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FRACTIONAL RELAXATION EQUATIONS AND BROWNIAN CROSSING PROBABILITIES OF A RANDOM BOUNDARY

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Abstract

In this paper we analyze different forms of fractional relaxation equations of order $\nu \in (0, 1)$, and we derive their solutions in both analytical and probabilistic forms. In particular, we show that these solutions can be expressed as random boundary crossing probabilities of various types of stochastic process, which are all related to the Brownian motion B. In the special case $\nu = \frac{1}{2}$, the fractional relaxation is shown to coincide with $\Pr\{\sup_{0\le s\le t} B(s) < U\}$ for an exponential boundary U. When we generalize the distributions of the random boundary, passing from the exponential to the gamma density, we obtain more and more complicated fractional equations.

Keywords: Fractional relaxation equation; generalized Mittag-Leffler function; processes with random time; reflecting and elastic Brownian motion; iterated Brownian motion; boundary crossing probability

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1. Introduction

The differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = -\lambda p(t), \qquad t \ge 0, \tag{1.1}$$

is known in the physics literature as the *relaxation equation*. The solution to (1.1), with initial condition $p(0) = 1$, is clearly equal to $p(t) = \mathrm{e}^{-\lambda t}$. Since the end of the 1990s intensive research has focused on the application of fractional calculus to mathematical physics: many classical equations have been modified by substituting the integer-order derivatives with fractional derivatives. Equation (1.1) has been extended in the following fractional sense:

$$\frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu} \psi(t) = -\lambda \psi(t), \qquad t \ge 0. \tag{1.2}$$

Here $\nu \in (0, 1)$ and $\mathrm{d}^\nu/\mathrm{d}t^\nu$ represents the fractional derivative according to the Caputo definition, i.e.

$$\frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu} u(t) = \begin{cases} \dfrac{1}{\Gamma(m-\nu)} \displaystyle\int_0^t \frac{1}{(t-s)^{1+\nu-m}} \frac{\mathrm{d}^m}{\mathrm{d}s^m} u(s)\,\mathrm{d}s & \text{for } m-1 < \nu < m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} u(t) & \text{for } \nu = m, \end{cases}$$

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with $m = \lfloor \alpha \rfloor + 1$. Obviously, for $\nu = 1$, the *fractional relaxation equation* (1.2) coincides with the standard equation (1.1).

Equation (1.2) has been studied in, e.g. [16] and [19], and its solution was given analytically in terms of the Mittag-Leffler function as

$$\psi_\nu(t) = E_{\nu,1}(-\lambda t^\nu), \tag{1.3}$$

where

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)}, \qquad \alpha, \beta \in \mathbb{C},\ \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

The analysis of the fractional relaxation equation is primarily physically motivated, e.g. to study the electromagnetic properties of a wide range of materials (which display a long-memory, instead of exponential, decay—see [30] and [31]) as well as the rheological models for the description of some viscoelastic materials (see [10], [21], [22], and [29]).

Moreover, the so-called Mittag-Leffler distribution has often been applied to statistics (see, e.g. [15] and [26]) or to queueing theory [28].

The solution $\psi_\nu(t),\ t \geq 0$, can actually be expressed in probabilistic terms in two interesting forms, which we will present and explore here. The first form represents the probability of no events up to time t (or survival probability) for the so-called *fractional Poisson process* $\mathcal{N}_\nu(t),\ t \geq 0$ (see, amongst others, [2], [4], [14], [18], and [32]). Indeed, the equality

$$\psi_\nu(t) = p_0^\nu(t) = \Pr\{\mathcal{N}_\nu(t) = 0\} \tag{1.4}$$

holds and, thus, we can apply the results obtained in the above-cited articles to $\psi_\nu(t)$. For example, we will resort to the equality of the one-dimensional distribution between $\mathcal{N}_\nu$ and a composition of the standard Poisson process $N(t)$ with a random time process $\mathcal{T}_\nu(t)$, i.e. $N(\mathcal{T}_\nu(t)),\ t \geq 0$. Thus, thanks to (1.4), we can write

$$\psi_\nu(t) = \int_0^\infty \mathrm{e}^{-\lambda y} q_\nu(y,t)\,\mathrm{d}y = \Pr\{\mathcal{T}_\nu(t) < U\}, \tag{1.5}$$

where $q_\nu(y,t)$ is the density of $\mathcal{T}_\nu$ (which is itself a solution to a fractional diffusion equation) and U is an exponential random variable with parameter $\lambda > 0$. Equation (1.5) is particularly interesting in the special case where $\nu = \frac{1}{2}$, since it becomes

$$\psi_{1/2}(t) = \int_0^\infty \mathrm{e}^{-\lambda y} \frac{\mathrm{e}^{-y^2/4t}}{\sqrt{\pi t}}\,\mathrm{d}y = \Pr\{|B(t)| < U\}, \tag{1.6}$$

where B is a Brownian motion starting from 0 and with variance $2t$.

As a consequence, a second probabilistic interpretation of the solution to the fractional relaxation equation (1.2) can be given in terms of the *crossing probability* of a random boundary by a standard Brownian motion, for $\nu = \frac{1}{2}$. Indeed, it is well known that the relationship

$$\Pr\{|B(t)| < z\} = \Pr\Big\{\sup_{0 \leq s \leq t} B(s) < z\Big\} = \Pr\{B(s) < z \text{ for all } s \in (0,t)\}$$

holds, where the expression on the right-hand side is commonly referred to as the *crossing probability*. For other values of ν, e.g. $\nu = \frac{1}{3}$, a related result holds, as we will see in the next section.

Moreover, (1.5) shows that the solution to (1.2) can be expressed as a standard relaxation with random time represented by $\mathcal{T}_\nu$, i.e. as $\psi(\mathcal{T}_\nu(t))$. The results given in [20] also allow us to express the solution as a time-changed relaxation via an inverse stable subordinator $E(t)$, i.e. as $\psi_\nu(t) = \psi(E(t))$. In fact, $\psi(\mathcal{T}_\nu(t))$ and $\psi(E(t))$ share one-dimensional distributions and, therefore, the two approaches can be considered equivalent.

In the successive sections we analyze some extensions of result (1.5) in the following directions.

- We consider other random time processes in place of $\mathcal{T}_\nu$ and, therefore, in (1.6) instead of Brownian motion: for example, the sojourn time of a Brownian motion on the positive half-line, the first passage time of a Brownian motion through a certain level, and the elastic Brownian motion (by analogy with the analysis carried out for the fractional Poisson process in [5]).
- We consider a different random variable (i.e. the gamma) instead of U in (1.6).
- We introduce in (1.2) the assumption of a *distributed fractional derivative* (see [1] and [19]).

2. Fractional relaxation equation of order ν

A first probabilistic expression of the solution $\psi_\nu(t)$, $t \geq 0$, to (1.2) can be found by considering that the latter coincides with the fractional equation satisfied by the survival probability (i.e. the probability of no events up to time t) of a fractional Poisson process of order $\nu \in (0, 1)$. Let $\mathcal{N}_\nu(t)$, $t \geq 0$, denote the process with probabilities $p_k^\nu(t)$ solving the recursive differential equation

$$\frac{\mathrm{d}^\nu p_k^\nu}{\mathrm{d}t^\nu} = -\lambda(p_k^\nu - p_{k-1}^\nu), \qquad k \geq 0,\ t \geq 0, \tag{2.1}$$

with initial conditions

$$p_k^\nu(0) = \begin{cases} 1, & k = 0, \\ 0, & k \geq 1, \end{cases}$$

and $p_{-1}^\nu(t) = 0$. The process $\mathcal{N}_\nu$ has been studied in a series of papers (see, e.g. [2], [14], and [18] for the homogeneous case, and [32] for the nonhomogeneous case) and its distribution has been expressed in analytic forms in terms of derivatives of the Mittag-Leffler function or as generalized Mittag-Leffler (GML) functions

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j z^j}{j!\,\Gamma(\alpha j + \beta)}, \qquad \alpha, \beta, \gamma \in \mathbb{C},\ \mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0, \tag{2.2}$$

where $(\gamma)_j = \gamma(\gamma+1)\cdots(\gamma+j-1)$ for $j = 1, 2, \dots$ and $\gamma \neq 0$, and $(\gamma)_0 = 1$ (see [4]). Moreover, in [2] a probabilistic expression for the process was given as the composition of a standard Poisson process N with a random time argument $\mathcal{T}_\nu$, independent of N. The following equality for one-dimensional distributions was proved to hold in [2]:

$$\mathcal{N}_\nu(t) \overset{\mathrm{D}}{=} N(\mathcal{T}_\nu(t)). \tag{2.3}$$

Here $\mathcal{T}_\nu(t)$ possesses the transition density $q_\nu(y,t)$ coinciding with the folded solution to the fractional diffusion equation

$$\frac{\partial^{2\nu} v}{\partial t^{2\nu}} = \frac{\partial^2 v}{\partial y^2}, \quad t \ge 0,\ y \in \mathbb{R}; \qquad v(y,0) = \delta(y), \qquad v_t(y,0) = 0, \tag{2.4}$$

i.e. with

$$q_\nu(y,t) = \begin{cases} 2v(y,t), & y \ge 0, \\ 0, & y < 0. \end{cases} \tag{2.5}$$

It should be noted that, since the process $\mathcal{N}_\nu$ is non-Markovian, identity (2.3) does not extend to finite-dimensional distributions of order larger than 1.

Alternatively, it was also proved in [20] and [25] that $q_\nu(y,t)$ solves the equation

$$\frac{\partial^\nu q}{\partial t^\nu} = -\frac{\partial q}{\partial y}, \quad t \ge 0; \qquad q(y,0) = \delta(y), \tag{2.6}$$

where, in this case, $y \ge 0$. In any case we can write

$$p_k^\nu(t) = \Pr\{\mathcal{N}_\nu(t) = k\} = \frac{\lambda^k}{k!}\int_0^{+\infty} y^k \mathrm{e}^{-\lambda y} q_\nu(y,t)\,\mathrm{d}y,$$

so that we immediately have, in view of (2.1) for $k=0$,

$$\psi_\nu(t) = p_0^\nu(t) = \Pr\{\mathcal{N}_\nu(t) = 0\} = \int_0^{+\infty} \mathrm{e}^{-\lambda y} q_\nu(y,t)\,\mathrm{d}y. \tag{2.7}$$

Therefore, in view of (2.3), the fractional relaxation ψ_ν can be expressed as the composition of the standard relaxation with the random time $\mathcal{T}_\nu$:

$$\psi_\nu(t) = \psi(\mathcal{T}_\nu(t)), \qquad t \ge 0.$$

2.1. Exponential boundary crossing probabilities of Brownian motion

Owing to (2.7), we give a second probabilistic form of the solution to (1.2) in terms of boundary crossing probabilities in the following result.

Theorem 2.1. *Let U be a random boundary, exponentially distributed (with parameter $\lambda > 0$). Then the crossing probability of U by the independent random process $\mathcal{T}_\nu(t)$ with transition density $q_\nu(y,t)$, i.e.*

$$\psi_\nu(t) = \Pr\{\mathcal{T}_\nu(t) < U\}, \tag{2.8}$$

satisfies the fractional relaxation equation (1.2), with initial condition $\psi_\nu(0) = 1$.

Proof. We consider the analytic expression of the folded solution $q_\nu(y,t)$ to problem (2.4) in terms of the Wright function

$$\mathcal{W}_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!\,\Gamma(\alpha j + \beta)}, \qquad \alpha \ge -1,\ \beta > 0,\ x \in \mathbb{R},$$

which reads

$$q_\nu(y,t) = 2v(y,t) = \frac{1}{t^\nu} W_{-\nu,1-\nu}\left(-\frac{y}{t^\nu}\right), \qquad y, t \ge 0$$

(see, e.g. [16]). Therefore, we can rewrite (2.8) as

$$\begin{aligned}\psi_\nu(t) &= \Pr\{\mathcal{T}_\nu(t) < U\} \\ &= \lambda \int_0^\infty \mathrm{e}^{-\lambda y} \Pr\{\mathcal{T}_\nu(t) < y\}\,\mathrm{d}y \\ &= \frac{\lambda}{t^\nu} \int_0^\infty \mathrm{e}^{-\lambda y} \int_0^y W_{-\nu,1-\nu}\left(-\frac{z}{t^\nu}\right) \mathrm{d}z\,\mathrm{d}y \\ &= \frac{1}{t^\nu} \int_0^\infty \mathrm{e}^{-\lambda z} W_{-\nu,1-\nu}\left(-\frac{z}{t^\nu}\right) \mathrm{d}z \\ &= E_{\nu,1}(-\lambda t^\nu), \end{aligned} \tag{2.9}$$

by the well-known formula for the Laplace transform of the Wright function (see [27, Equation (1.165), p. 39]). The last expression in (2.9) coincides with the solution to (1.2) given in (1.3).

