

ON THE DIOPHANTINE EQUATION $x^2 + d^{2l+1} = y^n$

ATTILA BÉRCZES

*Institute of Mathematics, University of Debrecen, Number Theory Research Group, Hungarian Academy of Sciences and University of Debrecen, H-4010 Debrecen, P.O. Box 12, Hungary
e-mail: berczes@math.klte.hu*

and ISTVÁN PINKI

*Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O. Box 12, Hungary
e-mail: pinki@math.klte.hu*

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Abstract. Let $d > 0$ be a squarefree integer and denote by $h = h(-d)$ the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. It is well known (see e.g. [25]) that for a given positive integer N there are only finitely many squarefree d 's for which $h(-d) = N$. In [45], Saradha and Srinivasan and in [28] Le and Zhu considered the equation in the title and solved it completely under the assumption $h(-d) = 1$ apart from the case $d \equiv 7 \pmod{8}$ in which case y was supposed to be odd. We investigate the title equation in unknown integers (x, y, l, n) with $x \geq 1, y \geq 1, n \geq 3, l \geq 0$ and $\gcd(x, y) = 1$. The purpose of this paper is to extend the above result of Saradha and Srinivasan to the case $h(-d) \in \{2, 3\}$.

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1. Introduction. There are many results concerning the generalised Ramanujan–Nagell equation

$$x^2 + D = y^n, \tag{1}$$

where $D > 0$ is a given integer and x, y, n are positive integer unknowns with $n \geq 3$. Results obtained for general superelliptic equations clearly provide effective finiteness results for this equation, too (see for example [9, 47, 49] and the references given there).

The first result concerning the above equation was due to V. A. Lebesgue [29] who proved that there are no solutions for $D = 1$. Ljunggren [30] solved (1) for $D = 2$, and Nagell [40, 41] solved it for $D = 3, 4$ and 5 . In his elegant paper [20], Cohn gave a fine summary of the earlier results on equation (1). Further, he developed a method by which he found all solutions of the above equation for 77 positive values of $D \leq 100$. For $D = 74$ and $D = 86$, equation (1) was solved by Mignotte and de Weger [37]. By using the theory of Galois representations and modular forms, Bennett and Skinner [8] solved (1) for $D = 55$ and $D = 95$. On combining the theory of linear forms in logarithms with Bennett and Skinner's method and with several additional ideas, Bugeaud, Mignotte and Siksek [14] gave all the solutions of (1) for the remaining 19 values of $D \leq 100$.

Let $S = \{p_1, \dots, p_s\}$ denote a set of distinct primes and \mathbf{S} the set of non-zero integers composed only of primes from S . Put $P := \max\{p_1, \dots, p_s\}$ and denote by Q the product of the primes of S . In recent years, equation (1) has been considered also in

the more general case when D is no longer fixed but $D \in \mathbf{S}$ with $D > 0$. It follows from Theorem 2 of [48] that in (1) n can be bounded from above by an effectively computable constant depending only on P and s . In [24], an effective upper bound was derived for n which depends only on Q . Cohn [19] showed that if $D = 2^{2k+1}$ then equation (1) has solutions only when $n = 3$ and in this case, there are three families of solutions. The case $D = 2^{2k}$ was considered by Arif and Abu Muriefah [1]. They conjectured that the only solutions are given by $(x, y) = (2^k, 2^{2k+1})$ and $(x, y) = (11 \cdot 2^{k-1}, 5 \cdot 2^{2(k-1)/3})$, with the latter solution existing only when $(k, n) = (3M + 1, 3)$ for some integer $M \geq 0$. Partial results towards this conjecture were obtained in [1] and [18] and it was finally proved by Arif and Abu Muriefah [4]. Arif and Abu Muriefah [2] proved that if $D = 3^{2k+1}$ then (1) has exactly one infinite family of solutions. The case $D = 3^{2k}$ has been solved by Luca [32] under the additional hypothesis that x and y are coprime. In fact in [33], Luca solved completely equation (1) if $D = 2^a 3^b$ and $\gcd(x, y) = 1$. Abu Muriefah [38] established that equation (1) with $D = 5^{2k}$ may have a solution only if 5 divides x and p does not divide k for any odd prime p dividing n . The case $D = 2^a 3^b 5^c 7^d$ with $\gcd(x, y) = 1$, where a, b, c, d are non-negative integers was studied by Pink [42]. The cases when $D = 7^{2k}$ and $D = 2^a 5^b$ were also considered by Luca and Togbe [34, 35]. For the case $D = 2^a 5^b 13^c$ see Goins, Luca and Togbe [23], for $D = 5^a 17^b$ see [43], while if $D = 5^a 13^b$ see [39]. The cases $D = 2^a 11^b$ and $D = 5^a 11^b$ have been recently considered in [16] and [17], respectively. Let $p \geq 5$ be an odd prime with $p \not\equiv 7 \pmod{8}$. Arif and Abu Muriefah [5] determined all solutions of the equation $x^2 + p^{2k+1} = y^n$, where $\gcd(n, 3h_0) = 1$ and $n \geq 3$. Here, h_0 denotes the class number of the field $\mathbb{Q}(\sqrt{-p})$. They also obtained partial results [3] if $D = p^{2k}$, where p is an odd prime. In the particular case when $\gcd(x, y) = 1$, $D = p^2$, p prime with $3 \leq p < 100$ Le [27] gave all the solutions of equation (1). The equation $x^2 + p^m = y^n$ was considered by Le [26] and for infinitely many odd primes by Zhu [54], $D = p^{2k}$ with $2 \leq p < 100$ prime and $\gcd(x, y) = 1$ was solved by Bérczes and Pink [10]. If in (1) $D = a^2$ with $3 \leq a \leq 501$ and a odd Tengely [50] solved completely equation (1) under the assumption $(x, y) \in \mathbb{N}^2$, $\gcd(x, y) = 1$. The equation $A^4 + B^2 = C^n$ for $AB \neq 0$ and $n \geq 4$ was completely solved by Bennett, Ellenberg and Nathan [7] (see also Ellenberg [22]). For related results concerning equation (1), see [36, 45, 46, 51, 52] and the references given there. For a survey concerning equation (1), see [15].

2. Results. Let $d > 0$ be a squarefree integer. Consider equation (1) in the case when

$$D = d^{2l+1} \text{ with } h(-d) = 1. \quad (2)$$

In [45], Saradha and Srinivasan and in [28] Le and Zhu solved completely equation (2) under the additional assumption that y is odd if $d \equiv 7 \pmod{8}$.

