THE NUMBER OF CIRCULAR PATTERNS COMPATIBLE WITH A PSEUDO-SYMMETRIC CONNECTED GRAPH

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In this paper we prove:

THEOREM. *Let $\mathfrak{G}$ be an ordered pseudo-symmetric connected graph with l lines and v vertices. Let there be a_{ij} lines directed from vertex i to vertex j, $i, j = 1, 2, \ldots, v$. Let* $\gcd(a_{ij}) = d$, *and define* $\sum_{j=1}^{v} a_{ij} = a_i$, $i = 1, 2, \ldots, v$. *The number of distinct l long circular arrangements of the v vertices arising from circuits of the graph is:*

$$\frac{\nabla}{\prod_{i=1}^{v} a_i} \sum_{x|d} \phi(x) \prod_{i=1}^{v} \begin{bmatrix} \frac{a_i}{x} \\ \frac{a_{i1}}{x}, \frac{a_{i2}}{x}, \ldots, \frac{a_{iv}}{x} \end{bmatrix},$$

where ϕ is the Euler phi function, the large bracket indicates a multinomial coefficient, and ∇ can be taken as the $(v-1) \times (v-1)$ determinant:

$$\det \begin{bmatrix} a_2 - a_{22} & -a_{23} & -a_{24} & \ldots & -a_{2,v-1} & -a_{2v} \\ -a_{32} & a_3 - a_{33} & -a_{34} & \ldots & -a_{3,v-1} & -a_{3v} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdot & \cdots \\ -a_{v2} & -a_{v3} & & & -a_{v,v-1} & a_v - a_{vv} \end{bmatrix}.$$

(For all graph-theoretic undefined terms and unproved theorems see (**1**, especially Chapters 16 and 17).

Proof. We say that an l long circle of the v vertices is of frequency p if it is composed of a sequence of p identical $1/p$ long stretches but is not composed of a sequence of more than p identical stretches. Each frequency p circular pattern that arises from a circuit of $\mathfrak{G}$ arises from exactly $(\prod_{i,j=1}^{v} a_{ij}!)/p$ Euler circuits of $\mathfrak{G}$. Moreover, if $d = \gcd(a_{ij})$, the only frequencies p that can arise are divisors of d. We thus have the fundamental relation:

$$(1) \qquad \sum_{p|d} \frac{n_p(\mathfrak{G})}{p} = \frac{\text{no. of Euler circuits of } \mathfrak{G}}{\prod_{i,j=1}^{v} a_{ij}!} = E(\mathfrak{G}),$$

where $n_p(\mathfrak{G})$ is the number of circular patterns of frequency p that arise from Euler circuits of $\mathfrak{G}$.

For each p dividing d we define a new graph $\mathfrak{G}_p$ with the same number v of

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vertices as before but with a_{ij}/p lines directed from vertex i to vertex j. Obviously

$$(\mathfrak{G}_p)_q = \mathfrak{G}_{pq}.$$

Every l/p long sequence of vertices arising from an Euler circuit of $\mathfrak{G}_p$ defines by a p-fold repetition an l-long circular sequence of frequency $\geqslant p$. If, in keeping with our previous notation, $n_1(\mathfrak{G}_p)$ is the number of frequency 1 circular patterns of length l/p defined by Euler circuits of $\mathfrak{G}_p$, we have the relation:

$$n_p(\mathfrak{G}_1) = n_1(\mathfrak{G}_p).$$

Shortening $n_1(\mathfrak{G}_p)$ to $n(\mathfrak{G}_p)$, (1) becomes

$$\sum_{p|d} \frac{n(\mathfrak{G}_p)}{p} = E(\mathfrak{G}). \tag{2}$$

Note that if q is some fixed divisor of d, then (2) implies that

$$\sum_{p|d/q} \frac{n([\mathfrak{G}_q]_p)}{p} = E(\mathfrak{G}_q).$$

Since $(\mathfrak{G}_q)_p = \mathfrak{G}_{qp}$, letting $qu = d$, we have

$$E(\mathfrak{G}_{d/u}) = \sum_{p|u} \frac{n(\mathfrak{G}_{pd/u})}{p} = \sum_{p|u} \frac{n(\mathfrak{G}_{d/p})}{u/p},$$