The previous results can be particularly relevant in the special case where $\nu = \frac{1}{2}$, since the random process $\mathcal{T}_\nu$ reduces to a reflecting Brownian motion; indeed, in this case the equation governing the process, (2.4), coincides with the heat equation and $q_{1/2}(y,t)$ becomes the Gaussian with variance $2t$, folded with respect to the origin. Therefore, the fractional relaxation equation of order $\frac{1}{2}$ is solved by

$$\psi_{1/2}(t) = \frac{1}{\sqrt{\pi t}} \int_0^{+\infty} \mathrm{e}^{-\lambda y} \mathrm{e}^{-y^2/4t}\mathrm{d}y = \Pr\{|B(t)| < U\} = \Pr\Big\{\sup_{0\le s\le t} B(s) < U\Big\}. \tag{2.10}$$

The previous expression can be checked directly by applying (1.3):

$$\begin{aligned}\psi_{1/2}(t) &= E_{1/2,1}(-\lambda\sqrt{t}) \\ &= \sum_{j=0}^\infty \frac{(-2\lambda\sqrt{t})^j\Gamma(j/2+1/2)}{\Gamma(j+1)\sqrt{\pi}} \quad \text{(by the duplication property of the gamma function)} \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-z} z^{-1/2} \sum_{j=0}^\infty \frac{(-2\lambda\sqrt{zt})^j}{j!}\,\mathrm{d}z \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-z} z^{-1/2} \mathrm{e}^{-2\lambda\sqrt{zt}}\,\mathrm{d}z, \end{aligned} \tag{2.11}$$

which gives (2.10) after a change of variable.

Also, for $\nu = 1/2^n$, $n \ge 1$, the solution can be expressed in terms of the boundary crossing probability of known processes. Indeed, the random process $\mathcal{T}_\nu$ coincides in this case with the $(n-1)$-times *iterated reflecting Brownian motion* defined as $I_{n-1}(t) = |B_1(|B_2(\ldots(|B_n(t)|)\ldots)|)|$, where the $B_j(t)$ are independent Brownian motions with variance $2t$ for any j. The transition density $q_{1/2^n}(y,t)$ of I_{n-1} is given by

$$q_{1/2^n}(y,t) = \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{\mathrm{e}^{-y^2/4s_1}}{\sqrt{\pi s_1}} \frac{\mathrm{e}^{-s_1^2/4s_2}}{\sqrt{\pi s_2}} \cdots \frac{\mathrm{e}^{-s_{n-1}^2/4t}}{\sqrt{\pi t}}\,\mathrm{d}s_1\cdots\mathrm{d}s_{n-1}, \qquad y,t \ge 0,$$

which coincides with the folded solution to the following fractional diffusion equation:

$$\frac{\partial^{1/2^n} q}{\partial t^{1/2^n}} = \frac{\partial^2 q}{\partial y^2}, \quad y \in \mathbb{R},\ t \ge 0; \qquad q(y,0) = \delta(y)$$

(see [23] for $n = 1$ and [24] for $n > 1$). Therefore, in this case, the solution to the fractional relaxation equation can be expressed in terms of the exponential boundary crossing probability of an iterated Brownian motion, i.e.

$$\psi_{1/2^n}(t) = \psi(I_{n-1}(t)) = \int_0^{+\infty} \mathrm{e}^{-\lambda y} q_{1/2^n}(y,t)\,\mathrm{d}y = \Pr\{I_{n-1}(t) < U\}.$$

For other rational values of the fractional order ν, such as, e.g. $\nu = \frac{1}{3}$, the solution can still be represented as the boundary crossing probability, but of less known processes.

For $\nu = \frac{1}{3}$, the random process $\mathcal{T}_\nu$ in (2.8) reduces to the process $A(t)$, whose transition function is given by

$$q_{1/3}(y,t) = \sqrt[3]{\frac{3^2}{t}}\,\mathrm{Ai}\biggl(\frac{y}{\sqrt[3]{3t}}\biggr), \qquad y, t \ge 0, \tag{2.12}$$

where

$$\mathrm{Ai}(w) = \frac{1}{\pi}\int_0^\infty \cos\biggl(aw + \frac{\alpha^3}{3}\biggr)\,\mathrm{d}\alpha, \qquad w \in \mathbb{R}, \tag{2.13}$$

is the Airy function (see [24]). By exploiting the relationship between (2.13) and the modified Bessel function

$$I_\nu(w) = \sum_{k=0}^\infty \frac{(w/2)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}, \qquad w \in \mathbb{R},$$

i.e.

$$\mathrm{Ai}(w) = \frac{\sqrt{w}}{3}\biggl[I_{-1/3}\biggl(\frac{2\sqrt{w^3}}{3}\biggr) - I_{1/3}\biggl(\frac{2\sqrt{w^3}}{3}\biggr)\biggr], \qquad w > 0,$$

we can rewrite the transition density (2.12) of the process $A(t)$, $t \ge 0$, as

$$q_{1/3}(y,t) = \sqrt{\frac{y}{3t}}\biggl[I_{-1/3}\biggl(2\sqrt{\frac{y}{3^3 t}}\biggr) - I_{1/3}\biggl(2\sqrt{\frac{y}{3^3 t}}\biggr)\biggr], \qquad y, t \ge 0.$$

Therefore, in this case, the fractional relaxation can be written as

$$\psi_{1/3}(t) = \psi(A(t)) = \int_0^{+\infty} \mathrm{e}^{-\lambda y} q_{1/3}(y,t)\,\mathrm{d}y = \Pr\{A(t) < U\}.$$

For the process $A(t)$, the following relationship between the crossing probability and the maximum distribution has been proved in [6, Equation (1.16)]:

$$\Pr\Bigl\{\sup_{0\le s\le t} A(t) < u\Bigr\} = \Pr\{A(t) < u\} - \Pr\{A(t) > u\} + K(t);$$

here

$$K(t) = \frac{1}{\Gamma(2/3)}\int_0^t \frac{1}{\sqrt[3]{t-s}}\,\frac{\partial}{\partial u} q_{1/3}(u,s)\,\mathrm{d}s.$$

It is worth comparing the asymptotic behavior of the different crossing probabilities introduced so far. By using the well-known integral representation of the Mittag-Leffler function (see [1] or [2] for $c = 1$),

$$E_{\nu,\beta}(-ct^\nu) = \frac{t^{1-\beta}}{\pi}\int_0^{+\infty} r^{\nu-\beta}\mathrm{e}^{-rt}\frac{r^\nu \sin(\beta\pi) + c\sin((\beta-\nu)\pi)}{r^{2\nu} + 2r^\nu c\cos(\nu\pi) + c^2}\,\mathrm{d}r, \tag{2.14}$$

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we obtain the following asymptotic behavior of the solution ψ_ν:

$$\psi_\nu(t) \simeq \begin{cases} 1 - \dfrac{\lambda t^\nu}{\Gamma(1+\nu)}, & 0 < t \ll 1, \\ \dfrac{1}{\lambda t^\nu \Gamma(1-\nu)}, & t \to \infty. \end{cases} \tag{2.15}$$

Therefore, the boundary crossing probability of Brownian motion exhibits a power decay for $t \to \infty$ of exponent $\frac{1}{2}$, instead of the usual exponential decay of the standard relaxation ψ. For the n-times iterated Brownian motion, the exponent $1/2^n$ of t is smaller than $\frac{1}{2}$ and decreases as n becomes larger. This is intuitively explained by the fact that the number of compositions increases in the definition of the process I_n: this strays the fractional relaxation more and more away from the standard (exponential) behavior as n increases, and makes the tail of the relaxation more and more heavy.

For the process $A(t)$, the crossing probability possesses a power decay for $t \to \infty$ with exponent $\frac{1}{3}$, which is between the Brownian case and the iterated case (for any $n > 1$).

2.2. Exponential boundary crossing probabilities of more general processes

We now present some extensions of the previous results, obtained by considering the crossing probabilities of different kinds of process. This corresponds to substituting the random process $\mathcal{T}_\nu(t)$ in (2.8) with some other process, linked to the Brownian motion by various relationships, such as the elastic Brownian motion, the Bessel process (or its square), the first passage time through a level t by a standard Brownian motion, or its sojourn time on the positive half-line.

We start with the sojourn time of a Brownian motion on the positive half-line, which is a positive, nondecreasing process. Let $\Gamma_t^+(t) = \operatorname{meas}\{s < t \colon B(t) > 0\}$ be the *sojourn time* on the positive half-line of a standard Brownian motion B. Then its density $q^+(s,t)$ is given by

$$q^+(s,t) = \Pr\{\Gamma_t^+ \in \mathrm{d}s\} = \frac{\mathrm{d}s}{\pi\sqrt{s(t-s)}}, \qquad 0 < s < t.$$

Theorem 2.2. *Let U be a random boundary, exponentially distributed with parameter $\lambda > 0$. Then the crossing probability of U by the random process $\Gamma_t^+(t)$ with transition density $q^+(s,t)$ is given by*

$$\psi^+(t) = \psi(\Gamma^+(t)) = \Pr\{\Gamma^+(t) < U\} = \mathrm{e}^{-\lambda t/2} I_0\left(\frac{\lambda t}{2}\right) \tag{2.16}$$

and (2.16) solves the following second-order differential equation

$$\frac{\mathrm{d}^2\psi^+}{\mathrm{d}t^2} + \left(\lambda + \frac{1}{t}\right)\frac{\mathrm{d}\psi^+}{\mathrm{d}t} = -\frac{\lambda}{2t}\psi^+, \qquad \psi^+(0) = 1. \tag{2.17}$$

Proof. We write the crossing probability as

$$\psi^+(t) = \int_0^t \mathrm{e}^{-\lambda s} \frac{\mathrm{d}s}{\pi\sqrt{s(t-s)}} = {}_1F_1\left(\frac{1}{2}; 1; -\lambda t\right) \quad \text{([11, Equation 3.383.1, p. 365])}, \tag{2.18}$$

where ${}_1F_1(\alpha, \gamma; x)$ denotes the confluent hypergeometric function, defined as

$${}_1F_1(\alpha; \gamma; x) = 1 + \sum_{j=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+j-1)}{\gamma(\gamma+1)\cdots(\gamma+j-1)} \frac{x^j}{j!}$$

for $x, \alpha \in \mathbb{C}$ and $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

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By applying the relationship with the Bessel functions (see Equation 9.215.2 of [11, p. 1086]), after some computations, we obtain the final form (2.16). As far as the equation satisfied by (2.16) is concerned, we recall that $I_0(\lambda x)$ coincides with the solution to the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} I_0(\lambda x) + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x} I_0(\lambda x) = \lambda^2 I_0(\lambda x), \tag{2.19}$$

as can easily be checked. Therefore, by the transformation $I_0(\lambda t/2) = \mathrm{e}^{\lambda t/2}\psi^+(t)$, from (2.19) we obtain (2.17), since

$$\frac{\mathrm{d}}{\mathrm{d}t} I_0\left(\frac{\lambda t}{2}\right) = \frac{\lambda}{2}\mathrm{e}^{\lambda t/2}\psi^+(t) + \mathrm{e}^{\lambda t/2}\frac{\mathrm{d}}{\mathrm{d}t}\psi^+(t),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_0\left(\frac{\lambda t}{2}\right) = \frac{\lambda^2}{4}\mathrm{e}^{\lambda t/2}\psi^+(t) + \lambda \mathrm{e}^{\lambda t/2}\frac{\mathrm{d}}{\mathrm{d}t}\psi^+(t) + \mathrm{e}^{\lambda t/2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\psi^+(t).$$

Alternatively, we can resort to form (2.18) and exploit the fact that the confluent hypergeometric function ${}_1F_1(\alpha;\gamma;x)$ satisfies the following equation:

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,{}_1F_1 + (\gamma - x)\frac{\mathrm{d}}{\mathrm{d}x}\,{}_1F_1 = \alpha\,{}_1F_1.$$

By taking into account the facts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_1F_1\left(\frac{1}{2};1;-\lambda t\right) = -\lambda\frac{\mathrm{d}}{\mathrm{d}(-\lambda t)}\,{}_1F_1\left(\frac{1}{2};1;-\lambda t\right),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,{}_1F_1\left(\frac{1}{2};1;-\lambda t\right) = \lambda^2\frac{\mathrm{d}}{\mathrm{d}(-\lambda t)^2}\,{}_1F_1\left(\frac{1}{2};1;-\lambda t\right),$$

we again obtain (2.17).

The asymptotic behavior of $\psi^+(t)$ can be deduced by noting that $I_\nu(x) \simeq (x/2)^\nu/\Gamma(\nu+1)$ as $x \to 0$ and that

$${}_1F_1(\alpha;\gamma,x) \simeq \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\mathrm{e}^{-\mathrm{i}\pi\alpha}x^{-\alpha}, \qquad \mathrm{Re}(x) \to -\infty$$

(see [13, p. 29]); thus, we obtain

$$\psi^+(t) \simeq \begin{cases} 1 - \dfrac{\lambda t}{2}, & 0 < t \ll 1, \\ \dfrac{1}{\sqrt{\lambda \pi t}}, & t \to \infty. \end{cases} \tag{2.20}$$

The limiting behavior of $\psi^+(t)$ is the same as that of the standard relaxation for $t \to 0$, while it coincides with that of $\psi_{1/2}(t)$ for $t \to \infty$ (up to multiplicative constants).