In the present paper, we extend the result of [45] to the case $h(-d) \in \{2, 3\}$. Namely, let $d > 0$ be a squarefree integer and consider the following equation:

$$x^2 + d^{2l+1} = y^n \text{ with } h(-d) \in \{2, 3\} \quad (3)$$

in integer unknowns x, y, l, n satisfying

$$x \geq 1, y > 1, n \geq 3, l \geq 0 \text{ and } \gcd(x, y) = 1. \quad (4)$$

THEOREM 1. *Consider equation (3) satisfying (4). Further, if $d \equiv 7 \pmod{8}$ also suppose that y is odd. Then all solutions of equation (3) are $(x, y, n, d, l) \in \{(2, 3, 3, 23, 0), (588, 71, 3, 23, 1), (6, 7, 3, 307, 0), (32, 11, 3, 307, 0), (598, 71, 3, 307, 0), (911054064, 939787, 3, 307, 0), (28, 11, 3, 547, 0), (70, 17, 3, 13, 0), (36, 11, 3, 35, 0), (702, 79, 3, 235, 0), (322, 47, 3, 139, 0), (2158, 167, 3, 499, 0)\}$.*

REMARK 1. The proof of our Theorem is organised as follows. Without loss of generality, we may assume that in (3) $n \geq 5$ prime or $n \in \{3, 4\}$. If in (3) $n \geq 5$ is an odd prime, we use the primitive prime divisor theorem of Bilu, Hanrot and Voutier to conclude that (3) does not have a solution. For the case $n = 4$, we reduce the problem to several ternary equations of signature $(m, m, 2)$ for which the modular method works. Namely, in this case we apply some results of Bennett and Skinner [8] to list all solutions of equation (3). Finally, if in (3) $n = 3$ there is a well-known method for solving equation (3). Namely, we may transform our equation (3) to several equations of the form

$$w^2 = t^3 - d^i, \quad (5)$$

where $2l + 1 = 6l_1 + i$, $l_1 \geq 0$, $i \in \{1, 3, 5\}$, $w = x/d^{3l_1}$ and $t = y/d^{2l_1}$. Now, we have to search for all S -integral points on the above elliptic curves, where S consists of the prime divisors of d . This method works well but in some cases the computation of the Mordell-Weil group becomes very time consuming. Therefore, we need another approach, too. In the case $n = 3$, we distinguish two cases according to $h(-d) = 2$ and $h(-d) = 3$. If in (3) $(h(-d), n) = (2, 3)$ then we combine the parametrisation provided by Lemma 1 with the modular method and with the method of Chabauty concerning the determination of all rational points of a hyperelliptic curve of genus 2. If in (3) $(h(-d), n) = (3, 3)$, we see that the parametrisation provided by Lemma 1 cannot be applied. Hence, we use an idea of Mignotte and de Weger [37] to reduce the problem to the resolution of several Thue-Mahler equations of degree 3. Then these equations are considered locally to get a contradiction.

3. Auxiliary results. Let $S = \{p_1, \dots, p_s\}$ be a set of distinct primes and denote by \mathbf{S} , the set of non-zero integers composed only of primes from S . Equation (3) is a special case of an equation of the type

$$X^2 + D = Y^n, \quad (6)$$

where

$$\gcd(X, Y) = 1 \quad (7)$$

and

$$D \in \mathbf{S}, D > 0, X \geq 1, Y > 1, n \geq 3. \quad (8)$$

The next lemma provides a parametrisation for the solutions of equation (6).

LEMMA 1. *Suppose that equation (6) has a solution under the assumptions (7) and (8) with $n \geq 3$ prime. Denote by $d > 0$ the square-free part of $D = dc^2$ and let h be the class number of the field $\mathbb{Q}(\sqrt{-d})$. Then equation (6) has a solution with $d \not\equiv 7 \pmod{8}$ or with $d \equiv 7 \pmod{8}$ and Y odd in one of the following cases:*

- (a) *there exist $u, v \in \mathbb{Z}$ such that $X + c\sqrt{-d} = (u + v\sqrt{-d})^n$ and $Y = u^2 + dv^2$.*
- (b) *$d \equiv 3 \pmod{8}$ and there exist $u, v \in \mathbb{Z}$ with $u \equiv v \equiv 1 \pmod{2}$ such that $X + c\sqrt{-d} = (\frac{u+v\sqrt{-d}}{2})^3$ and $Y = \frac{u^2+dv^2}{4}$.*
- (c) *$n = 3$ if $D = 3u^2 \pm 8$ or if $D = 3u^2 \pm 1$ for some $u \in \mathbb{Z}$.*
- (d) *$n = 5$ if $D \in \{19, 341\}$.*
- (e) *$n \mid h$.*

Proof. If $d \not\equiv 7 \pmod{8}$ then the Lemma is a reformulation of a theorem of Cohn [21]. So, it remains the case when in (3) $d \equiv 7 \pmod{8}$ and Y is odd. In this case, we may apply a result of Ljunggren [31] (pp. 593–594) to conclude that if in equation (3) $n \nmid h$ then there exist $u_1, v_1 \in \mathbb{Z}$ such that

$$X + c\sqrt{-d} = \left(\frac{u_1 + v_1\sqrt{-d}}{2}\right)^n, \quad u_1 \equiv v_1 \pmod{2}. \tag{9}$$

If in (9), u_1 and v_1 are both odd then since $d \equiv 7 \pmod{8}$, we get

$$u_1^2 + dv_1^2 \equiv 0 \pmod{8},$$

whence, by

$$Y = \frac{u_1^2 + dv_1^2}{4},$$

it follows that Y is even, a contradiction. So, u_1 and v_1 are both even and the Lemma is proved. □

Recall that a *Lucas-pair* is a pair (α, β) of algebraic integers such that $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime rational integers and α/β is not a root of unity. Given a Lucas-pair (α, β) one defines the corresponding sequence of *Lucas numbers* by

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (n = 0, 1, 2, \dots).$$

A prime number p is called a *primitive divisor* of L_n if p divides L_n but does not divide $(\alpha - \beta)^2 L_1 \cdots L_{n-1}$.

The next Lemma is a deep result of Bilu, Hanrot and Voutier [11] concerning the existence of primitive prime divisors in a Lucas sequence.

LEMMA 2. *Let $L_n = L_n(\alpha, \beta)$ be a Lucas sequence. If $n \geq 5$ is a prime then L_n has a primitive prime divisor except for finitely many pairs (α, β) which are explicitly determined in Table 1 of [11].*

Proof. This follows from Theorem 1.4 of [11] and Theorem 1 of [53]. □

The next lemma gives all the squarefree values of d for which the class number $h(-d)$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ is 2 or 3.

