Bull. Aust. Math. Soc. **110** (2024), 271–279
doi:10.1017/S0004972723001351

DIVISIBILITY OF SUMS OF PARTITION NUMBERS BY MULTIPLES OF 2 AND 3

NAYANDEEP DEKA BARUAH

(Received 17 October 2023; accepted 13 November 2023; first published online 22 December 2023)

Abstract

We show that certain sums of partition numbers are divisible by multiples of 2 and 3. For example, if $p(n)$ denotes the number of unrestricted partitions of a positive integer n (and $p(0) = 1$, $p(n) = 0$ for $n < 0$), then for all nonnegative integers m,

$$\sum_{k=0}^{\infty} p(24m + 23 - \omega(-2k)) + \sum_{k=1}^{\infty} p(24m + 23 - \omega(2k)) \equiv 0 \pmod{144},$$

where $\omega(k) = k(3k + 1)/2$.

2020 *Mathematics subject classification*: primary 11P81; secondary 05A17, 11P83, 05A19.

Keywords and phrases: partition function, sum of partition numbers, ℓ-regular overpartitions, singular overpartitions.

1. Introduction

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is a nonincreasing sequence of positive integers that sum to n, that is, $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The numbers λ_j are known as the *parts* of λ. The partition function $p(n)$ enumerates the partitions of n. For example, $p(4) = 5$, since there are 5 partitions of 4, namely,

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1 \quad \text{and} \quad 1+1+1+1+1.$$

By convention, we take $p(0) = 1$ and $p(n) = 0$ if n is not a nonnegative integer.

The generating function of $p(n)$, found by Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} \tag{1.1}$$

(see [3]), where for complex numbers a and q with $|q| < 1$, the standard q-product $(a;q)_\infty$ is defined by

$$(a;q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j).$$

Work on the arithmetic properties of $p(n)$ started when Ramanujan [20], [21, pages 210–213] discovered his famous congruences for $p(n)$: for every nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Since then, these congruences have been studied and generalised extensively in many directions. Ono [18] proved that for every prime $M \geq 5$ there exist infinitely many nonnested arithmetic progressions $An + B$ such that

$$p(An+B) \equiv 0 \pmod M.$$

Ahlgren [1] extended this result for arbitrary integers M with $\gcd(6, M) = 1$. However, for the primes 2 and 3, we have different results on the divisibility of the partition function. Subbarao [24] conjectured that every arithmetic progression contains infinitely many integers n for which $p(n)$ is odd as well as infinitely many integers m for which $p(m)$ is even. The conjecture has been settled by Ono [17] and Radu [19]. Suppose that A and B are integers with $A > B \geq 0$. If $\nu = 2$ or 3, then Radu [19] proved that there are infinitely many integers n such that

$$p(An+B) \not\equiv 0 \pmod \nu.$$

Recently, Ballantine and Merca [5] explored the parity of sums of partition numbers at certain places in arithmetic progressions. In particular, they proved that if

$$(a,b) \in \{(6,8),(8,12),(12,24),(15,40),(16,48),(20,120),(21,168)\},$$

then

$$\sum_{ak+1 \text{ square}} p(n-k) \equiv 1 \pmod 2 \quad \text{if and only if } bn+1 \text{ is a square}$$

(see [12, 13] for further results of this type).

In this paper, we show that certain sums of partition numbers are divisible by multiples of 2 and 3. Unlike the results of Ballantine and Merca [5], our results do not depend on the squares in arithmetic progressions. To state the main results in the next section, we now recall some more partition functions in the remainder of this section.

An *overpartition* of a positive integer n is a nonincreasing sequence of positive integers that sum to n, where the first occurrence of parts of each size may be overlined (see [10]). The overpartition function $\overline{p}(n)$ counts the overpartitions of n. For example, $\overline{p}(4) = 14$ and the overpartitions of 4 are

$$4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ \overline{2}+2,\ 2+1+1,$$
$$\overline{2}+1+1,\ 2+\overline{1}+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1.$$

https://doi.org/10.1017/S0004972723001351 Published online by Cambridge University Press

Since the overlined parts form a partition into unequal parts and the nonoverlined parts form an ordinary partition, the generating function of $\overline{p}(n)$, as noted by Corteel and Lovejoy [10], is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.2}$$