Another process that can be considered instead of the random time $\mathcal{T}_\nu(t)$ in (2.8) is the *first passage time through a level t* by a standard Brownian motion, denoted as

$$T(t) = \inf\{s > 0\colon B(s) = t\}.$$

We are interested here in the crossing probability

$$\psi_T(t) = \psi(T(t)) = \int_0^\infty \mathrm{e}^{-\lambda s} q_T(s,t)\,\mathrm{d}s = \Pr\{T(t) < U\}, \tag{2.21}$$

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where the density of $T(t),\ t \geq 0$, is the well-known stable law of index $\frac{1}{2}$, i.e.

$$q_T(s,t) = \frac{t\mathrm{e}^{-t^2/2s}}{\sqrt{2\pi s^3}}, \qquad s, t \geq 0.$$

As is well known, $T(t)$ is a nondecreasing Lévy process and so is a subordinator with Laplace exponent $\Phi(\lambda) = \sqrt{2\lambda}$. Therefore, (2.21) coincides with the Laplace transform of the first passage time, i.e.

$$\psi_T(t) = \mathrm{e}^{-t\Phi(\lambda)} = \mathrm{e}^{-t\sqrt{2\lambda}}$$

(see [7] for further details on this topic).

Clearly, $\psi_T(t)$ satisfies the standard relaxation equation, even if with a different constant:

$$\frac{\mathrm{d}\psi_T}{\mathrm{d}t} = -\sqrt{2\lambda}\psi_T, \qquad \psi_T(0) = 1.$$

We note that time changing the relaxation ψ using the $\frac{1}{2}$-stable subordinator $T(t)$ again produces a standard relaxation, while performing the same operation using an inverse stable subordinator $E(t)$ yields the fractional relaxation $\psi_{1/2}$ (as mentioned in the introduction).

If we now consider n independent Brownian motions $B_j,\ j = 1, \ldots, n$, and use them to construct the n-times iterated process $T_1(T_2(\ldots T_n(t)\ldots)),\ t \geq 0$, where $T_j = \inf\{s > 0\colon B_j(s) = t\},\ j = 1, \ldots, n$, then its crossing probability can be evaluated as follows:

$$\begin{aligned}
\psi_T^n(t) &= \Pr\{T_1(T_2(\ldots T_n(t)\ldots)) < U\} \\
&= \int_0^\infty \mathrm{e}^{-\lambda s}\left(\int_0^{+\infty} \mathrm{d}z_1 \cdots \int_0^{+\infty} \mathrm{d}z_{n-1} \frac{t\mathrm{e}^{-t^2/2z_1}}{\sqrt{2\pi z_1^3}} \cdots \frac{z_{n-1}\mathrm{e}^{-z_{n-1}^2/2z_n}}{\sqrt{2\pi z_n^3}} \frac{z_n\mathrm{e}^{-z_n^2/2s}}{\sqrt{2\pi s^3}}\right)\mathrm{d}s \\
&= \int_0^{+\infty} \mathrm{d}z_1 \cdots \int_0^{+\infty} \mathrm{d}z_{n-1} \frac{t\mathrm{e}^{-t^2/2z_1}}{\sqrt{2\pi z_1^3}} \cdots \frac{z_{n-1}\mathrm{e}^{-z_{n-1}^2/2z_n}}{\sqrt{2\pi z_n^3}} \int_0^\infty \mathrm{e}^{-\lambda s}\frac{z_n\mathrm{e}^{-z_n^2/2s}}{\sqrt{2\pi s^3}}\,\mathrm{d}s \\
&= \int_0^{+\infty} \mathrm{d}z_1 \cdots \frac{t\mathrm{e}^{-t^2/2z_1}}{\sqrt{2\pi z_1^3}} \cdots \int_0^{+\infty} \frac{z_{n-1}\mathrm{e}^{-z_{n-1}^2/2z_n}}{\sqrt{2\pi z_n^3}}\mathrm{e}^{-z_n\sqrt{2\lambda}}\,\mathrm{d}z_{n-1} \\
&= \int_0^{+\infty} \mathrm{d}z_1 \cdots \frac{t\mathrm{e}^{-t^2/2z_1}}{\sqrt{2\pi z_1^3}} \cdots \int_0^{+\infty} \frac{z_{n-2}\mathrm{e}^{-z_{n-2}^2/2z_{n-1}}}{\sqrt{2\pi z_{n-1}^3}}\mathrm{e}^{-z_{n-1}\sqrt{2\sqrt{2\lambda}}}\,\mathrm{d}z_{n-2} \\
&= \mathrm{e}^{-\lambda^{1/2^n}2^{1-1/2^n}t}.
\end{aligned}$$

Again, the probability ψ_T^n satisfies (for any n) the standard relaxation equation with constant $\lambda^{1/2^n}2^{1-1/2^n}$ and displays an asymptotic behavior similar to the standard relaxation, despite the complicated construction via the n-times composition.

We now analyze the crossing probability of an exponential boundary U for a *squared Bessel process*. Let us denote by $R_\gamma^2(t) = (R_\gamma(t))^2,\ t \geq 0$, the square of a γ-Bessel process, starting at 0. It is well known that, for $\gamma = n$, this process can be expressed as

$$R_n^2(t) = \sum_{j=1}^n B_j^2(t), \qquad t \geq 0,$$

where the $B_j(t),\ j = 1, \ldots, n$, are independent Brownian motions in $\mathbb{R}^n$. Moreover, the density of R_γ^2 can be written as

$$p_\gamma^2(s,t) = \frac{s^{\gamma/2-1}\mathrm{e}^{-s/2t}}{(2t)^{\gamma/2}\Gamma(\gamma/2)}, \qquad s,t \geq 0$$

(see, e.g. [9]), which is a more tractable form (for our aims) than that of R_γ. Thus, the crossing probability of this process can be easily evaluated as

$$\psi_\gamma(t) = \Pr\{R_\gamma^2(t) < U\} = \int_0^\infty \mathrm{e}^{-\lambda s}\frac{s^{\gamma/2-1}\mathrm{e}^{-s/2t}}{(2t)^{\gamma/2}\Gamma(\gamma/2)}\,\mathrm{d}s = \frac{1}{(2\lambda t+1)^{\gamma/2}}, \tag{2.22}$$

which satisfies the following first-order differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_\gamma = \frac{\gamma\lambda}{2\lambda t+1}\psi_\gamma, \qquad \psi_\gamma(0) = 1.$$

In this case, the behavior of $\psi_\gamma(t)$ for increasing (but still finite) values of t can be represented as $\psi_\gamma(t) \simeq (k/t)^{\gamma/2}$ (for some constant k and $0 < \gamma < 2$), and, thus, it coincides with the one described as 'algebraic decay' and displayed by relaxation processes in complex material (see, e.g. [29]). However, for the other fractional relaxations, this is true only in the limit for $t \to \infty$. Indeed, function (2.22) coincides with the so-called Nutting law, which is commonly used to fit experimental data of materials featuring nonstandard (i.e. non-Debye) relaxation (see [21] and the references therein).

As we have seen, the generalizations analyzed so far in this section are not linked to fractional equations; on the other hand, in the following case, we consider crossing probabilities governed again by fractional equations. Let $B_\alpha^{\mathrm{el}}(t),\ t \geq 0$, be the so-called *elastic Brownian motion* with absorbing rate $\alpha > 0$ (see [3] and [12]), defined as

$$B_\alpha^{\mathrm{el}}(t) = \begin{cases} |B(t)|, & t < T_\alpha, \\ 0, & t \geq T_\alpha, \end{cases} \tag{2.23}$$

where T_α is a random time with distribution

$$\Pr\{T_\alpha > t \mid \mathcal{B}_t\} = \mathrm{e}^{-\alpha L(0,t)}, \qquad \alpha > 0,$$

$\mathcal{B}_t = \sigma\{B(s),\ s \leq t\}$ is the natural filtration, and $L(0,t) = \lim_{\varepsilon\downarrow 0}(1/2\varepsilon)\mathrm{meas}\{s \leq t\colon |B(t)| < \varepsilon\}$ is the local time in the origin of B. It is well known that its distribution can be expressed as

$$q_\alpha^{\mathrm{el}}(s,t) = 2\mathrm{e}^{\alpha s}\int_s^{+\infty} w\mathrm{e}^{-\alpha w}\frac{\mathrm{e}^{-w^2/2t}}{\sqrt{2\pi t^3}}\,\mathrm{d}w + q_\alpha(t)\delta(s), \qquad s,t \geq 0, \tag{2.24}$$

where $\delta(s)$ is Dirac's delta function with the pole at the origin and

$$q_\alpha(t) = 1 - \Pr\{B_\alpha^{\mathrm{el}}(t) > 0\} = 1 - 2\mathrm{e}^{\alpha^2 t/2}\int_{\alpha\sqrt{t}}^{+\infty}\frac{\mathrm{e}^{-w^2/2}}{\sqrt{2\pi}}\,\mathrm{d}w$$

is the probability that the process is absorbed by the barrier at 0 up to time t. Thus, we define the crossing probability of an exponential boundary U for the process B_α^{el} as

$$\psi_\alpha^{\mathrm{el}}(t) = \Pr\{B_\alpha^{\mathrm{el}}(t) < U\} = \int_0^\infty \mathrm{e}^{-\lambda s} q_\alpha^{\mathrm{el}}(s,t)\,\mathrm{d}s. \tag{2.25}$$

Theorem 2.3. *Let U be a random boundary, exponentially distributed with parameter $\lambda > 0$. Then the crossing probability of U for the random process $B_\alpha^{\mathrm{el}}(t)$ with transition density $q_\alpha^{\mathrm{el}}(s,t)$*

is given by

$$\psi_\alpha^{\mathrm{el}}(t) = \Pr\{B_\alpha^{\mathrm{el}}(t) < U\} = 1 - \frac{\lambda}{\lambda-\alpha}\bigg[E_{1/2,1}\bigg(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\bigg) - E_{1/2,1}\bigg(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\bigg)\bigg] \tag{2.26}$$

for any $\lambda \neq \alpha$ *and by*

$$\psi_\lambda^{\mathrm{el}}(t) = \Pr\{B_\lambda^{\mathrm{el}}(t) < U\} = 1 - \lambda\sqrt{2t}\,E_{1/2,1/2}\bigg(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\bigg) \tag{2.27}$$

for $\alpha = \lambda$*. The crossing probability* $\psi_\alpha^{\mathrm{el}}(t)$ *satisfies the following fractional differential equation for any* $\alpha, \lambda > 0$*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_\alpha^{\mathrm{el}} + \frac{\alpha+\lambda}{\sqrt{2}}\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\psi_\alpha^{\mathrm{el}} = \frac{\alpha\lambda}{2}(1-\psi_\alpha^{\mathrm{el}}) - \frac{\lambda}{\sqrt{2\pi t}}, \qquad \psi_\alpha^{\mathrm{el}}(0) = 1. \tag{2.28}$$

Proof. We take the Laplace transform of (2.25), which reads, for any $\alpha, \lambda > 0$,

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\eta t}\psi_\alpha^{\mathrm{el}}(t)\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-\eta t}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-\lambda s} q_\alpha^{\mathrm{el}}(s,t)\,\mathrm{d}s \\
&= 2\int_0^\infty \mathrm{e}^{-\eta t}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-\lambda s+\alpha s}\,\mathrm{d}s \int_s^{+\infty} w\mathrm{e}^{-\alpha w}\frac{\mathrm{e}^{-w^2/2t}}{\sqrt{2\pi t^3}}\,\mathrm{d}w \\
&\quad + \frac{1}{\eta} - 2\int_0^\infty \mathrm{e}^{-\eta t+\alpha^2 t/2}\,\mathrm{d}t \int_{\alpha\sqrt{t}}^{+\infty}\frac{\mathrm{e}^{-w^2/2}}{\sqrt{2\pi}}\,\mathrm{d}w \\
&= 2\int_0^\infty \mathrm{e}^{-\lambda s+\alpha s}\,\mathrm{d}s \int_s^{+\infty}\mathrm{e}^{-(\alpha+\sqrt{2\eta})w}\,\mathrm{d}w + \frac{1}{\eta} \\
&\quad - \frac{2}{2\eta-\alpha^2} + \frac{2\alpha}{\sqrt{2\pi}(2\eta-\alpha^2)}\frac{1}{\sqrt{\eta}}\int_0^{+\infty}\mathrm{e}^{-z}\frac{1}{\sqrt{z}}\,\mathrm{d}z \\
&= \frac{2}{\sqrt{2\eta}+\alpha}\int_0^\infty \mathrm{e}^{-\lambda s-\sqrt{2\eta}s}\,\mathrm{d}s + \frac{2\eta-\alpha^2-2\eta+\sqrt{2\eta}\alpha}{\eta(2\eta-\alpha^2)} \\
&= \frac{2}{(\sqrt{2\eta}+\alpha)(\sqrt{2\eta}+\lambda)} + \frac{\alpha(\sqrt{2\eta}-\alpha)}{\eta(2\eta-\alpha^2)} \\
&= \frac{\alpha\lambda\eta^{-1}+\sqrt{2}\alpha\eta^{-1/2}+2}{(\sqrt{2\eta}+\alpha)(\sqrt{2\eta}+\lambda)}.
\end{aligned} \tag{2.29}$$

We can check that (2.29) coincides with the Laplace transform of (2.26) for $\alpha \neq \lambda$, i.e.

$$\begin{aligned}
\mathcal{L}\{\psi_\alpha^{\mathrm{el}};\eta\} &= \int_0^\infty \mathrm{e}^{-\eta t}\psi_\alpha^{\mathrm{el}}(t)\,\mathrm{d}t \\
&= \frac{1}{\eta} - \frac{\lambda}{\lambda-\alpha}\sum_{j=0}^\infty \frac{1}{\Gamma(j/2+1)}\bigg[\bigg(-\frac{\alpha}{\sqrt{2}}\bigg)^j - \bigg(-\frac{\lambda}{\sqrt{2}}\bigg)^j\bigg]\int_0^\infty \mathrm{e}^{-\eta t}t^{j/2}\,\mathrm{d}t \\
&= \frac{1}{\eta} - \frac{\lambda}{\lambda-\alpha}\frac{1}{\eta}\sum_{j=0}^\infty\bigg[\bigg(-\frac{\alpha}{\sqrt{2\eta}}\bigg)^j - \bigg(-\frac{\lambda}{\sqrt{2\eta}}\bigg)^j\bigg] \\
&= \frac{1}{\eta} - \frac{\lambda}{\lambda-\alpha}\frac{1}{\eta}\bigg(\frac{\sqrt{2\eta}}{\sqrt{2\eta}+\alpha} - \frac{\sqrt{2\eta}}{\sqrt{2\eta}+\lambda}\bigg),
\end{aligned}$$

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which easily gives (2.29). As a further check of (2.26), it is easy to see that, for $\alpha = 0$ (in the case of no absorption), it reduces to $\psi_{1/2}(t) = E_{1/2,1}(-\lambda\sqrt{t})$, since in this case $B^{\mathrm{el}}(t) = |B(t)|$, $t \geq 0$.