LEMMA 3. *Let $d > 0$ be a squarefree integer. Denote by $h(-d)$, the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Then $h(-d) = 2$ if and only if $d \in T_2$, where*

$$T_2 = \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\},$$

and $h(-d) = 3$ if and only if $d \in T_3$, where

$$T_3 = \{23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907\}.$$

Proof. See [6] and [44]. □

The following lemma is a special case of a result of Bennett and Skinner [8].

LEMMA 4. *Suppose that x, y, z are non-zero, pairwise coprime integers. If $m \geq 5$ is an integer then the equation*

$$x^m + y^m = 2z^2$$

has the only solutions $(m, x, y, z) = \{(5, 3, -1, \pm 11), (5, -1, 3, \pm 11)\}$.

If $m \geq 4$ then the equation

$$x^m + y^m = 3z^2$$

has no solutions.

If $m \geq 7$ prime then the equation

$$x^m + 8y^m = 3z^2$$

has no solutions with $xy \neq \pm 1$.

4. Proof of Theorem 1. Consider equation (3) satisfying assumption (4). Without loss of generality, we may distinguish three cases according that in (3) $n \in \{3, 4\}$ or $n \geq 5$ prime. Denote by T the set $T = T_2 \cup T_3$.

CASE 1. $n \geq 5$ prime.

By a result of Arif and Muriefah [5], we have to consider only those square-free values of $d \in T$ which are not primes. In what follows we work in the field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ for $d \in T$. Denote by $h(-d)$ the class number of \mathbb{K} . One can see that equation (3) fulfils the assumptions of Lemma 1. Since for $d \in T$ we have $d \neq 19, 341$ and since $h(-d) \in \{2, 3\}$, by $n \geq 5$ prime, we conclude that equation (3) can have a solution only in case (a) of Lemma 1. Namely, applying to (3) the parametrisation provided by Lemma 1 and taking complex conjugation, we get

$$(x + d^l \sqrt{-d}) = (u + v \sqrt{-d})^n \text{ and } (x - d^l \sqrt{-d}) = (u - v \sqrt{-d})^n \tag{10}$$

for some $u, v \in \mathbb{Z}$. Further, we also have $y = u^2 + dv^2$.

If in (3) $y > 1$ is even and d is also even then it is clear that x has to be even, too. But this is a contradiction since x and y are coprime. If in (3) $y > 1$ is even and d is odd, we obviously have that x is odd. Since for any odd integer a we have $a^2 \equiv 1 \pmod{8}$ and since $n \geq 5$ we get that $1 + d \equiv 0 \pmod{8}$ by reducing (3) modulo 8. This leads to $d \equiv 7 \pmod{8}$ for $d \in T$. But for these values of d , we consider only odd values of y . Hence, in what follows we may assume that in (3) $y > 1$ is odd (and hence $x \geq 1$ is even). By (a) of Lemma 1, we see that $u \mid x$ and since $y > 1$ is odd and $\gcd(x, y) = 1$ we get that $\gcd(2u, y) = 1$.

Let $\alpha = u + v \sqrt{-d}$ and $\beta = u - v \sqrt{-d}$. Then $\gcd(\alpha\beta, \alpha + \beta) = \gcd(y, 2u) = 1$. If α/β is a root of unity then by $d \in T$ we have $\alpha/\beta \in \{\pm 1\}$. This leads to $u = 0$ or $v = 0$.

By (10) $u = 0$ yields $x = 0$, while $v = 0$ yields $d^l = 0$, which is a contradiction by (3) and (4). Thus,

$$L_n := \frac{(u + v\sqrt{-d})^n - (u - v\sqrt{-d})^n}{2v\sqrt{-d}} \tag{11}$$

is a Lucas sequence. Further, by (a) of Lemma 1 we obtain

$$L_n = \frac{d^l}{v}. \tag{12}$$

Since $(\alpha - \beta)^2 = -4dv^2$, we see by (12) and by the definition of the primitive prime divisor that L_n cannot have a primitive prime divisor. But for $d \in T$, Lemma 2 implies that L_n has a primitive prime divisor for $n \geq 5$ prime. This is a contradiction. Therefore, our equation (3) does not have any solution for $n \geq 5$ prime and $d \in T$ (recall, that y is odd if $d \equiv 7 \pmod{8}$).

CASE 2. $n = 4$

By factorising (3), we get

$$(y^2 + x)(y^2 - x) = d^{2l+1}. \tag{13}$$

Since in this case we can also suppose that y is odd and x is even, we obtain by $\gcd(x, y) = 1$ that $\gcd(y^2 + x, y^2 - x) = 1$ also holds. Further, since x, y and d are positive (13) implies that $y^2 + x$ and $y^2 - x$ are positive, too. Hence from (13), we have for some $d_1, d_2 \in \mathbb{Z}$

$$y^2 + x = d_1^{2l+1} \text{ and } y^2 - x = d_2^{2l+1},$$

where $d_1 > 0$ and $d_2 > 0$, $\gcd(d_1, d_2) = 1$ and $d_1d_2 = d > 0$. Hence, we get

$$d_1^{2l+1} + d_2^{2l+1} = 2y^2, \tag{14}$$

which is a ternary equation. By applying Lemma 4 to (14), we get that $(d_1, d_2) \in \{(3, -1), (-1, 3)\}$ for $l \geq 2$. However, this contradicts the fact that d_1 and d_2 are positive. Hence, there are no solutions to (14) for $l \geq 2$, which implies that there are no solutions to our equation (3) in this case with $l \geq 2$. Finally, if in (14) $l \in \{0, 1\}$ then we can easily enumerate the solutions of (14) (and hence the solutions of our original equation (3)) since d_1 and d_2 are positive divisors of d . We get $(x, y, n, d, l) \in \{(1, 2, 4, 15, 0), (15, 4, 4, 31, 0), (6083, 78, 4, 23, 1)\}$, and we see that in the above solutions, we have $d \equiv 7 \pmod{8}$ and y is even, which is excluded. Thus, we do not have any solution to (3) fulfilling the assumptions of Theorem 1.

