2, \ldots$, such that $q_0 < 1$ and $\sum_{k=0}^{\infty} q_k = 1$) we have $p = 1 - q_0$ and

$$Q = \sum_{k=1}^{\infty} \frac{q_k}{1 - q_0} D^{*k}.$$

Now one of the basic results in mathematical risk theory is the fact that the convolution

$$G = P_1 * \ldots * P_n$$

of the distributions (1) can be approximated by a compound Poisson distribution, i.e. by a distribution P of the form (2) with

$$q_k = \frac{\lambda^k}{k!}\, e^{-\lambda}, \qquad \left(k = 0, 1, 2, \ldots, \lambda = \sum_{i=1}^{n} p_i\right)$$

and claim amount distribution

$$D = \sum_{i=1}^{n} p_i Q_i \Big/ \sum_{i=1}^{n} p_i.$$

Concerning bounds for the error of this approximation we have (see e.g. Gerber (1984), theorem 1a, p. 192)

$$d(G, P) \leqslant \sum_{i=1}^{n} p_i^2,$$

where

$$d(G, P) = \sup_A |G(A) - P(A)|$$

is the maximal difference of the probabilities for events A.

It is the purpose of this paper to improve (for $\lambda \geqslant 1$) the above bound for portfolios which are homogeneous in as much as the Q_i in the representation (1) are all identical to Q, say. As result we have the following.

THEOREM. *For the distribution*

$$G = [(1 - p_1)\delta_0 + p_1 Q] * \cdots * [(1 - p_n)\delta_0 + p_n Q]$$

and the compound Poisson distribution P with intensity $\lambda = \sum_{i=1}^{n} p_i$ and claim amount distribution $D = Q$, we have

$$d(G, P) \leqslant \sum_{i=1}^{n} p_i^2 \Big/ \sum_{i=1}^{n} p_i. \tag{3}$$

REMARK 1. Obviously,

$$\sum_{i=1}^{n} p_i^2 \Big/ \sum_{i=1}^{n} p_i \leqslant \max\{p_i : i = 1, \ldots, n\}.$$

REMARK 2. By the following device the result of the theorem may also be applied to portfolios that can be divided into subportfolios, which are homogeneous in the above sense. It is standard that

$$d(G_1 * G_2, P_1 * P_2) \leqslant d(G_1, P_1) + d(G_2, P_2).$$

Hence, if one calculates the bound in (3) for each subportfolio, then the sum of these bounds yields an error bound for the compound Poisson approximation of the given portfolio.

PROOF OF THE THEOREM. (i) For easier derivation of the result, we observe that

$$d(G, P) = \sup_A (G(A) - P(A)).$$

(This follows from

$$|G(A) - P(A)| = \max\{G(A) - P(A), G(A^c) - P(A^c)\},$$

where A^c denotes the complement of A.) In the following we set

$$B = B_{p_1} * \cdots * B_{p_n},$$

where B_p is the Bernoulli distribution with parameter p. Furthermore, let

$$b_k = B\{k\} \qquad (k = 0, 1, \ldots, n),$$

and

$$q_k = P_\lambda\{k\} = e^{-\lambda} \frac{\lambda^k}{k!} \qquad (k = 0, 1, 2, \ldots)$$

with

$$\lambda = \sum_{i=1}^{n} p_i.$$

(ii) We have

$$d(G, P) \leqslant d(B, P_\lambda). \tag{4}$$

(This means that the proof of our result may be reduced to the case $Q = \delta_1$, where δ_1 is the distribution concentrated in 1.)

For any distribution D, we have

$$D*[(1-p)\delta_0 + pQ] = (1-p)D + p(D*Q).$$

Hence, by induction over n,

$$G = \sum_{k=0}^{n} b_k Q^{*k}$$

and, with

$$A_0 = \{k \in \{0, 1, \ldots, n\} : b_k > q_k\},$$

we obtain for an arbitrary event A:

$$\begin{aligned} G(A) - P(A) &= \sum_{k=0}^{n} b_k Q^{*k}(A) - \sum_{k=0}^{\infty} q_k Q^{*k}(A) \\ &\leqslant \sum_{k=0}^{n} (b_k - q_k) Q^{*k}(A) \leqslant \sum_{k \in A_0} (b_k - q_k) \\ &= B(A_0) - P_\lambda(A_0) \leqslant d(B, P_\lambda). \end{aligned}$$

This gives (4).