In 2003, Lovejoy [15] considered the function $\overline{A}_\ell(n)$, known as the *ℓ-regular overpartition* function [22], that enumerates the overpartitions of n, which have no parts being a multiple of ℓ. For example, $\overline{A}_3(4) = 10$ and the 3-regular overpartitions of 4 are given by

$$4,\ \overline{4},\ 2+2,\ \overline{2}+2,\ 2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,$$
$$\overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1.$$

Andrews [4] extended the idea of ℓ-regular overpartitions by considering the enumeration $\overline{C}_{k,i}(n)$ of so-called *singular overpartitions* of n that correspond to ℓ-regular overpartitions of n in which only parts $\equiv \pm i \pmod{k}$ may be overlined. Clearly, $\overline{A}_3(n) = \overline{C}_{3,1}(n)$ for all $n \geq 1$.

Since the multiples of ℓ cannot appear in an ℓ-regular overpartition, it follows from (1.2) that the generating function of $\overline{A}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{A}_\ell(n)q^n = \frac{(-q;q)_\infty/(q;q)_\infty}{(-q^\ell;q^\ell)_\infty/(q^\ell;q^\ell)_\infty} = \frac{(-q;q)_\infty(q^\ell;q^\ell)_\infty}{(q;q)_\infty(-q^\ell;q^\ell)_\infty}. \tag{1.3}$$

2. Main results

Euler's famous recurrence relation for $p(n)$ is given by

$$\sum_{k=-\infty}^{\infty} (-1)^k p(n-\omega(k)) = \delta_{0,n}, \tag{2.1}$$

where $\omega(k) = k(3k+1)/2$, for integers k, are the generalised pentagonal numbers and $\delta_{i,j}$ is the Kronecker delta. For integers $n \geq 1$, it easily follows from (2.1) that

$$\sum_{k=0}^{\infty} p(n-\omega(-2k)) + \sum_{k=1}^{\infty} p(n-\omega(2k))$$
$$= \sum_{k=1}^{\infty} p(n-\omega(-2k+1)) + \sum_{k=1}^{\infty} p(n-\omega(2k-1)). \tag{2.2}$$

In this paper, we show divisibility of the above sums of partition numbers by multiples of 2 and 3. The following main result arises from (2.2) and Jacobi's triple product identity [3, page 21, Theorem 2.8].

https://doi.org/10.1017/S0004972723001351 Published online by Cambridge University Press

THEOREM 2.1. *Let $\overline{A}_3(n)$ denote the number of 3-regular overpartitions of n, which is also equal to Andrews' singular overpartition function $\overline{C}_{3,1}(n)$. Then, for all integers $n \geq 1$,*

$$\sum_{k=0}^{\infty} p(n-\omega(-2k)) + \sum_{k=1}^{\infty} p(n-\omega(2k)) = \sum_{k=1}^{\infty} p(n-\omega(-2k+1)) + \sum_{k=1}^{\infty} p(n-\omega(2k-1)) = \frac{\overline{A}_3(n)}{2}. \quad (2.3)$$

There are several recent papers that studied the arithmetical properties of $\overline{A}_\ell(n)$ and $\overline{C}_{k,i}(n)$. For results on $\overline{A}_3(n)$ and $\overline{C}_{3,1}(n)$, see [2, 4, 6–9, 11, 14, 16, 22, 23, 26]. Employing Theorem 2.1 and congruences for $\overline{A}_3(n)$, that is, for $\overline{C}_{3,1}(n)$, one can easily deduce divisibility properties of the sums of the partition numbers in (2.2). For example, Barman and Ray [7, Theorems 1.1–1.3] proved that for a fixed positive integer k, $\overline{C}_{3,1}(n)$ is divisible by 2^k and $2 \cdot 3^k$ for almost all n. Therefore, it follows that the above sums of partition numbers are divisible by 3^k for almost all n. In the following corollary, we present selected congruences for the sums in nondecreasing order of the moduli that arise from the congruences for $\overline{A}_3(n)$ or $\overline{C}_{3,1}(n)$, which either appeared in [2, 4, 6, 8, 9, 11, 14, 16, 22, 23, 26] or are easily deduced from these results.