CASE 3. $n = 3$

Suppose first that $h(-d) = 2$, i.e. $d \in T_2$. The cases $d \in \{5, 6, 10, 13, 15, 22, 35\}$ were considered earlier as a consequence of some results concerning the resolution of equations of the form $x^2 + p^a q^b = y^3$. See for example, [17, 23, 32, 42]. Therefore, we may assume that $d \in T_2$ and $d \geq 37$. As we mentioned in Remark 1, the well-known method for solving our equation (3) by searching S -integral points on some elliptic curves does not work in some cases. Namely, if $d = 427$ the computation of the

Mordell-Weil group of the curve

$$w^2 = t^3 - 427^5$$

becomes very time consuming. Hence, we need another approach, namely we use the parametrisation provided by Lemma 1. We see that equation (3) can have a solution in the cases (a), (b), (c) of Lemma 1. We consider these cases separately. By equating imaginary parts in case (a) (of Lemma 1), we obtain

$$v(3u^2 - dv^2) = d^l \tag{15}$$

Since in (3) we have $\gcd(x, y) = 1$ then by $u \mid x$ and $y = u^2 + dv^2$ we see that $\gcd(u, v) = 1$ also holds. Now, if in (15) we have $p \mid v$ and $p \mid 3u^2 - dv^2$ for some prime p then since $\gcd(u, v) = 1$ we get that $p = 3$. By (15) we get that $p = 3$ has to divide d . This yields

$$\gcd(v, 3u^2 - dv^2) = 1 \text{ if } 3 \nmid d \text{ or } \gcd(v, 3u^2 - dv^2) = 3 \text{ if } 3 \mid d.$$

Suppose that $\gcd(v, 3u^2 - dv^2) = 3$. Hence, $3 \mid d$ and therefore by $d \in T_2$ and $d \geq 37$, we see that $d \in \{51, 123, 267\}$. For these values of d , we transform our equation (3) to the form

$$w^2 = t^3 - d^i, \tag{16}$$

where

$$d \in \{51, 123, 267\}, 2l + 1 = 6l_1 + i, l_1 \in \mathbb{Z}_{\geq 0}, i \in \{1, 3, 5\}, w = x/d^{3l_1} \text{ and } t = y/d^{2l_1}. \tag{17}$$

We have to give all the S -integral points of the above curves, where S consists of the prime divisors of 51, 123 and 267 respectively. Now, using the computer algebra package MAGMA (see [12]), we see that for $d \in \{51, 123, 267\}$ and $i \in \{3, 5\}$ the rank of the Mordell-Weil group of the curves (16) is 0. Therefore, the computation of S -integral points on these elliptic curves is trivial. If in (16) we have $i = 1$, MAGMA was able to compute the Mordell-Weil group and hence we can list all the S -integral points in this case, too. By using transformation (17), we see that we do not get any solution of our original equation (3).

If in (15) we have $\gcd(v, 3u^2 - dv^2) = 1$ (which occurs if $3 \nmid d$) then by (15) we obtain some systems of equations of the form

$$\begin{cases} 3u^2 - dv^2 = \pm d_1^l \\ v = \pm d_2^l, \end{cases} \tag{18}$$

where $d_1 > 0, d_2 > 0$ are positive divisors of d with $d = d_1 d_2$ and $\gcd(d_1, d_2) = 1$. If in (18) $l = 0$ then we obtain $v = \pm 1$ and $3u^2 = d \pm 1$. For $d \in T_2$ and $3 \nmid d$ this yields $(u, v, d) = (\pm 2, \pm 1, 13)$. Since $y = u^2 + dv^2$ we obtain the solution $(x, y, n, d, l) = (70, 17, 3, 13, 0)$ to our equation (3), which is already known (see e.g. [23]). Hence, we may suppose that in (18) $l \geq 1$. Now, if $d_1 \neq 1$ then d_1 has a prime factor $q \geq 2$. Since $3 \nmid d$ and $d_1 \mid d$ we get by (18) that $q \mid u$. But by $q \mid d, u \mid x$ and $y = u^2 + dv^2$ we infer that q is a common prime factor of x and y . This is a contradiction in view of (4). Finally, if in (18) $l \geq 1$ and $d_1 = 1$ we obviously have that $d_2 = d$. Hence by (18), we

obtain the equation

$$3u^2 = d^{2l+1} \pm 1, \tag{19}$$

which is a ternary equation for which we can apply Lemma 4. We obtain that if $l \geq 2$ then (19) has no solutions. If in (19) $l = 1$ we obtain $3u^2 = d^3 \pm 1$ and $v = \pm d$. But for $d \in T_2$ and $3 \nmid d$ we do not obtain any solution.

Consider now the case (b) of Lemma 1. By equating imaginary parts, we obtain

$$v(3u^2 - dv^2) = 8d^l, \tag{20}$$

where $d \equiv 3 \pmod{8}$, u and v are odd integers and $y = \frac{u^2 + dv^2}{4}$. We work as in the previous case. Since $\gcd(v, 3u^2 - dv^2) = 1$ and v is odd equation (20) leads to systems of equations of the form

$$\begin{cases} 3u^2 - dv^2 = \pm 8d_1^l \\ v = \pm d_2^l, \end{cases} \tag{21}$$

where $d_1 > 0, d_2 > 0$ are divisors of d with $d = d_1 d_2$ and $\gcd(d_1, d_2) = 1$. If in (21) $l = 0$ then we have $v = \pm 1$ and $3u^2 = d \pm 8$. For $d \in T_2$ and $d \equiv 3 \pmod{8}$, this yields $(u, v, d) \in \{(\pm 3, \pm 1, 35), (\pm 9, \pm 1, 235)\}$. Since $y = (u^2 + dv^2)/4$ we obtain two solutions to our equation (3), namely $(x, y, n, d, l) \in \{(36, 11, 3, 35, 0), (702, 79, 3, 235, 0)\}$. The case $l \geq 1$ and $d_1 \neq 1$ leads to a contradiction since x and y are coprime. Finally, if in (21) $l \geq 1$ and $d_1 = 1$ (and hence $d_2 = d$), we get

$$3u^2 = d^{2l+1} \pm 8 \cdot 1^{2l+1}, \tag{22}$$

which is a ternary equation of signature $(2l + 1, 2l + 1, 2)$. By applying Lemma 4 to (22), we see that equation (22) does not have a solution if $2l + 1 \geq 7$ is a prime. Hence, we may suppose that in (22) $2l + 1 = 3^a 5^b$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Now, if $a \geq 1$ then we may transform (22) to an elliptic equation

$$w^2 = t^3 \pm 8 \cdot 3^3, \tag{23}$$

where $w = 9u$ and $t = 3d^{3^{a-1}5^b}$. By using MAGMA, we get $(t, w) \in \{(-6, 0), (6, 0), (\pm 28, 10), (\pm 189, 33)\}$ which do not lead to a solution of the original equation (3).

Recall that $2l + 1 = 3^a 5^b$. Consider now the case if in (22) we have $b \geq 1$. Then we may transform (22) to the form

$$w^2 = t^5 \pm 8 \cdot 3^5, \tag{24}$$

where $w = 27u$ and $t = 3d^{3^a 5^{b-1}}$. We see that the curves occurring in (24) are hyperelliptic curves of genus 2. By using MAGMA, we infer that the Jacobian of the above curves has rank 0. Therefore, we may apply the method of Chabauty to give all the rational points of the above curves. Using MAGMA again we get that the curves in (24) do not have any rational points. Hence, there are no solutions to our original equation (3) in this case.