For $\alpha = \lambda$, the Laplace transform (2.29) becomes

$$\int_0^\infty \mathrm{e}^{-\eta t}\psi_\lambda^{\mathrm{el}}(t)\,\mathrm{d}t = \frac{\lambda^2\eta^{-1} + \sqrt{2}\lambda\eta^{-1/2} + 2}{(\sqrt{2\eta} + \lambda)^2}. \tag{2.30}$$

By comparing (2.30) with the formula holding for the Laplace transform of the GML function defined in (2.2) (see [13, p. 47]), i.e.

$$\mathcal{L}\{t^{\gamma-1}E_{\beta,\gamma}^{\delta}(\omega t^\beta); \eta\} = \frac{\eta^{\beta\delta-\gamma}}{(\eta^\beta - \omega)^\delta} \tag{2.31}$$

(where $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\delta) > 0$, and $\eta > |\omega|^{1/\mathrm{Re}(\beta)}$), we easily obtain

$$\psi_\lambda^{\mathrm{el}}(t) = 1 - \frac{\lambda\sqrt{t}}{\sqrt{2}} E_{1/2,3/2}^{2}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right), \tag{2.32}$$

which can be also rewritten as (2.27).

By taking the Laplace transform of (2.26) and considering the well-known expression for the Laplace transform of the Caputo derivative, i.e.

$$\mathcal{L}\left\{\frac{\mathrm{d}^\nu u}{\mathrm{d}t^\nu}; \eta\right\} = \int_0^\infty \mathrm{e}^{-\eta t}\frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu}u(t)\,\mathrm{d}t = \eta^\nu\mathcal{L}\{u;\eta\} - \sum_{r=0}^{m-1}\eta^{\nu-r-1}\frac{\mathrm{d}^r}{\mathrm{d}t^r}u(t)\bigg|_{t=0}, \tag{2.33}$$

we obtain

$$\begin{aligned}\eta\mathcal{L}\{\psi_\alpha^{\mathrm{el}};\eta\} - \psi_\alpha^{\mathrm{el}}(0) + \frac{\alpha+\lambda}{\sqrt{2}}\eta^{1/2}\mathcal{L}\{\psi_\alpha^{\mathrm{el}};\eta\} - \frac{\alpha+\lambda}{\sqrt{2}}\eta^{-1/2}\psi_\alpha^{\mathrm{el}}(0)&\\ = \frac{\alpha\lambda}{2}\left(\frac{1}{\eta} - \mathcal{L}\{\psi_\alpha^{\mathrm{el}};\eta\}\right) - \frac{\lambda\Gamma(1/2)}{\sqrt{2\pi\eta}}.&\end{aligned} \tag{2.34}$$

By taking into account the initial condition $\psi_\alpha^{\mathrm{el}}(0) = 1$, the solution of (2.34) coincides with (2.29).

In order to study the asymptotics of the solution $\psi_\alpha^{\mathrm{el}}(t)$ for $\alpha \neq \lambda$, we use the integral expansion for the Mittag-Leffler function (2.14), so that we obtain

$$\psi_\alpha^{\mathrm{el}}(t) = 1 - \frac{\lambda}{\lambda-\alpha}\frac{1}{\pi}\int_0^{+\infty} z^{-1/2}\mathrm{e}^{-z}\left[\frac{\alpha/\sqrt{2}}{z/\sqrt{t} + \alpha^2\sqrt{t}/2} - \frac{\lambda/\sqrt{2}}{z/\sqrt{t} + \lambda^2\sqrt{t}/2}\right]\mathrm{d}z. \tag{2.35}$$

Therefore, the limiting behavior of the crossing probability reads

$$\psi_\alpha^{\mathrm{el}}(t) \simeq \begin{cases} 1 - \dfrac{\lambda\sqrt{2t}}{\sqrt{\pi}}, & 0 < t \ll 1, \\[2ex] 1 - \dfrac{\sqrt{2}}{\alpha\sqrt{\pi t}}, & t \to \infty, \end{cases} \tag{2.36}$$

where the first line is obtained from (2.35) as follows:

$$\begin{aligned}\psi^{\mathrm{el}}_{\alpha}(t) &= 1+\frac{\lambda\sqrt{t}}{\sqrt{2}\pi}\int_0^{+\infty} z^{-3/2}\mathrm{e}^{-z}\,\mathrm{d}z\\ &= 1+\frac{\lambda\sqrt{t}}{\sqrt{2}\pi}\Gamma\Big(-\frac{1}{2}\Big)\\ &= 1-\frac{\lambda\sqrt{2t}}{\sqrt{\pi}} \quad \text{(by the reflection formula of the gamma function)}.\end{aligned}$$

Thus, in this case, the crossing probability maintains a limiting behavior similar to the previous ones for $t\to 0$, but is drastically different for $t\to\infty$ (see (2.15)). In the last case, instead of tending to 0, it tends to 1: this can be intuitively explained by noting that the absorbing effect is stronger as t increases and, in the limit, the process B^{el} will be absorbed with probability 1. This effect is directly correlated with the absorbing rate α. Thus, it is evident from (2.36) that $\psi^{\mathrm{el}}_{\alpha}$ loses the usual property of complete monotonicity that characterizes the standard and also the fractional relaxations (see, e.g. [19]).

In the case $\alpha=\lambda$ we must apply the integral expansion of GML functions (see [1]),

$$E^k_{\nu,\beta}(-ct^{\nu}) = \frac{t^{1-\beta}}{2\pi\mathrm{i}}\int_0^{\infty}\mathrm{e}^{-rt}r^{\nu k-\beta}\bigg[\frac{\mathrm{e}^{\mathrm{i}\pi\beta}}{(r^{\nu}+c\mathrm{e}^{\mathrm{i}\pi\nu})^k}-\frac{\mathrm{e}^{-\mathrm{i}\pi\beta}}{(r^{\nu}+c\mathrm{e}^{-\mathrm{i}\pi\nu})^k}\bigg]\mathrm{d}r \tag{2.37}$$

(for $k=2$, $\nu=\frac{1}{2}$, $\beta=\frac{3}{2}$, and $c=\lambda/\sqrt{2}$), so that (2.32) can be developed as

$$\begin{aligned}\psi^{\mathrm{el}}_{\lambda}(t) &= 1+\frac{\lambda}{\sqrt{2}}\frac{1}{2\pi}\int_0^{\infty}\frac{\mathrm{e}^{-rt}r^{-1/2}}{(r+\lambda^2/2)^2}\bigg[\Big(\sqrt{r}-\frac{\mathrm{i}\lambda}{\sqrt{2}}\Big)^2+\Big(\sqrt{r}+\frac{\mathrm{i}\lambda}{\sqrt{2}}\Big)^2\bigg]\mathrm{d}r\\ &= 1+\frac{\lambda}{\sqrt{2t}}\frac{1}{\pi}\int_0^{\infty}\mathrm{e}^{-z}z^{-1/2}\frac{z/t-\lambda^2/2}{(z/t+\lambda^2/2)^2}\,\mathrm{d}z.\end{aligned}$$

Therefore, also for $\alpha=\lambda$, the asymptotic behavior is given exactly by (2.36).

Remark 2.1. An interesting relation can be found between the crossing probabilities $\psi^{\mathrm{el}}_{\alpha}(t)$ and $\psi_{1/2}(t)$: for $\lambda\neq\alpha$, $\psi^{\mathrm{el}}_{\alpha}(t)$ can be rewritten, in view of (2.26) and (2.11), as

$$\psi^{\mathrm{el}}_{\alpha}(t) = 1-\frac{\lambda}{\lambda-\alpha}[\psi^{\alpha}_{1/2}(t)-\psi^{\lambda}_{1/2}(t)],$$

where $\psi^{\alpha}_{1/2}(t)$ and $\psi^{\lambda}_{1/2}(t)$ denote the crossing probabilities $\Pr\{|B(t)|<U\}$ of an exponential boundary U of parameters α and λ, respectively, for Brownian motion. Thus, the identity

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\psi^{\mathrm{el}}_{\alpha} = -\frac{\lambda}{\lambda-\alpha}\bigg[\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\psi^{\alpha}_{1/2}(t)-\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\psi^{\lambda}_{1/2}(t)\bigg] = \frac{\lambda}{\lambda-\alpha}\bigg[\frac{\alpha}{\sqrt{2}}\psi^{\alpha}_{1/2}(t)-\frac{\lambda}{\sqrt{2}}\psi^{\lambda}_{1/2}(t)\bigg],$$

is also verified for the corresponding differential equations by applying Theorem 2.1 for $\nu=\frac{1}{2}$.

2.3. Crossing probabilities of a gamma-distributed boundary

We extend the previous results by considering the crossing probabilities of a random boundary, distributed with different laws, instead of an exponential boundary. In particular, we choose its natural generalization, i.e. the gamma distribution. Thus, we consider the probability, which extends (1.6),

$$\psi^k_{1/2}(t) = \Pr\{|B(t)| < G\} = \int_0^\infty [1 - F_G(y)] \frac{\mathrm{e}^{-y^2/4t}}{\sqrt{\pi t}}\,\mathrm{d}y, \tag{2.38}$$

where G is a gamma random variable (RV) with parameters $\lambda, k > 0$ and F_G denotes its cumulative distribution function. For our convenience, we write the latter as

$$F_G(y) = \frac{\lambda^k}{\Gamma(k)} \int_0^y \mathrm{e}^{-\lambda z} z^{k-1}\,\mathrm{d}z = \frac{(\lambda y)^k}{\Gamma(k)} \sum_{j=0}^\infty \frac{(-\lambda y)^j}{j!\,(j+k)}.$$

Theorem 2.4. *Let G be a gamma-distributed random boundary with parameters $\lambda, k > 0$. Then the crossing probability of G by a standard Brownian motion is given by*

$$\psi^k_{1/2}(t) = \Pr\{|B(t)| < G\} = 1 - (\lambda\sqrt{t})^k E^k_{1/2,k/2+1}(-\lambda\sqrt{t}), \tag{2.39}$$

which satisfies the fractional relaxation equation

$$\sum_{j=1}^k \binom{k}{j} \lambda^{-j} \frac{\mathrm{d}^{j/2}}{\mathrm{d}t^{j/2}} \psi^k_{1/2}(t) = -\psi^k_{1/2}(t), \tag{2.40}$$

with initial condition $\psi^k_{1/2}(0) = 1$ for $k \geq 1$ and the additional conditions

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r} \psi^k_{1/2}(t)\bigg|_{t=0} = 0, \qquad r = 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor \text{ for any odd } k > 1,$$

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r} \psi^k_{1/2}(t)\bigg|_{t=0} = 0, \qquad r = 1, \dots, \frac{k}{2} - 1 \text{ for any even } k > 2.$$