COROLLARY 2.2. *For brevity, set*

$$S(n) := \sum_{k=0}^{\infty} p(n-\omega(-2k)) + \sum_{k=1}^{\infty} p(n-\omega(2k)) = \sum_{k=1}^{\infty} p(n-\omega(-2k+1)) + \sum_{k=1}^{\infty} p(n-\omega(2k-1)).$$

For any nonnegative integers k and n,

$$\begin{aligned}
S(3n+2) &\equiv 0 \pmod{2},\\
S(4n+2) &\equiv 0 \pmod{2},\\
S(2^k(4n+3)) &\equiv 0 \pmod{3},\\
S(9n+3) &\equiv 0 \pmod{3},\\
S(2^{k+1}(6n+5)) &\equiv 0 \pmod{4},\\
S(4^k(16n+6)) &\equiv 0 \pmod{4},\\
S(4^k(16n+10)) &\equiv 0 \pmod{4},\\
S(4^k(16n+14)) &\equiv 0 \pmod{4},\\
S(8n+7) &\equiv 0 \pmod{6},\\
S(36n+21) &\equiv 0 \pmod{6},\\
S(6n+5) &\equiv 0 \pmod{8},
\end{aligned}$$

https://doi.org/10.1017/S0004972723001351 Published online by Cambridge University Press

$$
\begin{aligned}
S(4^k(72n+42)) &\equiv 0 \pmod 8,\\
S(4^k(144n+78)) &\equiv 0 \pmod 8,\\
S(48n+12) &\equiv 0 \pmod 9,\\
S(8n+6) &\equiv 0 \pmod{12},\\
S(9n+6) &\equiv 0 \pmod{12},\\
S(36n+30) &\equiv 0 \pmod{12},\\
S(24n+17) &\equiv 0 \pmod{16},\\
S(4^k(72n+60)) &\equiv 0 \pmod{16},\\
S(2^k(12n+7)) &\equiv 0 \pmod{18},\\
S(144n+102) &\equiv 0 \pmod{24},\\
S(9^k(48n+28)) &\equiv 0 \pmod{27},\\
S(9^k(48n+44)) &\equiv 0 \pmod{27},\\
S(72n+51) &\equiv 0 \pmod{32},\\
S(72n+69) &\equiv 0 \pmod{32},\\
S(2^{k+1}(12n+11)) &\equiv 0 \pmod{36},\\
S(24n+14) &\equiv 0 \pmod{36},\\
S(18n+15) &\equiv 0 \pmod{48},\\
S(12n+11) &\equiv 0 \pmod{72},\\
S(24n+23) &\equiv 0 \pmod{144}.
\end{aligned}
$$

Note that the last congruence is equivalent to the example stated in the abstract.

The powers of 2 and 3 in the modulus in each of the above congruences are sharp. However, there might be sub-progressions of the given arithmetic progression along which the powers of 2 and 3 in the modulus may be higher. Furthermore, combining two congruences may also give congruences for higher modulus.

There are congruences for $\overline{A}_3(n)$ or $\overline{C}_{3,1}(n)$ that depend on specific properties of the integer n. For example, Li and Yao [14] show that if $p \equiv 3 \pmod 4$ and $p \nmid n$, then for any $k \geq 0$,

$$\overline{A}_3(108p^{2k+1}(4n+p)) \equiv 0 \pmod{27}. \tag{2.4}$$

Noting also that $\overline{A}_3(n) \equiv 0 \pmod 2$ for all integers $n \geq 1$ (see [8, Theorem 2.9]), it readily follows from (2.4) that

$$S(108p^{2k+1}(4n+p)) \equiv 0 \pmod{27}.$$

There are several other results like (2.4), which can be derived from results in [8, 9, 14, 16, 23, 25, 26].

We prove Theorem 2.1 and Corollary 2.2 in the next two sections.

https://doi.org/10.1017/S0004972723001351 Published online by Cambridge University Press

3. Proof of Theorem 2.1

We use Jacobi's triple product identity and (2.1) to prove Theorem 2.1.

Jacobi's triple product identity [3, page 21, Theorem 2.8] can be stated as follows. For $z \neq 0$ and $|q| < 1$,

$$\sum_{k=-\infty}^{\infty} z^k q^{k^2} = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty.$$

Replacing q by $q^{3/2}$ and z by $\sqrt{q}$ and then manipulating the q-products,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} &= (-q^2; q^3)_\infty (-q; q^3)_\infty (q^3; q^3)_\infty \\ &= \frac{(-q; q)_\infty (q^3; q^3)_\infty}{(-q^3; q^3)_\infty} \\ &= \frac{(-q; q)_\infty (q^3; q^3)_\infty}{(q; q)_\infty (-q^3; q^3)_\infty} \cdot (q; q)_\infty.\end{aligned}$$