Finally, we have to consider the case when in (3) $(h(-d), n) = (3, 3)$ holds (and hence $d \in T_3$). In this case, the parametrisation provided by Lemma 1 is not enough

for our purposes. Hence, we follow the idea of Mignotte and de Weger [37] to reduce the problem to several Thue-Mahler equations of degree 3. Then we consider these equations locally to prove the non-solvability. This approach works well except three cases, namely if $d \in \{23, 307, 547\}$. In these cases, we reduce the problem to finding S -integral points on some elliptic curves.

We illustrate the reduction of (3) to some Thue-Mahler equations if $d = 883$ (see e.g. [37]). For other values of $d \in T_3$, the reduction can be done in the same way. Consider equation

$$x^2 + 883^{2l+1} = y^3. \tag{25}$$

By factorising in the field $\mathbb{Q}(\sqrt{-883})$ we get the following ideal equation

$$\langle x + 883^l \sqrt{-883} \rangle \langle x - 883^l \sqrt{-883} \rangle = \langle y \rangle^3. \tag{26}$$

Hence, we have

$$\langle x + 883^l \sqrt{-883} \rangle = \mathcal{A}^3 \mathcal{B} \tag{27}$$

for some ideals \mathcal{A} and \mathcal{B} , where \mathcal{B} is the third-power-free part of $\langle x + 883^l \sqrt{-883} \rangle$. Let $\bar{\mathcal{B}}$ denote the conjugate ideal of \mathcal{B} . Then by (26) we infer that $\mathcal{B}\bar{\mathcal{B}} = \mathcal{C}^3$ for some ideal \mathcal{C} . Since $\mathcal{B} \mid \langle -2 \cdot 883^l \sqrt{-883} \rangle$ this implies that in (27) $\mathcal{B} = \langle 1 \rangle$. Hence, we get

$$\langle x + 883^l \sqrt{-883} \rangle = \mathcal{A}^3. \tag{28}$$

Pick a rational prime p which splits in $\mathbb{Q}(\sqrt{-883})$. We see that $\langle 13 \rangle = \mathcal{P}\bar{\mathcal{P}}$ holds. The ideal class of \mathcal{P} has order 3 in the class group and hence \mathcal{P}^3 is a principal ideal. Further, there exists an integer $k \in \{-1, 0, 1\}$ such that $\mathcal{P}^{-k}\mathcal{A}$ is principal. Hence, we infer

$$\mathcal{P}^{-k}\mathcal{A} = \left\langle u + v \frac{1 + \sqrt{-883}}{2} \right\rangle, \tag{29}$$

where $u, v \in 13^{-\max\{0,k\}}\mathbb{Z}$. Thus, (29) implies

$$\mathcal{P}^{-3k}\mathcal{A}^3 = \left\langle u + v \frac{1 + \sqrt{-883}}{2} \right\rangle^3. \tag{30}$$

Denote by γ the generator of the principal ideal \mathcal{P}^3 . Since $\mathcal{P}^3 = \left\langle \frac{29+3\sqrt{-883}}{2} \right\rangle$ (i.e $\gamma = \frac{29+3\sqrt{-883}}{2}$) we get by (28) and (30) that

$$\langle x + 883^l \sqrt{-883} \rangle = \left\langle \frac{29 + 3\sqrt{-883}}{2} \right\rangle^k \left\langle u + v \frac{1 + \sqrt{-883}}{2} \right\rangle^3. \tag{31}$$

Since the only units in $\mathbb{Q}(\sqrt{-883})$ are $\{\pm 1\}$ by (31) we may write

$$x + 883^l \sqrt{-883} = \left(\frac{29 + 3\sqrt{-883}}{2} \right)^k (U + V\sqrt{-883}), \tag{32}$$

where $k \in \{-1, 0, 1\}$ and

$$U = \frac{1}{2}(2u^3 + 3u^2v - 1323uv^2 - 662v^3), \quad V = \frac{1}{2}(3u^2v + 3uv^2 - 220v^3). \quad (33)$$

Hence, we obtain three Thue-Mahler equations, according to $k = 0, 1$ and -1 . Table 1 contains the values of $d \in T_3$, our choice for the prime p which splits in $\mathbb{Q}(\sqrt{-d})$, the generator γ of \mathcal{P}^3 we choose and the resulting Thue-Mahler equations.

We see that the equations corresponding to the case $k = 0$ in Table 1 are reducible. In this case, we reduce the problem to some ternary equations for which the modular method can be applied. We present the method in the case $d = 883$. For the other values of $d \in T_3$, we can apply the same approach. Consider the equation for $d = 883$ with $k = 0$, i.e.

$$3u^2v + 3uv^2 - 220v^3 = 2 \cdot 883^l. \quad (34)$$

If in equation (34) $l = 0$ then we obtain a Thue-equation, which is easy to solve. Hence, we may suppose that in (34) $l \geq 1$. It is clear that in (34) u and v cannot be both even, since otherwise we get a contradiction by reducing (34) modulo 4. Now, if in (34) $\gcd(u, v) \neq 1$ then it follows obviously that $883 \mid u$ and $883 \mid v$. But, using (32) with $k = 0$ then by (33) we get that $883 \mid x$. Since $l \geq 1$ our original equation (3) implies that $883 \mid y$. This is a contradiction in view of $\gcd(x, y) = 1$. Hence, we may assume that in (34) $\gcd(u, v) = 1$ (and $l \geq 1$). Thus, we have to solve four systems of equations of the form

$$\begin{cases} 3u^2 + 3uv - 220v^2 = \pm f \\ v = \pm g, \end{cases} \quad (35)$$

where $(f, g) \in \{(2 \cdot 883^l, 1), (883^l, 2), (2, 883^l), (1, 2 \cdot 883^l)\}$. If $(f, g) \in \{(2 \cdot 883^l, 1), (883^l, 2)\}$, we get by (35) that

$$\begin{cases} 3(2u \pm 1)^2 = 883 \pm 8 \cdot 883^l, \text{ if } (f, g) = (2 \cdot 883^l, 1) \\ 3(2u \pm 1)^2 = 883 \pm 883^l, \text{ if } (f, g) = (883^l, 2). \end{cases} \quad (36)$$