Proof. We can rewrite (2.38) as

$$\begin{aligned}
\psi^k_{1/2}(t) &= \int_0^\infty \left(1 - \frac{(\lambda y)^k}{\Gamma(k)} \sum_{j=0}^\infty \frac{(-\lambda y)^j}{j!\,(j+k)}\right) \frac{\mathrm{e}^{-y^2/4t}}{\sqrt{\pi t}}\,\mathrm{d}y \\
&= 1 - \frac{1}{\Gamma(k)\sqrt{\pi t}} \sum_{j=0}^\infty \frac{(-1)^j \lambda^{j+k}}{j!\,(j+k)} \int_0^\infty y^{j+k} \mathrm{e}^{-y^2/4t}\,\mathrm{d}y \\
&= 1 - \frac{1}{\Gamma(k)\sqrt{\pi}} \sum_{j=0}^\infty \frac{(-1)^j (2\lambda\sqrt{t})^{j+k}}{j!\,(j+k)} \Gamma\left(\frac{j}{2} + \frac{k}{2} + \frac{1}{2}\right) \\
&= 1 - \frac{2}{\Gamma(k)} \sum_{j=0}^\infty \frac{(-1)^j (\lambda\sqrt{t})^{j+k}}{j!\,(j+k)} \frac{\Gamma(j+k)}{\Gamma(j/2 + k/2)} \\
&= 1 - \frac{(\lambda\sqrt{t})^k}{\Gamma(k)} \sum_{j=0}^\infty \frac{\Gamma(j+k)(-\lambda\sqrt{t})^j}{j!\,\Gamma(j/2 + k/2 + 1)}.
\end{aligned} \tag{2.41}$$

If we now assume that k is an integer, we can recognize in (2.41) the GML function (2.2), so that we obtain (2.39). As a further check, it is easy to ascertain that, in the special case

$k=1$ (where the RV G reduces to the exponential RV U), the crossing probability $\psi^k_{1/2}$ given in (2.39) coincides with the fractional relaxation $\psi_{1/2}$ in (2.11):

$$\begin{aligned}\psi^k_{1/2}(t) &= 1-\lambda\sqrt{t}E_{1/2,3/2}(-\lambda\sqrt{t})\\ &= 1+\sum_{l=1}^{\infty}\frac{(-\lambda\sqrt{t})^l}{\Gamma(l/2+1)}\\ &= E_{1/2,1}(-\lambda\sqrt{t})\\ &= \psi_{1/2}(t).\end{aligned}$$

In order to derive (2.40), we resort to the Laplace transform of (2.39), which reads

$$\mathcal{L}\{\psi^k_{1/2};\eta\}=\frac{(\sqrt{\eta}+\lambda)^k-\lambda^k}{\eta(\sqrt{\eta}+\lambda)^k}, \tag{2.42}$$

by again applying formula (2.31) for $\gamma=k/2+1$, $\beta=\frac{1}{2}$, and $\delta=k$. We now rewrite (2.42) as

$$\sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}[\eta^{j/2}\mathcal{L}\{\psi^k_{1/2};\eta\}-\eta^{j/2-1}]=-\frac{\lambda^k}{\eta}. \tag{2.43}$$

By simplifying this expression, we obtain the Laplace transform of (2.40). We can check that the initial conditions are satisfied by using the series expression of $E^k_{\nu,\beta}(-ct^\nu)$, and noting that, for $t=0$, $E^k_{\nu,\beta}(-\lambda\sqrt{t})=1/\Gamma(\beta)$; thus, we obtain

$$\psi^k_{1/2}(t)\big|_{t=0}=1-\frac{(\lambda\sqrt{t})^k}{\Gamma(k/2+1)}\bigg|_{t=0}=1.$$

For the other conditions, we can apply the formula for the rth-order derivative of a GML function (see Equation (1.9.6), of [13, p. 46]),

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}[z^{\beta-1}E^\rho_{\alpha,\beta}(\lambda z^\alpha)]=z^{\beta-r-1}E^\rho_{\alpha,\beta-r}(\lambda z^\alpha),\qquad \lambda\in\mathbb{C},\ r\in\mathbb{N}, \tag{2.44}$$

so that we obtain

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r}\psi^k_{1/2}(t)=-\lambda^k t^{k/2-r}E^k_{1/2,k/2-r+1}(-\lambda\sqrt{t}),\qquad r\in\mathbb{N}. \tag{2.45}$$

By recalling (2.33), we note that the Laplace form (2.43) holds if the derivatives of order r of $\psi^k_{1/2}$ vanish for $r=1,\dots,\lfloor k/2\rfloor$ if $k>1$ is odd and for $r=1,\dots,k/2-1$ if $k>2$ is even; this is verified by (2.45).

Finally, we check that (2.40) becomes, for $k=1$, the fractional relaxation equation $\mathrm{d}^{1/2}\psi_{1/2}(t)/\mathrm{d}t^{1/2}=-\lambda\psi_{1/2}(t)$.

Remark 2.2. By comparing (2.39) with the results in [4], we can deduce that the crossing probability $\psi^k_{1/2}(t)$ can be written in terms of the fractional Poisson process of order $\nu=\frac{1}{2}$ as

$$\psi^k_{1/2}(t)=\Pr\{T_k>t\}=\Pr\{\mathcal{N}_{1/2}(t)<k\}, \tag{2.46}$$

where $T_k=\inf\{t\geq 0\colon \mathcal{N}_{1/2}(t)=k\}$ is the waiting probability of the kth event. On the other hand, we can prove that the following relationship holds between the crossing probabilities given in (2.38) for a gamma boundary of parameters (λ, k) and $(\lambda, k-1)$ (respectively denoted as $\psi^k_{1/2}(t)$ and $\psi^{k-1}_{1/2}(t)$):

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\psi^k_{1/2}(t) = -\lambda[\psi^k_{1/2}(t) - \psi^{k-1}_{1/2}(t)]. \tag{2.47}$$

Indeed, we can evaluate the fractional derivative of order $\frac{1}{2}$ of $\psi^k_{1/2}$ by considering (2.45):

$$\begin{aligned}\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\psi^k_{1/2}(t) &= -\frac{\lambda^k}{\sqrt{\pi}(k-1)!}\sum_{j=0}^{\infty}\frac{(j+k-1)!(-\lambda)^j}{j!\,\Gamma(j/2+k/2)}\int_0^t (t-s)^{-1/2}s^{k/2+j/2-1}\,\mathrm{d}s \\ &= -\lambda^k t^{k/2-1/2}E^k_{1/2,k/2+1/2}(-\lambda\sqrt{t}).\end{aligned} \tag{2.48}$$

By applying the recursive formula for the GML function proved in [4], i.e.

$$x^n E^m_{\nu,n\nu+z}(-x) + x^{n+1}E^m_{\nu,(n+1)\nu+z}(-x) = x^n E^{m-1}_{\nu,n\nu+z}(-x), \qquad n, m > 0,\ z \geq 0,\ x > 0,$$

for $m = n = k$, $x = -\lambda\sqrt{t}$, $\nu = \frac{1}{2}$, and $z = \frac{1}{2}$ to (2.48), we obtain

$$\begin{aligned}\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\psi^k_{1/2}(t) &= -t^{-1/2}(\lambda^k t^{k/2}E^k_{1/2,k/2+1/2}(-\lambda\sqrt{t})) \\ &= -t^{-1/2}[\lambda^k t^{k/2}E^{k-1}_{1/2,k/2+1/2}(-\lambda\sqrt{t}) - \lambda^{k+1}t^{(k+1)/2}E^k_{1/2,k/2+1}(-\lambda\sqrt{t})] \\ &= -\lambda^k t^{k/2-1/2}E^{k-1}_{1/2,k/2+1/2}(-\lambda\sqrt{t}) + \lambda(1-\psi^k_{1/2}(t)),\end{aligned}$$

which gives (2.47). The latter could alternatively be obtained by observing that

$$p_k^{1/2}(t) = \Pr\{\mathcal{N}_{1/2}(t) = k\} = \psi^k_{1/2}(t) - \psi^{k-1}_{1/2}(t)$$

satisfies (2.1) with $\nu = \frac{1}{2}$ and taking into account (2.46).

The asymptotic behavior of the crossing probability $\psi^k_{1/2}$ for small t can be deduced by the series expression of the GML function,

$$E^k_{\nu,\beta}(-ct^\nu) \simeq \frac{1}{\Gamma(\beta)} - \frac{ct^\nu k}{\Gamma(\beta+\nu)}, \qquad 0 < t \ll 1, \tag{2.49}$$

so that we obtain

$$\psi^k_{1/2}(t) \simeq 1 - \frac{(\lambda\sqrt{t})^k}{\Gamma(k/2+1)}. \tag{2.50}$$

The same result can be obtained by resorting to the Laplace transform and to the Tauberian theory, which allows us to infer (formally) the asymptotic behavior of a function $f(t)$ for $t \to \infty$ and $t \to 0^+$ from the limiting behavior of its Laplace transform $\mathcal{L}\{f;\eta\}$ for $\eta \to 0^+$ and $\eta \to \infty$, respectively (see also [19] for details). To this end, we rewrite (2.42) as

$$\mathcal{L}\{\psi^k_{1/2};\eta\} = \frac{1}{\eta} - \frac{\lambda^k}{\eta(\sqrt{\eta}+\lambda)^k},$$

which, for $\eta \to \infty$, can be approximated as

$$\mathcal{L}\{\psi^k_{1/2};\eta\} = \frac{1}{\eta} - \frac{\lambda^k}{\eta^{k/2+1}} + o(\eta^{-k/2-1}),$$

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so we again obtain (2.50). For $t \to \infty$, it is worth writing (2.42) as

$$\mathcal{L}\{\psi_{1/2}^k; \eta\} = \frac{\sum_{j=1}^k \binom{k}{j} \eta^{j/2-1/2} \lambda^{-j}}{\sum_{j=0}^k \binom{k}{j} \eta^{j/2+1/2} \lambda^{-j}} \simeq \frac{k}{\lambda \eta^{1/2}} \quad \text{as } \eta \to 0^+,$$

so that we obtain $\psi_{1/2}^k(t) \simeq k/\lambda\sqrt{\pi t}$. Thus, the limiting behavior of $\psi_{1/2}^k$ can be summarized as

$$\psi_{1/2}^k(t) \simeq \begin{cases} 1 - \dfrac{(\lambda\sqrt{t})^k}{\Gamma(k/2+1)}, & 0 < t \ll 1, \\ \dfrac{k}{\lambda\sqrt{\pi t}}, & t \to +\infty, \end{cases}$$

which, of course, coincides with (2.15) for $k = 1$ and $\nu = \frac{1}{2}$. We can deduce that, while passing from an exponential boundary to a gamma-distributed boundary makes a significant difference for small t, this effect fades away for large t. Indeed, the rate of decrease to 0 for $t \to \infty$ of the crossing probability is exactly the same for any $k \geq 1$.

Analogously, we can generalize the results of Theorem 2.3 by considering the crossing probability of a gamma-distributed boundary for the elastic Brownian motion defined in (2.23).

Theorem 2.5. *Let G be a gamma-distributed random boundary with parameters $\lambda, k > 0$. Then the crossing probability of G, or the random process $B_\alpha^{\mathrm{el}}(t)$ with transition density $q^{\mathrm{el}}(s,t)$ (given in (2.24)) for any $\lambda, \alpha > 0$ is equal to*

$$\psi_{k,\alpha}^{\mathrm{el}}(t) = \Pr\{B_\alpha^{\mathrm{el}}(t) < G\} = 1 - \left(\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^k \sum_{l=0}^{\infty} \left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)^l E_{1/2,(l+k)/2+1}^k\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right), \tag{2.51}$$

which in the particular case $\alpha = \lambda$ reduces to

$$\psi_{k,\lambda}^{\mathrm{el}}(t) = \Pr\{B_\lambda^{\mathrm{el}}(t) < G\} = 1 - \left(\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^k E_{1/2,k/2+1}^{k+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right). \tag{2.52}$$

Proof. By following some steps similar to those in the proof of Theorem 2.3 we can write the Laplace transform of $\psi_{k,\alpha}^{\mathrm{el}}(t)$ as

$$\begin{aligned}
\mathcal{L}\{\psi_{k,\alpha}^{\mathrm{el}}; \eta\} &= \int_0^\infty \mathrm{e}^{-\eta t}\, \mathrm{d}t \int_0^\infty [1 - F_G(s)] q_\alpha^{\mathrm{el}}(s,t)\, \mathrm{d}s \\
&= 2\int_0^\infty [1 - F_G(s)] \mathrm{e}^{\alpha s}\, \mathrm{d}s \int_s^{+\infty} \mathrm{e}^{-(\alpha+\sqrt{2\eta})w}\, \mathrm{d}w + \frac{1}{\eta} \\
&\quad - \frac{2}{2\eta - \alpha^2} + \frac{2\alpha}{\sqrt{2\eta}(2\eta - \alpha^2)} \\
&= \frac{2}{(\sqrt{2\eta}+\alpha)\sqrt{2\eta}} - \frac{2\lambda^2}{\sqrt{2\eta}^{k+1}(\sqrt{2\eta}+\alpha)} \sum_{j=0}^{\infty} \binom{k+j-1}{j} \left(-\frac{\lambda}{\sqrt{2\eta}}\right)^j \\
&\quad + \frac{\alpha(\sqrt{2\eta}-\alpha)}{\eta(2\eta-\alpha^2)} \\
&= \frac{2(\sqrt{2\eta}+\lambda)^k - \lambda^k}{\sqrt{2\eta}(\sqrt{2\eta}+\alpha)(\sqrt{2\eta}+\lambda)^k} + \frac{\alpha}{\eta(\sqrt{2\eta}+\alpha)} \\
&= \frac{1}{\eta} - \frac{\sqrt{2}\lambda^k}{\sqrt{\eta}(\sqrt{2\eta}+\alpha)(\sqrt{2\eta}+\lambda)^k}.
\end{aligned} \tag{2.53}$$