It follows that

$$\frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} = \frac{(-q; q)_\infty (q^3; q^3)_\infty}{(q; q)_\infty (-q^3; q^3)_\infty},$$

which, with the aid of (1.1) and (1.3), may be rewritten as

$$\bigg(\sum_{n=0}^{\infty} p(n) q^n\bigg)\bigg(\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2}\bigg) = \sum_{n=0}^{\infty} \overline{A}_3(n) q^n.$$

Equating the coefficients of q^n on both sides of this equation yields

$$\sum_{k=-\infty}^{\infty} p(n - \omega(k)) = \overline{A}_3(n),$$

which may be rewritten as

$$\begin{aligned}\sum_{k=0}^{\infty} p(n - \omega(-2k)) + \sum_{k=1}^{\infty} p(n - \omega(2k)) + \sum_{k=1}^{\infty} p(n - \omega(-2k+1)) \\ + \sum_{k=1}^{\infty} p(n - \omega(2k-1)) = \overline{A}_3(n).\end{aligned} \tag{3.1}$$

From (2.2) and (3.1) it readily follows that

$$2\bigg(\sum_{k=0}^{\infty} p(n - \omega(-2k)) + \sum_{k=1}^{\infty} p(n - \omega(2k))\bigg) = \overline{A}_3(n)$$

https://doi.org/10.1017/S0004972723001351 Published online by Cambridge University Press

and

$$2\Big(\sum_{k=1}^{\infty} p(n-\omega(-2k+1)) + \sum_{k=1}^{\infty} p(n-\omega(2k-1))\Big) = \overline{A}_3(n);$$

which is equivalent to (2.3). This completes the proof of Theorem 2.1.

4. Proof of Corollary 2.2

Most of the congruences follow easily from the corresponding congruences and generating function representations of $\overline{A}_3(n)$ or $\overline{C}_{3,1}(n)$ in [2, 4, 6, 8, 9, 11, 14, 16, 22, 23, 26] and Theorem 2.1. Therefore, we only prove the last three congruences in Corollary 2.2, that is,

$$S(18n+15) \equiv 0 \pmod{48}, \tag{4.1}$$

$$S(12n+11) \equiv 0 \pmod{72}, \tag{4.2}$$

and

$$S(24n+23) \equiv 0 \pmod{144}. \tag{4.3}$$

Andrews [4, Theorem 2] and Yao [26, Theorem 1.1, (1.8)] proved that

$$\overline{A}_3(9n+6) \equiv 0 \pmod{3} \quad \text{and} \quad \overline{A}_3(18n+15) \equiv 0 \pmod{32},$$

from which it follows that

$$\overline{A}_3(18n+15) \equiv 0 \pmod{96}.$$

Now (4.1) is apparent from Theorem 2.1 and the above congruence.

Next, Barman and Ray [6, Section 3] showed that

$$\sum_{n=0}^{\infty} \overline{A}_3(12n+11)q^n = 144\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty^{12}}{(q;q)_\infty^{22}(q^6;q^6)_\infty^{3}} + 576q\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^{3}(q^6;q^6)_\infty^{6}}{(q;q)_\infty^{19}}. \tag{4.4}$$

Therefore,

$$\overline{A}_3(12n+11) \equiv 0 \pmod{144},$$

which, by Theorem 2.1, readily implies (4.2).

It also follows from (4.4) that

$$\sum_{n=0}^{\infty} \overline{A}_3(12n+11)q^n \equiv 144\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty^{12}}{(q;q)_\infty^{22}(q^6;q^6)_\infty^{3}} \pmod{288}. \tag{4.5}$$

But, by the binomial theorem, $(q^j;q^j)_\infty^2 \equiv (q^{2j};q^{2j})_\infty \pmod{2}$ for any integer $j \geq 1$. Therefore, it follows from (4.5) that

$$\sum_{n=0}^{\infty} \overline{A}_3(12n+11)q^n \equiv 144 f_4 f_6^3 \pmod{288}.$$

https://doi.org/10.1017/S0004972723001351 Published online by Cambridge University Press

Equating the coefficients of q^{2n+1} on both sides of this congruence yields

$$\overline{A}_3(24n + 23) \equiv 0 \pmod{288},$$

which, by Theorem 2.1, readily gives (4.3).