so

$$uE(\mathfrak{G}_{d/u}) = \sum_{p|u} pn(\mathfrak{G}_{d/p}). \tag{3}$$

Because of the form of (3) we can now use the Möbius inversion formula to express $pn(\mathfrak{G}_{d/p})$ in terms of $uE(\mathfrak{G}_{d/u})$; see (**2**, Chapter 6) for all the needed number-theoretic terms and definitions. Summing over all divisors r of d, we obtain:

$$\sum_{u|d} n(\mathfrak{G}_{d/u}) = \sum_{u|d} \frac{1}{r} \sum_{s|r} \mu(s) \frac{r}{s} E(\mathfrak{G}_{ds/r}) = \sum_{x|d} E(\mathfrak{G}_x) \sum_{s|x} \frac{\mu(s)}{s}, \tag{4}$$

where μ is the Möbius function. Since by a standard identity:

$$\sum_{s|x} \frac{\mu(s)}{s} = \frac{\phi(x)}{x},$$

where ϕ is the Euler phi function, (4) yields:

$$\sum_{r|d} n(\mathfrak{G}_r) = \sum_{x|d} \frac{E(\mathfrak{G}_x)}{x} \phi(x). \tag{5}$$

By a standard theorem of Tutte, Bott, Aardenne-Ehrenfest, de Bruijn (**1**, p. 169) for a graph $\mathfrak{G}$:

$$\text{no. of Euler circuits} = \prod_{i=1}^{v} (a_i - 1)!\, \nabla \tag{6}$$

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where ∇ can be taken as:

$$\nabla = \det \begin{bmatrix} a_2 - a_{22} & -a_{23} & -a_{24} & \dots & -a_{2,v-1} & -a_{2v} \\ -a_{32} & a_3 - a_{33} & -a_{34} & \dots & -a_{3,v-1} & -a_{3v} \\ \dots\dots & \dots\dots & \dots\dots & \dots & \dots\dots & \dots \\ -a_{v2} & -a_{v3} & & \dots & -a_{v,v-1} & a_v - a_{vv} \end{bmatrix}.$$

Since the relevant determinant $\nabla\mathfrak{G}_x$ for $\mathfrak{G}_x$ has each element divided by x, we have for each x dividing d:

$$E(\mathfrak{G}_x) = \frac{\nabla}{x^{n-1}} \frac{\prod_{i=1}^{v} (a_i/x - 1)!}{\prod_{i,j=1}^{v} (a_{ij}/x)!}. \tag{7}$$

Substituting (7) in (5) we find that the number of distinct circular patterns of the v vertices defined by Euler circuits of the graph $\mathfrak{G}$ is:

$$\sum_{x|d} \frac{\phi(x)}{x^v} \frac{\prod_{i=1}^{v} (a_i/x - 1)!}{\prod_{i,j=1}^{v} (a_{ij}/x)!} = \frac{\nabla}{\prod_{i=1}^{v} a_i} \sum_{x|d} \phi(x) \prod_{i=1}^{v} \begin{bmatrix} \dfrac{a_i}{x} \\ \dfrac{a_{i1}}{x}, \dfrac{a_{i2}}{x}, \dfrac{a_{iv}}{x} \end{bmatrix}, \tag{8}$$

where the large brackets indicate a multinomial coefficient.

A special case of formula (8) gives the number of distinct circular patterns of 0's and 1's compatible with a given frequency count f_i, $i = 0, \dots, 2^n - 1$ of n bit words. The associated graph is similar to the Good diagram (3) with lines corresponding to n bit words and vertices to $n - 1$ bit words. If $i = \sum_{j=0}^{n-1} a_j 2^j$, then the graph has f_i lines extending from vertex $\sum_{j=0}^{n-2} a_j 2^j$ to vertex $\sum_{j=0}^{n-1} a_j 2^{j-1}$. Since each vertex has at most 2 distinct vertices as successor in these graphs, all the multinomial coefficients in formula (8) are binomial coefficients.

Of course, if $\gcd(f_i) = 1$, in particular if $l = \sum f_i$ is prime, the formula (8) reduces to a single term.

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