If in (36) we have $l = 1$ then it is clear that (36) leads to a contradiction. Hence, we may suppose that in (36) $l \geq 2$. Thus, $883 \mid (2u \pm 1)^2$ and therefore $883^2 \mid (2u \pm 1)^2$. But this is a contradiction by reducing (36) modulo 883^2 . Finally, if in (35) we have $(f, g) \in \{(2, 883^l), (1, 2 \cdot 883^l)\}$ we get by (35) the following equations

$$\begin{cases} 3\left(\frac{2u+v}{2}\right)^2 = 883^{2l+1} \pm 1^{2l+1}, \text{ if } (f, g) = (1, 2 \cdot 883^l) \\ 3(2u \pm v)^2 = 883^{2l+1} \pm 8 \cdot 1^{2l+1}, \text{ if } (f, g) = (2, 883^l). \end{cases} \quad (37)$$

For the first equation of (37), we can apply Lemma 4 to conclude that $l = 1$. (Note, that in the first equation of (37), $\frac{2u+v}{2}$ is a rational integer since v is even (i.e. $v = 2 \cdot 883^l$)). The case $l = 1$ does not lead to a solution. For the second equation of (37), we apply Lemma 4 to conclude that this equation does not have any solution if $2l + 1$ consists only of primes ≥ 7 . Finally, if $3 \mid 2l + 1$ or $5 \mid 2l + 1$ then we use the method already applied in the case $(h(-d), n) = (2, 3)$. Namely, we reduce the second equation of (37) to some elliptic and hyperelliptic equations of genus 2. Then we apply MAGMA to solve the equations under consideration. We see that there

Table 1. Resulting Thue-Mahler equations

d	p	γ	Thue-Mahler equations
23	2	$\frac{-3+\sqrt{-23}}{2}$	$3u^2v - 3uv^2 - 5v^3 = 2 \cdot 23^l$ $u^3 - 6u^2v - 12uv^2 + 16v^3 = 2^4 \cdot 23^l$ $-u^3 - 3u^2v + 21uv^2 - v^3 = 2^4 \cdot 23^l$
31	5	$1 - 2\sqrt{-31}$	$3u^2v - 3uv^2 - 7v^3 = 2 \cdot 31^l$ $-4u^3 + 9u^2v + 87uv^2 - 53v^3 = 2 \cdot 5^3 \cdot 31^l$ $4u^3 - 3u^2v - 93uv^2 + 39v^3 = 2 \cdot 5^3 \cdot 31^l$
59	3	$\frac{-7-\sqrt{-59}}{2}$	$3u^2v - 3uv^2 - 14v^3 = 2 \cdot 59^l$ $-u^3 - 9u^2v + 54uv^2 + 27v^3 = 2 \cdot 3^3 \cdot 59^l$ $u^3 - 12u^2v - 33uv^2 + 71v^3 = 2 \cdot 3^3 \cdot 59^l$
83	3	$\frac{5-\sqrt{-83}}{2}$	$3u^2v + 3uv^2 - 20v^3 = 2 \cdot 83^l$ $-u^3 + 6u^2v + 69uv^2 - 19v^3 = 2 \cdot 3^3 \cdot 83^l$ $u^3 + 9u^2v - 54uv^2 - 81v^3 = 2 \cdot 3^3 \cdot 83^l$
107	3	$\frac{1+\sqrt{-107}}{2}$	$3u^2v + 3uv^2 - 26v^3 = 2 \cdot 107^l$ $u^3 + 3u^2v - 78uv^2 - 53v^3 = 2 \cdot 3^3 \cdot 107^l$ $-u^3 + 81uv^2 + 27v^3 = 2 \cdot 3^3 \cdot 107^l$
139	5	$\frac{19-\sqrt{-139}}{2}$	$3u^2v + 3uv^2 - 34v^3 = 2 \cdot 139^l$ $-u^3 + 27u^2v + 132uv^2 - 271v^3 = 2 \cdot 5^3 \cdot 139^l$ $u^3 + 30u^2v - 75uv^2 - 375v^3 = 2 \cdot 5^3 \cdot 139^l$
211	5	$\frac{-17-\sqrt{-211}}{2}$	$3u^2v + 3uv^2 - 52v^3 = 2 \cdot 211^l$ $-u^3 - 27u^2v + 132uv^2 + 521v^3 = 2 \cdot 5^3 \cdot 211^l$ $u^3 - 24u^2v - 183uv^2 + 363v^3 = 2 \cdot 5^3 \cdot 211^l$
283	7	$\frac{33-\sqrt{-283}}{2}$	$3u^2v + 3uv^2 - 70v^3 = 2 \cdot 283^l$ $-u^3 + 48u^2v + 261uv^2 - 1049v^3 = 2 \cdot 7^3 \cdot 283^l$ $u^3 + 51u^2v - 162uv^2 - 1261v^3 = 2 \cdot 7^3 \cdot 283^l$
307	7	$\frac{6-\sqrt{-307}}{2}$	$3u^2v + 3uv^2 - 76v^3 = 2 \cdot 307^l$ $-2u^3 + 15u^2v + 477uv^2 - 226v^3 = 2 \cdot 7^3 \cdot 307^l$ $2u^3 + 21u^2v - 441uv^2 - 686v^3 = 2 \cdot 7^3 \cdot 307^l$
331	5	$\frac{-13+\sqrt{-331}}{2}$	$3u^2v - 3uv^2 - 82v^3 = 2 \cdot 331^l$ $u^3 - 21u^2v - 228uv^2 + 657v^3 = 2 \cdot 5^3 \cdot 331^l$ $-u^3 - 18u^2v + 267uv^2 + 409v^3 = 2 \cdot 5^3 \cdot 331^l$
379	5	$\frac{-11-\sqrt{-379}}{2}$	$3u^2v + 3uv^2 - 94v^3 = 2 \cdot 379^l$ $-u^3 - 18u^2v + 267uv^2 + 659v^3 = 2 \cdot 5^3 \cdot 379^l$ $u^3 - 15u^2v - 300uv^2 + 375v^3 = 2 \cdot 5^3 \cdot 379^l$
499	5	$\frac{1+\sqrt{-499}}{2}$	$3u^2v - 3uv^2 - 124v^3 = 2 \cdot 499^l$ $u^3 + 3u^2v - 372uv^2 - 249v^3 = 2 \cdot 5^3 \cdot 499^l$ $-u^3 + 375uv^2 + 125v^3 = 2 \cdot 5^3 \cdot 499^l$
547	11	$-28 + \sqrt{-547}$	$3u^2v - 3uv^2 - 136v^3 = 2 \cdot 547^l$ $2u^3 - 87u^2v - 735uv^2 + 4218v^3 = 2 \cdot 11^3 \cdot 547^l$ $-2u^3 - 81u^2v + 903uv^2 + 3398v^3 = 2 \cdot 11^3 \cdot 547^l$
643	7	$\frac{27-\sqrt{-643}}{2}$	$3u^2v + 3uv^2 - 160v^3 = 2 \cdot 643^l$ $-2u^3 + 78u^2v + 1044uv^2 - 3838v^3 = 4 \cdot 7^3 \cdot 643^l$ $2u^3 + 84u^2v - 882uv^2 - 4802v^3 = 4 \cdot 7^3 \cdot 643^l$
883	13	$\frac{29+3\sqrt{-883}}{2}$	$3u^2v + 3uv^2 - 220v^3 = 2 \cdot 883^l$ $3u^3 + 48u^2v - 1941uv^2 - 4183v^3 = 2 \cdot 13^3 \cdot 883^l$ $-3u^3 + 39u^2v + 2028uv^2 - 2197v^3 = 2 \cdot 13^3 \cdot 883^l$
907	13	$\frac{25+3\sqrt{-907}}{2}$	$3u^2v - 3uv^2 - 226v^3 = 2 \cdot 907^l$ $3u^3 + 33u^2v - 2076uv^2 - 1805v^3 = 2 \cdot 13^3 \cdot 907^l$ $-3u^3 + 42u^2v + 2001uv^2 - 3845v^3 = 2 \cdot 13^3 \cdot 907^l$