We can invert (2.53) by again applying (2.31), i.e.

$$\begin{aligned}
\psi_{k,\alpha}^{\mathrm{el}}(t) &= 1-\sqrt{2}\lambda^k \mathcal{L}\bigg\{\frac{1}{(\sqrt{2\eta}+\alpha)}\frac{\eta^{-1/2}}{(\sqrt{2\eta}+\lambda)^k};t\bigg\} \\
&= 1-\bigg(\frac{\lambda}{\sqrt{2}}\bigg)^k \int_0^t (t-s)^{-1/2} E_{1/2,1/2}\bigg(-\frac{\alpha\sqrt{t-s}}{\sqrt{2}}\bigg) s^{k/2-1/2} \\
&\qquad\qquad \times E_{1/2,k/2+1/2}^k\bigg(-\frac{\lambda\sqrt{s}}{\sqrt{2}}\bigg)\,\mathrm{d}s \\
&= 1-\bigg(\frac{\lambda}{\sqrt{2}}\bigg)^k \sum_{l=0}^{\infty}\frac{(-\alpha/\sqrt{2})^l}{\Gamma(l/2+1/2)}\sum_{j=0}^{\infty}\frac{(k+j-1)!\,(-\lambda/\sqrt{2})^j}{(k-1)!\,j!\,\Gamma(j/2+(k+1)/2)} \\
&\qquad\qquad \times \int_0^t (t-s)^{l/2-1/2} s^{(k-1)/2+j/2}\,\mathrm{d}s,
\end{aligned}$$

which, after some simplifications, coincides with (2.51). For $\alpha=\lambda$, we can rewrite the latter as

$$\begin{aligned}
\psi_{k,\alpha}^{\mathrm{el}}(t) &= 1-\bigg(\frac{\lambda\sqrt{t}}{\sqrt{2}}\bigg)^k \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{(k+j-1)!\,(-\lambda\sqrt{t}/\sqrt{2})^{j+l}}{(k-1)!\,j!\,\Gamma(j/2+(l+k)/2+1)} \\
&= 1-\bigg(\frac{\lambda\sqrt{t}}{\sqrt{2}}\bigg)^k \sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\frac{(k+m-l-1)!\,(-\lambda\sqrt{t}/\sqrt{2})^{m}}{(k-1)!\,(m-l)!\,\Gamma(m/2+k/2+1)} \\
&= 1-\bigg(\frac{\lambda\sqrt{t}}{\sqrt{2}}\bigg)^k \sum_{m=0}^{\infty}\frac{(-\lambda\sqrt{t}/\sqrt{2})^{m}}{\Gamma(m/2+k/2+1)}\sum_{l=0}^{m}\binom{k+m-l-1}{m-l} \\
&= 1-\bigg(\frac{\lambda\sqrt{t}}{\sqrt{2}}\bigg)^k \sum_{m=0}^{\infty}\frac{(-\lambda\sqrt{t}/\sqrt{2})^{m}}{\Gamma(m/2+k/2+1)}\binom{k+m}{k} \\
&\qquad\qquad\qquad\qquad \text{(by the identity proved in [5, p. 10])} \\
&= \psi_{k,\lambda}^{\mathrm{el}}(t).
\end{aligned}$$

As a final check, we can ascertain that, for $k=1$, (2.51) and (2.52) reduce to the corresponding expressions given for the exponential case in (2.26) and (2.27), respectively; indeed, (2.51) can be rewritten, for $k=1$, as

$$\begin{aligned}
\psi_{1,\alpha}^{\mathrm{el}}(t) &= 1+\sum_{l=0}^{\infty}\bigg(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\bigg)^l \sum_{j=0}^{\infty}\frac{(-\lambda\sqrt{t}/\sqrt{2})^{j+1}}{\Gamma((j+1)/2+l/2+1)} \\
&= 1-\sum_{l=0}^{\infty}\frac{(-\alpha\sqrt{t}/\sqrt{2})^{l}}{\Gamma(l/2+1)}+\sum_{l=0}^{\infty}\bigg(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\bigg)^l \sum_{m=0}^{\infty}\frac{(-\lambda\sqrt{t}/\sqrt{2})^{m}}{\Gamma((m+l)/2+1)} \\
&= 1-E_{1/2,1}\bigg(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\bigg)+\sum_{l=0}^{\infty}\bigg(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\bigg)^l \sum_{k=l}^{\infty}\frac{(-\lambda\sqrt{t}/\sqrt{2})^{k-l}}{\Gamma(k/2+1)} \\
&= 1-E_{1/2,1}\bigg(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\bigg)+\sum_{k=0}^{\infty}\frac{(-\lambda\sqrt{t}/\sqrt{2})^{k}}{\Gamma(k/2+1)}\sum_{l=0}^{k}\bigg(\frac{\alpha}{\lambda}\bigg)^l,
\end{aligned}$$

which coincides with (2.26). Formula (2.52) immediately reduces to expression (2.32) for $k = 1$.

Finally, putting $\alpha = 0$ and substituting $\lambda/\sqrt{2}$ with λ, (2.51) coincides with the corresponding crossing probability (2.39), which was obtained in the case of free Brownian motion (with no absorption).

The asymptotic behavior of $\psi^{\mathrm{el}}_{k,\lambda}$ for small t can be derived from (2.52) by again applying (2.49). Alternatively, we can use the Laplace transform (2.53), which can be approximated as

$$\mathcal{L}\{\psi^{\mathrm{el}}_{k,\alpha}; \eta\} \simeq \frac{1}{\eta} - \frac{\lambda^k}{2^{k/2}\eta^{k/2+1}} \quad \text{for } \eta \to \infty.$$

In both methods we obtain the first line of the following formula:

$$\psi^{\mathrm{el}}_{k,\lambda}(t) \simeq \begin{cases} 1 - \left(\dfrac{\lambda\sqrt{t}}{\sqrt{2}}\right)^k \dfrac{1}{\Gamma(k/2+1)}, & 0 < t \ll 1, \\ 1 - \dfrac{\sqrt{2}}{\alpha\sqrt{\pi t}}, & t \to +\infty. \end{cases} \tag{2.54}$$

The second line is obtained from (2.53), which can be rewritten as

$$\mathcal{L}\{\psi^{\mathrm{el}}_{k,\alpha}; \eta\} = \frac{1}{\eta} - \frac{\sqrt{2}}{\sqrt{\eta}[\sum_{j=0}^{k}\binom{k}{j}(2\eta)^{j/2+1/2}\lambda^{-j} + \alpha\sum_{j=0}^{k}\binom{k}{j}(2\eta)^{j/2}\lambda^{-j}]} \simeq \frac{1}{\eta} - \frac{\sqrt{2}}{\alpha\sqrt{\eta}}$$

as $\eta \to 0^+$. For $k = 1$, (2.54) coincides with (2.36), as expected. We finally note that, also in this case, as for the Brownian motion, the leading term in the expression obtained for $t \to \infty$ does not depend on k and, thus, for large values of t, considering an exponential or gamma-distributed boundary does not entail any consequence.

The fractional equations satisfied by the crossing probabilities obtained above can be derived by properly rewriting the Laplace transform in (2.53), as the following theorem shows.

Theorem 2.6. *The crossing probability $\psi^{\mathrm{el}}_{k,\alpha}$ given in (2.51) satisfies, for any $\lambda, \alpha > 0$, the fractional equation*

$$\begin{aligned} &\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\sqrt{2}}{\lambda}\right)^j \frac{\mathrm{d}^{j/2+1/2}}{\mathrm{d}t^{j/2+1/2}}\psi^{\mathrm{el}}_{k,\alpha} + \frac{\alpha}{\sqrt{2}}\sum_{j=1}^{k}\binom{k}{j}\left(\frac{\sqrt{2}}{\lambda}\right)^j \frac{\mathrm{d}^{j/2}}{\mathrm{d}t^{j/2}}\psi^{\mathrm{el}}_{k,\alpha} \\ &\qquad = \frac{\alpha}{\sqrt{2}}(1 - \psi^{\mathrm{el}}_{k,\alpha}) - \frac{c_k}{\sqrt{\pi t}}, \end{aligned} \tag{2.55}$$

where $c_k = 1$ for odd k and $c_k = 0$ for even k. The initial conditions are $\psi^{\mathrm{el}}_{k,\alpha}(0) = 1$ for any $k \geq 1$ and

$$\begin{aligned} \frac{\mathrm{d}^r}{\mathrm{d}t^r}\psi^{\mathrm{el}}_{k,\alpha}(t)\Big|_{t=0} &= 0, \qquad r = 1, \ldots, \frac{k-1}{2} \text{ for odd } k > 1, \\ \frac{\mathrm{d}^r}{\mathrm{d}t^r}\psi^{\mathrm{el}}_{k,\alpha}(t)\Big|_{t=0} &= 0, \qquad r = 1, \ldots, \frac{k}{2} - 1 \text{ for even } k > 1. \end{aligned} \tag{2.56}$$

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Proof. We rewrite (2.53) as

$$\begin{aligned}\mathcal{L}\{\psi^{\mathrm{el}}_{k,\alpha};\eta\}\eta(\sqrt{2\eta}+\alpha)\sum_{j=0}^{k}\binom{k}{j}2^{j/2}\lambda^{k-j}\eta^{j/2} \\ =(\sqrt{2\eta}+\alpha)\sum_{j=0}^{k}\binom{k}{j}2^{j/2}\lambda^{k-j}\eta^{j/2}-\sqrt{2\eta}\lambda^{k},\end{aligned}$$

so that we obtain

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}\bigg(\frac{\sqrt{2}}{\lambda}\bigg)^{j}[\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}\eta^{j/2+1/2}-\eta^{j/2-1/2}]+\frac{\alpha}{\sqrt{2}}\sum_{j=1}^{k}\binom{k}{j}\bigg(\frac{\sqrt{2}}{\lambda}\bigg)^{j}[\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}\eta^{j/2}-\eta^{j/2-1}] \\ =\frac{\alpha}{\sqrt{2}}\bigg[\frac{1}{\eta}-\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}\bigg]-\frac{1}{\sqrt{\eta}},\end{aligned}\tag{2.57}$$

where we have defined $\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}=\mathcal{L}\{\psi^{\mathrm{el}}_{k,\alpha};\eta\}$ for brevity. From the Laplace transform (2.57), taking into account (2.33) and the initial conditions (2.56), we can obtain (2.55) with $c_k=1$. For the initial conditions (2.56), we use an argument similar to that used in the proof of Theorem 2.4, with the only additional caveat that, in the case of even k, the highest order derivative, i.e. $\mathrm{d}^{k/2}\psi^{\mathrm{el}}_{k,\alpha}/\mathrm{d}t^{k/2}$, does not vanish at $t=0$, as can be ascertained by applying (2.44) to (2.51); indeed, we obtain

$$\frac{\mathrm{d}^{k/2}}{\mathrm{d}t^{k/2}}\psi^{\mathrm{el}}_{k,\alpha}(t)\bigg|_{t=0}=-\bigg(\frac{\lambda}{\sqrt{2}}\bigg)^{k}\sum_{l=0}^{\infty}\bigg(-\frac{\alpha}{\sqrt{2}}\bigg)^{l}t^{l/2}E^{k}_{1/2,l/2+1}\bigg(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\bigg)\bigg|_{t=0}=-\bigg(\frac{\lambda}{\sqrt{2}}\bigg)^{k}.$$

Therefore, (2.57), for even k, must be modified as

$$\begin{aligned}&\sum_{j=0}^{k-1}\binom{k}{j}\bigg(\frac{\sqrt{2}}{\lambda}\bigg)^{j}[\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}\eta^{j/2+1/2}-\eta^{j/2-1/2}] \\ &+\bigg(\frac{\sqrt{2}}{\lambda}\bigg)^{k}\bigg[\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}\eta^{k/2+1/2}-\eta^{k/2-1/2}-\frac{1}{\sqrt{\eta}}\frac{\mathrm{d}^{k/2}}{\mathrm{d}t^{k/2}}\psi^{\mathrm{el}}_{k,\alpha}(t)\bigg|_{t=0}\bigg] \\ &+\frac{\alpha}{\sqrt{2}}\sum_{j=1}^{k}\binom{k}{j}\bigg(\frac{\sqrt{2}}{\lambda}\bigg)^{j}[\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}\eta^{j/2}-\eta^{j/2-1}] \\ &\quad=\frac{\alpha}{\sqrt{2}}\bigg[\frac{1}{\eta}-\widetilde{\psi}^{\mathrm{el}}_{k,\alpha}\bigg],\end{aligned}$$

so that we obtain (2.55) with $c_k=0$.

As a further check, it is easy to see that, for $k=1$, the latter reduces to (2.28).

3. Fractional relaxation equation of distributed order

We now consider an extension of the fractional relaxation equation (1.2), obtained by adding the hypothesis that the fractional order ν is not a constant but a random variable with distribution $n(\nu)$. Thus, we will study the *distributed order fractional relaxation equation*, defined as

$$\int_0^1\frac{\mathrm{d}^{\nu}\psi}{\mathrm{d}t^{\nu}}n(\nu)\,\mathrm{d}\nu=-\lambda\psi,\qquad t\geq 0,\tag{3.1}$$

where, by assumption,

$$n(\nu) \geq 0, \qquad \int_0^1 n(\nu)\,\mathrm{d}\nu = 1, \qquad \nu \in (0, 1], \tag{3.2}$$

subject to the initial condition $\psi(0) = 1$. As a special case, for $n(\nu) = \delta(\nu - \bar{\nu})$ and a particular value of $\bar{\nu} \in (0, 1)$, (3.1) reduces to (1.2).