References

[1] S. Ahlgren, 'Distribution of the partition function modulo composite integers M', *Math. Ann.* **318** (2000), 795–803.

[2] Z. Ahmed and N. D. Baruah, 'New congruences for Andrews' singular overpartitions', *Int. J. Number Theory* **11** (2015), 2247–2264.

[3] G. E. Andrews, *The Theory of Partitions* (Addison-Wesley, Reading, MA, 1976).

[4] G. E. Andrews, 'Singular overpartitions', *Int. J. Number Theory* **11** (2015), 1523–1533.

[5] C. Ballantine and M. Merca, 'Parity of sums of partition numbers and squares in arithmetic progressions', *Ramanujan J.* **44** (2017), 617–630.

[6] R. Barman and C. Ray, 'Congruences for ℓ-regular overpartitions and Andrews' singular overpartitions', *Ramanujan J.* **45** (2018), 497–515.

[7] R. Barman and C. Ray, 'Divisibility of Andrews' singular overpartitions by powers of 2 and 3', *Res. Number Theory* **5** (2019), Article no. 22.

[8] S.-C. Chen, M. D. Hirschhorn and J. A. Sellers, 'Arithmetic properties of Andrews' singular overpartitions', *Int. J. Number Theory* **11** (2015), 1463–1476.

[9] S. Chern, 'New congruences for ℓ-regular overpartitions', *Integers* **17** (2017), Article no. A22.

[10] S. Corteel and J. Lovejoy, 'Overpartitions', *Trans. Amer. Math. Soc.* **356** (2004), 1623–1635.

[11] R. X. J. Hao and E. Y. Y. Shen, 'On the number of l-regular overpartitions', *Int. J. Number Theory* **17** (2021), 2153–2173.

[12] L. Hong and S. Zhang, 'Proof of the Ballantine–Merca conjecture and theta function identities modulo 2', *Proc. Amer. Math. Soc.*, to appear.

[13] W. Hu, O. X. M. Yao and T. Zhao, 'New parity results of sums of partitions and squares in arithmetic progressions', *Contrib. Discrete Math.* **14** (2019), 117–129.

[14] X. Li and O. X. M. Yao, 'New infinite families of congruences for Andrews' (k,i)-singular overpartitions', *Quaest. Math.* **41** (2018), 1005–1019.

[15] J. Lovejoy, 'Gordon's theorem for overpartitions', *J. Combin. Theory Ser. A* **103** (2003), 393–401.

[16] M. S. Mahadeva Naika and D. S. Gireesh, 'Congruences for Andrews' singular overpartitions', *J. Number Theory* **165** (2016), 109–130.

[17] K. Ono, 'On the parity of the partition function in arithmetic progressions', *J. reine angew. Math.* **472** (1996), 1–15.

[18] K. Ono, 'Distribution of the partition function modulo m', *Ann. Math.* **151** (2000), 293–307.

[19] C.-S. Radu, 'A proof of Subbarao's conjecture', *J. reine angew. Math.* **672** (2012), 161–175.

[20] S. Ramanujan, 'Some properties of $p(n)$, the number of partitions of n', *Proc. Cambridge Philos. Soc.* **19** (1919), 207–210.

[21] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan* (Cambridge University Press, Cambridge, 1927); reprinted by Chelsea, New York, 1962, and by the American Mathematical Society, Providence, RI, 2000.

[22] E. Y. Y. Shen, 'Arithmetic properties of l-regular overpartitions', *Int. J. Number Theory* **12** (2016), 841–852.

[23] E. Y. Y. Shen, 'Congruences modulo 9 for singular overpartitions', *Int. J. Number Theory* **13** (2017), 717–724.

[24] M. V. Subbarao, 'Remarks on the partition function', *Amer. Math. Monthly* **73** (1966), 851–854.

[25] H. S. Sumanth Bharadwaj, B. Hemanthkumar and M. S. Mahadeva Naika, 'On 3- and 9-regular overpartitions modulo powers of 3', *Colloq. Math.* **154** (2018), 121–130.
[26] O. X. M. Yao, 'Congruences modulo 16, 32, and 64 for Andrews's singular overpartitions', *Ramanujan J.* **43** (2017), 215–228.

NAYANDEEP DEKA BARUAH, Department of Mathematical Sciences,
Tezpur University, Napaam 784028, Assam, India
e-mail: nayan@tezu.ernet.in