(iii) The following part of the proof is a simplified version of the proof of theorem 1 in BARBOUR and HALL (1984, p. 474). For a set $A \subset \{0, 1, 2, \ldots\}$ and $U_k = \{0, 1, \ldots, k-1\}$ let

$$g(0) = 0, \qquad g(k) = \frac{1}{kq_k}\left[P_\lambda(A \cap U_k) - P_\lambda(A)P_\lambda(U_k)\right] \qquad (k = 1, 2, \ldots).$$

Then $(k+1)q_{k+1} = \lambda q_k$ gives

$$\lambda g(k+1) - kg(k) = \frac{1}{q_k}\left[P_\lambda(A \cap \{k\}) - q_k P_\lambda(A)\right],$$

i.e.

$$1_A(k) - P_\lambda(A) = \lambda g(k+1) - kg(k) \qquad (k = 0, 1, 2, \ldots), \tag{5}$$

where 1_A denotes the indicator-function of A.
For $i = 1, \ldots, n$, we set

$$B^{(i)} = B_{p_1} * \ldots * B_{p_{i-1}} * B_{p_{i+1}} * \ldots * B_{p_n}.$$

Let, furthermore, $X_1, \ldots, X_n$ be i.i.d. such that X_i is distributed according to B_{p_i}, $i = 1, \ldots, n$. If in the following we first integrate with respect to B_{p_i}, then for a function $h(x, y)$,

$$E\left[h\left(X_i, \sum_{j=1}^{n} X_j\right)\right] = \int \left[(1-p_i)h(0, y) + p_i h(1, y+1)\right] B^{(i)}(dy).$$

Using (5) we obtain with $h(x, y) = p_i g(y+1) - xg(y)$ that

$$\begin{aligned} B(A) - P_\lambda(A) &= \int \left[1_A(x) - P_\lambda(A)\right] B(dx) \\ &= \int \left[\lambda g(x+1) - xg(x)\right] B(dx) \\ &= \sum_{i=1}^{n} E\left[p_i g\left(\sum_{j=1}^{n} X_j + 1\right) - X_i g\left(\sum_{j=1}^{n} X_j\right)\right] \\ &= \sum_{i=1}^{n} p_i^2 \int \left[g(y+2) - g(y+1)\right] B^{(i)}(dy). \end{aligned} \tag{6}$$

In part (iv) of the proof we finally show that

$$g(k+1) - g(k) \leqslant 1/\lambda \qquad (k = 1, 2, \ldots). \tag{7}$$

Then (6) and (4) give the assertion of the theorem.

(iv) First we observe that (5) implies

$$\lambda\left[g(k+1) - g(k)\right] = 1_A(k) - P_\lambda(A) + (k - \lambda)g(k) \qquad (k = 1, 2, \ldots).$$

Hence, (7) follows from

$$(k - \lambda)g(k) \leqslant P_\lambda(A) \qquad (k = 1, 2, 3, \ldots). \tag{8}$$

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In order to prove this, we consider the cases $k > \lambda$ and $k \leqslant \lambda$. In the first case we use

$$P_\lambda(A \cap U_k) - P_\lambda(A)P_\lambda(U_k) \leqslant P_\lambda(A) - P_\lambda(A)P_\lambda(U_k) = P_\lambda(A)P_\lambda(U_k^c)$$

and

$$P_\lambda(U_k^c) = \sum_{m=k}^{\infty} q_m = q_k \sum_{m=0}^{\infty} \frac{\lambda^m}{(m+k)!}\, k!$$

$$\leqslant q_k \sum_{m=0}^{\infty} \left(\frac{\lambda}{k}\right)^m = \frac{kq_k}{k-\lambda}.$$

In the case $k \leqslant \lambda$ we have

$$(k-\lambda)g(k) = \frac{\lambda - k}{kq_k}\,[P_\lambda(A)P_\lambda(U_k) - P_\lambda(A \cap U_k)]$$

$$\leqslant \frac{\lambda - k}{kq_k}\, P_\lambda(A)P_\lambda(U_k).$$

Furthermore,

$$(\lambda - k)P_\lambda(U_k) \leqslant kq_k \qquad (k = 1, 2, \ldots).$$

For $k = 1$ this follows from $(\lambda - 1)e^{-\lambda} \leqslant \lambda e^{-\lambda}$. Under the assumption that the assertion is true for $k \geqslant 1$, we obtain

$$\begin{aligned}[\lambda - (k+1)]\,P_\lambda(U_{k+1}) &\leqslant (\lambda - k)P_\lambda(U_{k+1}) \\ &= (\lambda - k)[P_\lambda(U_k) + q_k] \\ &\leqslant kq_k + (\lambda - k)q_k = \lambda q_k = (k+1)q_{k+1}.\end{aligned}$$

Hence, (8) is proved.

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