We adopt here the following particular form for the density of the fractional order ν:

$$n(\nu) = n_1\delta(\nu - \nu_1) + n_2\delta(\nu - \nu_2), \qquad 0 < \nu_1 < \nu_2 \leq 1, \tag{3.3}$$

for $n_1, n_2 \geq 0$ and such that $n_1 + n_2 = 1$ (the conditions in (3.2) are trivially fulfilled). The density (3.3) has already been used in [8] and [17], in the analysis of the so-called double-order time-fractional diffusion equation, and corresponds to the case of a subdiffusion with retardation. Moreover, it was applied in [1] in the context of recursive equations of fractional order, where the equation governing the Poisson process was extended by introducing two fractional time derivatives.

Under assumption (3.3), (3.1) becomes

$$n_1\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}t^{\nu_1}}\psi + n_2\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}t^{\nu_2}}\psi = -\lambda\psi, \qquad t \geq 0, \tag{3.4}$$

and the corresponding solution ψ_{ν_1,ν_2} coincides with the so-called *double-order fractional relaxation* studied by Mainardi *et al*. [19], who provided an integral expression and some asymptotic representations for ψ_{ν_1,ν_2}. We present here an analytic form of the fundamental solution to (3.4) in terms of GML functions as well as a probabilistic representation in terms of crossing probabilities, in line with the results of the previous sections.

Theorem 3.1. *The solution to (3.4) with the initial condition $\psi(0) = 1$ can be written as*

$$\psi_{\nu_1,\nu_2}(t) = 1 - \frac{\lambda t^{\nu_2}}{n_2}\sum_{r=0}^{\infty}\biggl(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2}\biggr)^r E^{r+1}_{\nu_2,\nu_2+(\nu_2-\nu_1)r+1}\biggl(-\frac{\lambda t^{\nu_2}}{n_2}\biggr). \tag{3.5}$$

Proof. By taking the Laplace transform of (3.4) we obtain

$$n_1\eta^{\nu_1}\mathcal{L}\{\psi_{\nu_1,\nu_2};\eta\} - n_1\eta^{\nu_1-1} + n_2\eta^{\nu_2}\mathcal{L}\{\psi_{\nu_1,\nu_2};\eta\} - n_2\eta^{\nu_2-1} = -\lambda\mathcal{L}\{\psi_{\nu_1,\nu_2};\eta\},$$

whose solution can be written as

$$\begin{aligned}
\mathcal{L}\{\psi_{\nu_1,\nu_2};\eta\} &= \frac{n_1\eta^{\nu_1} + n_2\eta^{\nu_2}}{\eta(\lambda + n_1\eta^{\nu_1} + n_2\eta^{\nu_2})} \\
&= \frac{1}{\eta} - \frac{\lambda}{\eta}\frac{1}{\lambda + n_2\eta^{\nu_2}}\frac{1}{1 + n_1\eta^{\nu_1}/(\lambda + n_2\eta^{\nu_2})} \\
&= \frac{1}{\eta} - \frac{\lambda}{\eta}\frac{1}{\lambda + n_2\eta^{\nu_2}}\sum_{r=0}^{\infty}\biggl(-\frac{n_1\eta^{\nu_1}}{\lambda + n_2\eta^{\nu_2}}\biggr)^r \\
&= \frac{1}{\eta} - \frac{\lambda}{n_2}\sum_{r=0}^{\infty}\biggl(-\frac{n_1}{n_2}\biggr)^r\frac{\eta^{\nu_1 r-1}}{(\eta^{\nu_2} + \lambda/n_2)^{r+1}}.
\end{aligned}$$

By applying (2.31), we easily obtain (3.5). As a check, we can see that (3.5) reduces to (2.9) for $n_1 = 0$, $n_2 = 1$, and $\nu_2 = \nu$, since (3.4) becomes, in this case, the fractional relaxation equation (1.2).

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Despite the apparent similarity between (3.5) and (2.51), they are deeply different: while the sum is extended to the third (upper) parameter of the GML function for ψ_{ν_1,ν_2}, this is not the case for $\psi^{\mathrm{el}}_{k,\alpha}$. This is also reflected in the asymptotic behavior of the fractional relaxation of distributed order, which does not deviate from the usual relaxation behavior (unlike $\psi^{\mathrm{el}}_{k,\alpha}$). We can study the limit directly from (3.5), by applying (2.37):

$$\psi_{\nu_1,\nu_2}(t) = 1 - \frac{\lambda}{n_2}\sum_{r=0}^{\infty}\Big(-\frac{n_1}{n_2}\Big)^r \frac{1}{2\pi\mathrm{i}}\int_0^{\infty} \mathrm{e}^{-zt} z^{\nu_1 r-1}\bigg[\frac{\mathrm{e}^{-\mathrm{i}\pi\nu_2-\mathrm{i}\pi(\nu_2-\nu_1)r}}{(z^{\nu_2}+\lambda\mathrm{e}^{-\mathrm{i}\pi\nu_2}/n_2)^{r+1}} - \frac{\mathrm{e}^{\mathrm{i}\pi\nu_2+\mathrm{i}\pi(\nu_2-\nu_1)r}}{(z^{\nu_2}+\lambda\mathrm{e}^{\mathrm{i}\pi\nu_2}/n_2)^{r+1}}\bigg]\,\mathrm{d}z.$$

Thus, for $t\to 0$, we obtain

$$\begin{aligned}
\psi_{\nu_1,\nu_2}(t) &= 1 - \frac{\lambda}{n_2}\sum_{r=0}^{\infty}\Big(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2}\Big)^r \frac{t^{\nu_2}}{2\pi\mathrm{i}}\int_0^{\infty} \mathrm{e}^{-w} w^{\nu_1 r-1} \\
&\qquad\times\bigg[\frac{\mathrm{e}^{-\mathrm{i}\pi\nu_2-\mathrm{i}\pi(\nu_2-\nu_1)r}}{(w^{\nu_2}+\lambda t^{\nu_2}\mathrm{e}^{-\mathrm{i}\pi\nu_2}/n_2)^{r+1}} - \frac{\mathrm{e}^{\mathrm{i}\pi\nu_2+\mathrm{i}\pi(\nu_2-\nu_1)r}}{(w+\lambda t^{\nu_2}\mathrm{e}^{\mathrm{i}\pi\nu_2}/n_2)^{r+1}}\bigg]\,\mathrm{d}w \\
&\simeq 1 - \frac{\lambda t^{\nu_2}}{n_2}\sum_{r=0}^{\infty}\Big(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2}\Big)^r \frac{\sin(-\pi(\nu_1 r-\nu_2 r-\nu_2))}{\pi}\Gamma(\nu_1 r-\nu_2 r-\nu_2) \\
&= 1 - \frac{\lambda t^{\nu_2}}{n_2}\sum_{r=0}^{\infty}\Big(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2}\Big)^r \frac{1}{\Gamma(1+\nu_2 r+\nu_2-\nu_1 r)} \\
&\qquad\qquad\text{(by the reflection property of the gamma function)} \\
&= 1 - \frac{\lambda t^{\nu_2}}{n_2}\frac{1}{\Gamma(1+\nu_2)} + o(t^{\nu_2}),
\end{aligned}\tag{3.6}$$

while, for $t\to\infty$, we analogously obtain

$$\begin{aligned}
\psi_{\nu_1,\nu_2}(t) &= 1 - \frac{\lambda}{n_2}\sum_{r=0}^{\infty}\Big(-\frac{n_1}{n_2 t^{\nu_1}}\Big)^r \frac{1}{2\pi\mathrm{i}}\int_0^{\infty} \mathrm{e}^{-w} w^{\nu_1 r-1} \\
&\qquad\times\bigg[\frac{\mathrm{e}^{-\mathrm{i}\pi\nu_2-\mathrm{i}\pi(\nu_2-\nu_1)r}}{((w/t)^{\nu_2}+\lambda\mathrm{e}^{-\mathrm{i}\pi\nu_2}/n_2)^{r+1}} - \frac{\mathrm{e}^{\mathrm{i}\pi\nu_2+\mathrm{i}\pi(\nu_2-\nu_1)r}}{((w/t)^{\nu_2}+\lambda\mathrm{e}^{\mathrm{i}\pi\nu_2}/n_2)^{r+1}}\bigg]\,\mathrm{d}w \\
&\simeq 1 - \sum_{r=0}^{\infty}\Big(-\frac{n_1}{\lambda t^{\nu_1}}\Big)^r \frac{\sin(\pi\nu_1 r)}{\pi}\Gamma(\nu_1 r) \\
&= 1 - \sum_{r=0}^{\infty}\Big(-\frac{n_1}{\lambda t^{\nu_1}}\Big)^r \frac{1}{\Gamma(1-\nu_1 r)} \\
&= \frac{n_1}{\lambda t^{\nu_1}}\frac{1}{\Gamma(1-\nu_1)} + o(t^{-\nu_1}).
\end{aligned}\tag{3.7}$$

The above expression coincides with Equation (4.13) of [19], which were obtained in a different way, directly from the Laplace transform of ψ_{ν_1,ν_2}.

We now present a probabilistic form for the solution ψ_{ν_1,ν_2}, in line with the analysis carried out so far, in terms of the crossing probability of a random boundary for a stochastic process,

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which will be denoted, in this case, by $\mathcal{T}_{\nu_1,\nu_2}(t),\ t \geq 0$. To this end, we will compare (3.4) with the equation governing the probabilities $\widetilde{p}_k$ of the distributed order fractional Poisson process $\mathcal{N}_{\nu_1,\nu_2}(t),\ t \geq 0$, studied in [1], i.e.

$$\int_0^1 \frac{\mathrm{d}^\nu p_k}{\mathrm{d}t^\nu} n(\nu)\,\mathrm{d}\nu = -\lambda(p_k - p_{k-1}), \qquad k \geq 0,\ p_{-1}(t) = 0. \tag{3.8}$$

Indeed, (3.1) can be considered a special case of (3.8) for $k = 0$ and, if we add assumption (3.3), we obtain (3.4). Thus, we can use the results proved in [1] and write

$$\psi_{\nu_1,\nu_2}(t) = \widetilde{p}_0(t) = \Pr\{\mathcal{N}_{\nu_1,\nu_2}(t) = 0\} = \Pr\{N(\mathcal{T}_{\nu_1,\nu_2}(t)) = 0\}, \tag{3.9}$$

where N is the standard Poisson process (with intensity λ) and $\mathcal{T}_{\nu_1,\nu_2}$ is a random process (independent from N) with density

$$q_{\nu_1,\nu_2}(y,t) = n_1 \int_0^t \bar{p}_{\nu_2}(t-s;\,y) q_{\nu_1}(y,s)\,\mathrm{d}s + n_2 \int_0^t \bar{p}_{\nu_1}(t-s;\,y) q_{\nu_2}(y,s)\,\mathrm{d}s. \tag{3.10}$$

In (3.10), $\bar{p}_{\nu_j}(\cdot;\,z)$ denotes the density of a stable random variable X_{ν_j} of index $\nu_j \in (0,1]$ for $j = 1, 2$, with parameters $\beta = 1$, $\mu = 0$, and $\sigma = (n_j|y|\cos(\pi\nu_j/2))^{1/\nu_j}$, and q_{ν_j} for $j = 1, 2$ is defined in (2.5). Another form for the density q_{ν_1,ν_2} is given by the following series expression:

$$\begin{aligned} q_{\nu_1,\nu_2}(y,t) = {} & \frac{n_1}{\lambda t^{\nu_1}} \sum_{r=0}^{\infty} \frac{1}{r!}\left(-\frac{n_2|y|}{\lambda t^{\nu_2}}\right)^r \mathcal{W}_{-\nu_1,1-\nu_2 r-\nu_1}\left(-\frac{n_1|y|}{\lambda t^{\nu_1}}\right) \\ & + \frac{n_2}{\lambda t^{\nu_2}} \sum_{r=0}^{\infty} \frac{1}{r!}\left(-\frac{n_1|y|}{\lambda t^{\nu_1}}\right)^r \mathcal{W}_{-\nu_2,1-\nu_1 r-\nu_2}\left(-\frac{n_2|y|}{\lambda t^{\nu_2}}\right). \end{aligned}$$

From (3.9) we obtain

$$\psi_{\nu_1,\nu_2}(t) = \int_0^\infty \mathrm{e}^{-\lambda y} q_{\nu_1,\nu_2}(y,t)\,\mathrm{d}y = \Pr\{\mathcal{T}_{\nu_1,\nu_2}(t) < U\}.$$

It was also proved in [1] that the transition density q_{ν_1,ν_2} coincides with the folded solution

$$q_{\nu_1,\nu_2}(y,t) = \begin{cases} 2v(y,t), & y \geq 0, \\ 0, & y < 0, \end{cases}$$

of the fractional diffusion equation

$$\left(n_1 \frac{\partial^{\nu_1} v}{\partial t^{\nu_1}} + n_2 \frac{\partial^{\nu_2} v}{\partial t^{\nu_2}}\right)^2 = \frac{\partial^2 v}{\partial y^2}, \qquad y \in \mathbb{R},\ t \geq 0,\ n_1, n_2 > 0, \tag{3.11}$$

for $0 < \nu_1 < \nu_2 \leq 1$, with initial conditions

$$\begin{aligned} & v(y,0) = \delta(y) \quad \text{for } 0 < \nu_1 < \nu_2 \leq 1, \\ & \frac{\partial}{\partial t} v(y,t)\bigg|_{t=0} = 0 \quad \text{for } \tfrac{1}{2} < \nu_1 < \nu_2 \leq 1. \end{aligned} \tag{3.12}$$

Alternatively to (3.11)–(3.12), it can be proved (as we will see below in a special case) that q_{ν_1,ν_2} also solves the other equation

$$n_1 \frac{\partial^{\nu_1} v}{\partial t^{\nu_1}} + n_2 \frac{\partial^{\nu_2} v}{\partial t^{\nu_2}} = -\frac{\partial v}{\partial y}, \qquad y, t \geq 0,\ n_1, n_2 > 0,\ v(y, 0) = \delta(y),$$

which is the distributed order analogue of (2.6). In order to obtain a more explicit expression for the density q_{ν_1,ν_2}, we consider the special, but relevant, case where $\nu_1 = \frac{1}{2}$ and $\nu_2 = 1$.