are no solutions in the case when $d = 883$. We mention that the equations occurring in Table 1 corresponding to the choices $(k, l, d) \in \{(0, 0, 139), (0, 0, 499)\}$ have the solutions $(u, v) \in \{(\pm 3, \pm 1), (\pm 4, \pm 1)\}$ and $(u, v) \in \{(\pm 6, \pm 1), (\pm 7, \pm 1)\}$, respectively, which lead to the solutions $(x, y, n, d, l) \in \{(322, 47, 3, 139, 0), (2158, 167, 3, 499, 0)\}$ to our original equation (3).

Finally, we consider the “irreducible” case, i.e. the Thue-Mahler equations corresponding to the choice $k \in \{1, -1\}$. If $d \in \{31, 59, 83, 107, 139, 211, 283, 331, 379, 499, 643, 883, 907\}$ then we consider the Thue-Mahler equations locally to prove the unsolvability. Namely, for the above values of d the occurring Thue-Mahler equations lead to a contradiction by reducing them modulo 4. If $d \in \{23, 307, 547\}$, the local consideration does not work. Hence, in this case, we transform our original equation to several elliptic equations of the form

$$w^2 = t^3 - d^i, \quad (38)$$

where

$$d \in \{23, 307, 547\}, \quad 2l + 1 = 6l_1 + i, \quad l_1 \in \mathbb{Z}_{\geq 0}, \quad i \in \{1, 3, 5\}, \quad w = x/d^{3l_1} \text{ and } t = y/d^{2l_1}. \quad (39)$$

For the elliptic curves occurring in (38), MAGMA was able to compute the Mordell-Weil group and hence all the S -integral points, where $S = \{23\}, \{307\}$ and $\{547\}$, respectively. Namely, we get the following points on the curves (38): $(t, w, d, i) \in \{(3, \pm 2, 23, 1), (23, 0, 23, 3), (71, \pm 588, 23, 3), (7, \pm 6, 307, 1), (11, \pm 32, 307, 1), (71, \pm 598, 307, 1), (939787, \pm 911054064, 307, 1), (307, 0, 307, 3), (5219, \pm 376996, 307, 3), (11, \pm 28, 547, 1), (547, 0, 547, 3)\}$. Finally, using the substitution (39), we see that these solutions lead to the following solutions of our original equation (3) satisfying assumption (4): $(x, y, n, d, l) \in \{(2, 3, 3, 23, 0), (588, 71, 3, 23, 1), (6, 7, 3, 307, 0), (32, 11, 3, 307, 0), (598, 71, 3, 307, 0), (911054064, 939787, 3, 307, 0), (28, 11, 3, 547, 0)\}$.

This completes the proof of our theorem. \square

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REFERENCES

1. S. A. Arif and F. S. A. Muriefah, on the diophantine equation $x^2 + 2^k = y^n$, *Internat. J. Math. Math. Sci.* **20** (1997), 299–304.
2. S. A. Arif and F. S. A. Muriefah, the diophantine equation $x^2 + 3^m = y^n$, *Internat. J. Math. Math. Sci.* **21** (1998), 619–620.
3. S. A. Arif and F. S. A. Muriefah, The Diophantine equation $x^2 + q^{2k} = y^n$, *Arab. J. Sci. Eng. Sect. A Sci.* **26** (2001), 53–62.
4. S. A. Arif and F. S. A. Muriefah, On the Diophantine equation $x^2 + 2^k = y^n$ II, *Arab J. Math. Sci.* **7** (2001), 67–71.
5. S. A. Arif and F. S. A. Muriefah, On the Diophantine equation $x^2 + q^{2k+1} = y^n$, *J. Number Theory* **95** (2002), 95–100.
6. S. Arno, M. L. Robinson and F. S. Wheeler, Imaginary quadratic fields with small odd class number, *Acta. Arith.* **83** (1998), 295–330.