Theorem 3.2. *The solution to the fractional relaxation equation*

$$n_1 \frac{\mathrm{d}^{1/2}\psi}{\mathrm{d}t^{1/2}} + n_2 \frac{\mathrm{d}\psi}{\mathrm{d}t} = -\lambda\psi, \qquad t \geq 0,$$

with the initial condition $\psi(0) = 1$, can be expressed as

$$\psi_{1/2,1}(t) = \Pr\{\mathcal{T}_{1/2,1}(t) < U\}, \tag{3.13}$$

where U is an exponential RV with parameter λ and the transition density of $\mathcal{T}_{1/2,1}(t)$, $t \geq 0$, is given by

$$q_{1/2,1}(y, t) = \frac{n_1(t - n_2 y/2)}{\sqrt{\pi}} \frac{\mathrm{e}^{-n_1^2 y^2/4(t - n_2 y)}}{\sqrt{(t - n_2 y)^3}}, \qquad t \geq 0,\ 0 < y < \frac{t}{n_2}, \tag{3.14}$$

and satisfies the fractional equation

$$n_1 \frac{\partial^{1/2} q}{\partial t^{1/2}} + n_2 \frac{\partial q}{\partial t} = -\frac{\partial q}{\partial y}, \qquad q(y, 0) = \delta(y). \tag{3.15}$$

Proof. It has been proved in [1] that, for $\nu_2 = 1$ and $\nu_1 = \nu \in (0, 1)$, the density (3.10) can be expressed as

$$q_{\nu,1}(y, t) = n_1 I^{\nu}(\bar{\bar{p}}_{\nu}(\cdot; y))(t) + n_2 \bar{\bar{p}}_{\nu}(t; y), \tag{3.16}$$

where I^{ν} is the Riemann–Liouville fractional integral of order ν and $\bar{\bar{p}}_{\nu}$ denotes a stable law of index ν with parameters $\beta = 1$, $\mu = n_2|y|$, and $\sigma = (n_1|y|\cos(\pi\nu/2))^{1/\nu}$. If, moreover, we set $\nu = \frac{1}{2}$, we can recognize in $\bar{\bar{p}}_{1/2}$ the Lévy distribution, so that the density (3.16) becomes

$$\begin{aligned}
q_{1/2,1}(y, t) &= \frac{n_1}{\sqrt{\pi}} \int_0^t (t - s)^{-1/2} \bar{\bar{p}}_{1/2}(s; y)\,\mathrm{d}s + n_2 \bar{\bar{p}}_{1/2}(t; y) \\
&= \frac{n_1^2 y}{2\pi} \int_{n_2 y}^t (t - s)^{-1/2} \frac{\mathrm{e}^{-n_1^2 y^2/4(s - n_2 y)}}{\sqrt{(s - n_2 y)^3}}\,\mathrm{d}s + \frac{n_1 n_2 y}{2\pi} \frac{\mathrm{e}^{-n_1^2 y^2/4(t - n_2 y)}}{\sqrt{(t - n_2 y)^3}} \mathbf{1}_{\{0<y<t/n_2\}} \\
&= \frac{n_1^2 y}{2\pi} \int_0^{t - n_2 y} (t - n_2 y - z)^{-1/2} \frac{\mathrm{e}^{-n_1 y^2/4z}}{\sqrt{z^3}}\,\mathrm{d}z \\
&\quad + \frac{n_1 n_2 y}{2\pi} \frac{\mathrm{e}^{-n_1^2 y^2/4(t - n_2 y)}}{\sqrt{(t - n_2 y)^3}} \mathbf{1}_{\{0<y<t/n_2\}} \\
&= \left[\frac{n_1 \mathrm{e}^{-n_1^2 y^2/4(s - n_2 y)}}{\sqrt{\pi(t - n_2 y)}} + \frac{n_1 n_2 y}{2\pi} \frac{\mathrm{e}^{-n_1^2 y^2/4(t - n_2 y)}}{\sqrt{(t - n_2 y)^3}} \right] \mathbf{1}_{\{0<y<t/n_2\}}
\end{aligned}$$

(by Equation (3.8) of [23]),

which coincides with (3.14). In order to show that the latter satisfies the fractional relaxation equation (3.15), we evaluate its Laplace transform:

$$\begin{aligned}
\mathcal{L}\{q_{1/2,1}(y,\cdot);\eta\} &= \int_{n_2 y}^{\infty} \frac{n_1(t-n_2y/2)}{\sqrt{\pi}} \frac{\mathrm{e}^{-n_1^2y^2/4(t-n_2y)-\eta t}}{\sqrt{(t-n_2y)^3}}\,\mathrm{d}t \\
&= -\frac{n_1}{\sqrt{\pi}}\frac{\partial}{\partial\eta}\bigg(\int_{n_2y}^{\infty} \frac{\mathrm{e}^{-n_1^2y^2/4(t-n_2y)-\eta t}}{\sqrt{(t-n_2y)^3}}\,\mathrm{d}t\bigg) \\
&\quad - \frac{n_1n_2y}{2\sqrt{\pi}}\int_0^{\infty} \mathrm{e}^{-\eta z-\eta n_2 y}\frac{\mathrm{e}^{-n_1^2y^2/4z}}{\sqrt{z^3}}\,\mathrm{d}z \\
&= -\frac{\partial}{\partial\eta}\bigg(\frac{2}{y}\mathrm{e}^{-\eta n_2 y-\sqrt{\eta}n_1 y}\bigg) - n_2\mathrm{e}^{-\eta n_2 y-\sqrt{\eta}n_1 y} \\
&= (n_2+n_1\eta^{-1/2})\mathrm{e}^{-(n_2\eta+n_1\eta^{1/2})y}.
\end{aligned} \tag{3.17}$$

In (3.17) we applied the well-known formula for the Laplace transform of the first passage time of a Brownian motion. It is easy to check that

$$\int_0^{\infty} \mathrm{e}^{-\lambda y}\mathcal{L}\{q_{1/2,1}(y,\cdot);\eta\}\,\mathrm{d}y = \frac{n_2+n_1\eta^{-1/2}}{n_2\eta+n_1\eta^{1/2}+\lambda},$$

which is equal to the Laplace transform of ψ_{ν_1,ν_2} for $\nu_1=\frac{1}{2}$ and $\nu_2=1$ (given in Theorem 2.6 of [1]), thus proving result (3.13). If we now take the Fourier transform of (3.17), we obtain

$$\begin{aligned}
\mathcal{F}\{\mathcal{L}\{q_{1/2,1};\eta\};\beta\} &= \int_0^{\infty} \mathrm{e}^{\mathrm{i}\beta y}\mathcal{L}\{q_{1/2,1}(y,\cdot);\eta\}\,\mathrm{d}y \\
&= (n_2+n_1\eta^{-1/2})\int_0^{\infty}\mathrm{e}^{\mathrm{i}\beta y}\mathrm{e}^{-(n_2\eta+n_1\eta^{1/2})y}\,\mathrm{d}y \\
&= \frac{n_2+n_1\eta^{-1/2}}{n_2\eta+n_1\eta^{1/2}+\mathrm{i}\beta},
\end{aligned} \tag{3.18}$$

which coincides with the solution to (3.15) converted, via the Laplace–Fourier transform, into

$$(n_1\eta^{1/2}+n_2\eta)\mathcal{L}\{q_{1/2,1}(y,\cdot);\eta\} - (n_1\eta^{-1/2}+n_2)\delta(y) = -\frac{\partial}{\partial y}\mathcal{L}\{q_{1/2,1}(y,\cdot);\eta\}$$

and

$$(n_1\eta^{1/2}+n_2\eta+\mathrm{i}\beta)\mathcal{F}\{\mathcal{L}\{q_{1/2,1};\eta\};\beta\} = (n_1\eta^{-1/2}+n_2).$$

From (3.18), it is evident that (3.14) is well defined and integrates to 1, since, for $\beta=0$, we obtain $1/\eta$.

Remark 3.1. If we consider the two opposite special cases $n_2=0$ and $n_1=0$, the trajectories of the process $\mathcal{T}_{1/2,1}$ can be considered as an 'interpolation' between those of a free reflecting Brownian motion and the straight line $y=t/n_2$. Indeed, in the first case the density (3.14) becomes

$$q_{1/2,1}(y,t) = \frac{n_1\mathrm{e}^{-n_1^2y^2/4t}}{\sqrt{\pi t}}, \qquad y,t\geq 0,$$

while in the second case we can write (3.16) as $q_{1/2,1}(y,t)=n_2\bar{\bar{p}}_{1/2}(t;y)=n_2\delta(t-n_2y)$, since in this case $\sigma=0$. It is evident from (3.14) that the trajectories of $\mathcal{T}_{1/2,1}$ for any $n_1,n_2>0$

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TABLE 1: Limiting behaviors.

$t \to 0$	$t \to \infty$
$\psi(t) \simeq 1 - \lambda t$	$\psi(t) \simeq \mathrm{e}^{-\lambda t}$
$\psi_\nu(t) \simeq 1 - \lambda t^\nu/\Gamma(1+\nu)$	$\psi_\nu(t) \simeq 1/\lambda t^\nu \Gamma(1-\nu)$
$\psi^+(t) \simeq 1 - \lambda t/2$	$\psi^+(t) \simeq 1/\sqrt{\lambda \pi t}$
$\psi^T(t) \simeq 1 - \sqrt{2\lambda t}$	$\psi^T(t) \simeq \mathrm{e}^{-\sqrt{2\lambda t}}$
$\psi^\gamma(t) \simeq 1/(1+2\lambda t)^{\gamma/2}$	$\psi^\gamma(t) \simeq 1/(1+2\lambda t)^{\gamma/2}$
$\psi^{\mathrm{el}}(t) \simeq 1 - \lambda\sqrt{2t}/\sqrt{\pi}$	$\psi^{\mathrm{el}}(t) \simeq 1 - \sqrt{2}/\alpha\sqrt{\pi t}$
$\psi^k_{1/2}(t) \simeq 1 - (\lambda\sqrt{t})^k/\Gamma(k/2+1)$	$\psi^k_{1/2}(t) \simeq k/\lambda\sqrt{\pi t}$
$\psi^{\mathrm{el}}_{k,\alpha}(t) \simeq 1 - (\lambda\sqrt{t}/2)^k/\Gamma(k/2+1)$	$\psi^{\mathrm{el}}_{k,\alpha}(t) \simeq 1 - \sqrt{2}/\alpha\sqrt{\pi t}$
$\psi_{1/2,1}(t) \simeq 1 - \lambda t/n_2$	$\psi_{1/2,1}(t) \simeq n_1/\lambda\sqrt{\pi t}$

are forced under the line $y = t/n_2$ and this is reflected in the asymptotic behavior of the crossing probability $\psi_{1/2,1}$, which can be deduced from (3.6) and (3.7), and summarized as

$$\psi_{1/2,1}(t) \simeq \begin{cases} 1 - \dfrac{\lambda t}{n_2}, & 0 < t \ll 1, \\ \dfrac{n_1}{\lambda\sqrt{\pi t}}, & t \to \infty. \end{cases} \tag{3.19}$$

By comparing (3.19) with (2.15) we can conclude that $\psi_{1/2,1}$ displays the same limiting behavior as $\psi_{1/2}(t) = \Pr\{|B(t)| < U\}$ for $t \to \infty$. However, for $t \to 0$, it behaves as the standard relaxation (up to a constant) and, thus, tends to 1 much faster than $\psi_{1/2,1}$. We recall that similar limiting features were exhibited by the crossing probability ψ^+ of the Brownian sojourn time process (see (2.20)).

For the reader's convenience, we summarize the limiting behavior of the crossing probabilities analyzed in the previous sections in Table 1.

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