7. M. A. Bennett, J. S. Ellenberg and N. C. Ng, *The Diophantine equation $A^4 + 2^d B^2 = C^n$* submitted.
8. M. A. Bennett and C. M. Skinner, Ternary diophantine equations via Galois representations and modular forms, *Canad. J. Math.* **56**(1) (2004), 23–54.
9. A. Bérczes, B. Brindza and L. Hajdu, On power values of polynomials, *Publ. Math. Debrecen* **53** (1998), 375–381.
10. A. Bérczes and I. Pink, On the diophantine equation $x^2 + p^{2k} = y^n$, *Arch. Math.* **91** (2008), 505–517.
11. Y. Bilu, G. Hanrot and P. M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte, *J. Reine Angew. Math.* **539** (2001), 75–122.
12. W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, Computational algebra and number theory (London, 1993), *J. Symbolic Comput.* **24**(3–4) (1997), 235–265.
13. Y. Bugeaud, On the diophantine equation $x^2 - p^m = \pm y^n$, *Acta. Arith.* **80** (1997), 213–223.
14. Y. Bugeaud, M. Mignotte and S. Siksek, Classical and modular approaches to exponential and diophantine equations II. The Lebesgue-Nagell equation, *Compos. Math.* **142**(1) (2006), 31–62.
15. Y. Bugeaud and F. S. Abu Muriefah, The Diophantine equation $x^2 + c = y^n$: a brief overview, *Rev. Colombiana Mat.* **40** (2006), 31–37.
16. I. N. Cangül, M. Demirci, G. Soydan and N. Tzanakis, On the Diophantine equation $x^2 + 5^a 11^b = y^n$, *Funct. Approx. Comment. Math.* (2010), to appear.
17. N. Cangül, M. Demirci, F. Luca, Á. Pintér and G. Soydan, On the Diophantine equation $x^2 + 2^a 11^b = y^n$, *Fibonacci Quart.* **48** (2010), 39–46.
18. J. H. E. Cohn, The diophantine equation $x^2 + 2^k = y^n$. II., *Internat J. Math. Math. Sci.*, **22** (1999), 459–462.
19. J. H. E. Cohn, The diophantine equation $x^2 + 2^k = y^n$, *Arch. Math (Basel)* **59** (1992), 341–344.
20. J. H. E. Cohn, The diophantine equation $x^2 + C = y^n$, *Acta. Arith.* **65** (1993), 367–381.
21. J. H. E. Cohn, The diophantine equation $x^2 + C = y^n$ II., *Acta. Arith.* **109** (2003), 205–206.
22. J. S. Ellenberg, Galois representations to \mathbb{Q} -curves and the generalized Fermat Equation $A^4 + B^2 = C^p$, *Amer. J. Math.* **126**(4) (2004), 763–787.
23. E. Goins, F. Luca and A. Togbe, On the Diophantine Equation $x^2 + 2^a 5^b 13^c = y^n$, ANTS VIII Proceedings: A. J. van der Poorten and A. Stein (eds.), *ANTS VIII, Lecture Notes in Computer Science* **5011** (2008), 430–442.
24. K. Györy, I. Pink and Á. Pintér, Power values of polynomials and binomial Thue-Mahler equations, *Publ. Math. Debrecen* **65** (2004), 341–362.
25. H. Heilbronn, On the class number in imaginary quadratic fields, *Quart. J. Math. Oxford Ser.* **25** (1934), 150–160.
26. M. Le, On Cohn’s conjecture concerning the diophantine equation $x^2 + 2^m = y^n$, *Arch. Math. Basel* **78**(1) (2002), 26–35.
27. M. Le, On the diophantine equation $x^2 + p^2 = y^n$, *Publ. Math. Debrecen* **63** (2003), 27–78.
28. M. Le and H. Zhu, On some generalized Lebesgue-Nagell equations, *J. N. Th.* **131** (3) (2011), 458–469.
29. V. A. Lebesgue, Sur l’impossibilité en nombres entiers de l’équation $x^m = y^2 + 1$, *Nouvelle Annales des Mathématiques* **9**(1) (1850), 178–181.
30. W. Ljunggren, Über einige arcustangensgleichungen die auf interessante unbestimmte gleichungen führen, *Ark. Mat. Astr. Fys.* **29A** (1943), 13.
31. W. Ljunggren, On the diophantine equation $Cx^2 + D = y^n$, *Pacific J. Math.* **14** (1964), 585–596.
32. F. Luca, On a diophantine equation, *Bull. Austral. Math. Soc.* **61** (2000), 241–246.
33. F. Luca, On the equation $x^2 + 2^a 3^b = y^n$, *Int. J. Math. Sci.* **29** (2002), 239–244.
34. F. Luca and A. Togbe, On the diophantine equation $x^2 + 7^{2k} = y^n$, *Fibonacci Quart.* **54**(4) (2007), 322–326.

35. F. Luca and A. Togbe, On the diophantine equation $x^2 + 2^a 5^b = y^n$, *Int. J. Number Th.* **4**(6) (2008), 973–979.
36. F. Luca, Sz. Tengely and A. Togbe, On the Diophantine Equation $x^2 + C = 4y^n$, *Ann. Sci. Math. Québec* **33** (2) (2009), 171–184.
37. M. Mignotte and B. M. M de Weger, On the equations $x^2 + 74 = y^5$ and $x^2 + 86 = y^5$, *Glasgow Math. J.* **38**(1) (1996), 77–85.
38. F. S. A. Muriefah, On the Diophantine equation $x^2 + 5^{2k} = y^n$, *Demonstratio Math.* **319**(2) (2006), 285–289.
39. F. S. A. Muriefah, F. Luca and A. Togbe, On the diophantine equation $x^2 + 5^a 13^b = y^n$, *Glasgow Math. J.* **50** (2008), 175–181.
40. T. Nagell, Sur l'impossibilité de quelques équations a deux indéterminées, *Norsk. Mat. Forenings Skifter* **13** (1923), 65–82.
41. T. Nagell, Contributions to the theory of a category of diophantine equations of the second degree with two unknowns, *Nova Acta Reg. Soc. Upsal.* **IV 16**, Uppsala (1955), 1–38.
42. I. Pink, On the diophantine equation $x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n$, *Publ. Math. Debrecen* **70**(1–2) (2007), 149–166.
43. I. Pink and Zs. Rábai, On the diophantine equation $x^2 + 5^k 17^l = y^n$, *Commun. Math* **19** (2011), 1–9.
44. P. Ribenboim, *Classical Theory of Algebraic Numbers* (Springer, New York, 2001), 636.
45. N. Saradha and A. Srinivasan, Solutions of some generalized Ramanujan-Nagell equations, *Indag. Math. (N.S.)* **17**(1) (2006), 103–114.
46. N. Saradha and A. Srinivasan, Solutions of some generalized Ramanujan-Nagell equations via binary quadratic forms, *Publ. Math. Debrecen* **71**(3–4) (2007), 349–374.
47. A. Schinzel and R. Tijdeman, On the equation $y^m = P(x)$, *Acta. Arith.* **31** (1976), 199–204.
48. T. N. Shorey, A. J. van der Poorten, R. Tijdeman and A. Schinzel, Applications of the Gel'fond-Baker method to diophantine equations, in *Transcendence Theory: Advances and Applications* (Academic Press, London-New York, San Francisco, 1977), 59–77.
49. T. N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, Cambridge Tracts in Mathematics, 87. (Cambridge University Press, Cambridge, UK, 1986) x+240 pp.
50. Sz. Tengely, On the diophantine equation $x^2 + a^2 = 2y^p$, *Indag. Math. (N.S.)* **15** (2004), 291–304.
51. Sz. Tengely, On the Diophantine equation $x^2 + q^{2m} = 2y^p$, *Acta Arith.* **127** (2007), 71–86.
52. Sz. Tengely, On the Diophantine Equation $x^2 + C = 2y^n$, *Int. J. N. Th.* **5** (6) (2009), 1117–1128.
53. P. M. Voutier, Primitive divisors of Lucas and Lehmer sequences, *Math. Comp.* **64** (1995), 869–888.
54. H. Zhu, A note on the Diophantine equation $x^2 + q^m = y^3$, *Acta. Arith.* **146** (2) (2011), 195–202.