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SHARP STARLIKENESS CONDITIONS FOR ANALYTIC FUNCTIONS WITH BOUNDED DERIVATIVE

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Abstract

We develop sharp conditions for various types of starlikeness for functions analytic in the unit disk with bounded derivatives. We also describe the precise range $\{zf'(z)/f(z) : z \in \mathbb{D}, f \in \mathscr{T}_\lambda\}$, where $f \in \mathscr{T}_\lambda$ means $f(0) = 0, f'(0) = 1$, and $|f'(z) - 1| \leq \lambda$ in the unit disc $\mathbb{D}$, and draw some conclusions from that.

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1. Introduction and statement of results

We denote the set of analytic functions in the unit disk by $\mathscr{H}(\mathbb{D})$ and by $\mathscr{B}$ the subset of functions $f \in \mathscr{H}(\mathbb{D})$ with $|f(z)| \leq 1$ in $\mathbb{D}$, while

$$\mathscr{B}_0 := \left\{ f \in \mathscr{B} \cap C^0(\overline{\mathbb{D}}) : f(0) = 0, \sup_{z \in \mathbb{D}} |f(z)| = 1 \right\}.$$

$\mathscr{A}$ consists of the functions $f \in \mathscr{H}(\mathbb{D})$ with the normalization $f(0) = f'(0) - 1 = 0$.

For a given $b \in \mathscr{B}_0$, and $0 < \lambda \leq 1$, define $\mathscr{T}_\lambda(b)$ to be the set of functions $f \in \mathscr{A}$ such that

$$|f'(z) - 1| \leq \lambda |b(z)|, \quad z \in \mathbb{D}, \tag{1.1}$$

that is

$$f(z) = z + \lambda \int_0^z b(t) w(t)\, dt, \quad w \in \mathscr{B}.$$

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In particular, for $b(z) = z$ we write $\mathscr{T}_\lambda(b) =: \mathscr{T}_\lambda$.

In [8] the following result was obtained.

THEOREM 1.1. *Let $b \in \mathscr{B}_0$ and*

$$c := \sup_{z\in\mathbb{D}} \int_0^1 |b(tz)|\,dt,$$

$\mu := 1/\sqrt{1+c^2}$. Then $c \leq 1/2$ and $\mathscr{T}_\mu(b) \subset \mathscr{S}^*$, where $\mathscr{S}^*$ is the set of starlike univalent functions in $\mathscr{A}$. If

$$b(t) = \max_{0\leq\varphi\leq 2\pi} \left|b\left(te^{i\varphi}\right)\right|, \quad 0 \leq t \leq 1, \tag{1.2}$$

then the constant μ cannot be replaced by any larger number without violating the conclusion.

REMARKS. (1) Various choices of b lead to function classes which are commonly investigated. For instance the choices $b(z) = z^n$ restrict the functions under consideration to those with

$$f(z) = z + a_{n+1}z^{n+1} + \cdots . \tag{1.3}$$

(2) An interesting feature of this result is the sharpness part. It can be shown that there does not exist extremal functions, that is for every single function

$$f(z) = z + \mu \int_0^z b(t)w(t)\,dt \in \mathscr{T}_\mu(b)$$

there exists $\beta_f > 0$, so that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \beta_f, \quad z \in \mathbb{D},$$

or similarly, there exists $\mu_f > \mu$ so that $z + \mu_f \int_0^z b(t)w(t)\,dt \in \mathscr{S}^*$ holds as well. This kind of result has first been obtained by Fournier [1], who dealt with the classical case $b(z) = z$, where μ turns out to be $2/\sqrt{5}$. This bound, without the sharpness statement, has also been known before [4, 10]. In [9] Samaris extended Fournier's result to functions f with the property

$$\left| f'(z)\left(\frac{f(z)}{z}\right)^{\gamma-1} - 1 \right| \leq \lambda|z|^n, \quad \gamma > 0. \tag{1.4}$$

In this paper we choose to deal only with functions of the form (1.1), that is, $\gamma = 1$ in (1.4).

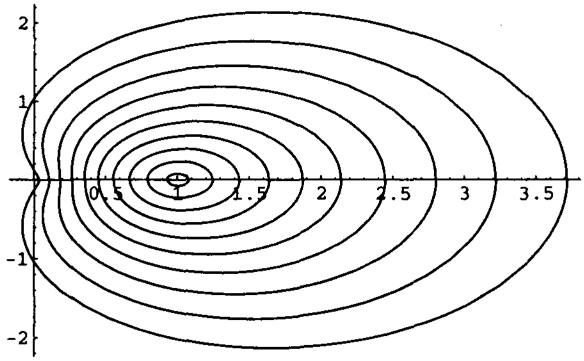

FIGURE 1. $\partial\Omega_{\lambda,1/2}$ for $\lambda = 0.05, (0.1), 0.95$

Our aim in this paper is to generalize Theorem 1.1 in various ways that enable us to obtain sharp bounds for other types of starlikeness. One obvious generalization of Theorem 1.1, which we do not include, is to get bounds for starlikeness of order α ($\operatorname{Re} zf'(z)/f(z) > \alpha$). For the case $b(z) = z$ this was already done by Fournier [2].

We begin with the study of the sets

$$\Gamma_\lambda(b) := \left\{ \frac{zf'(z)}{f(z)} : f \in \mathcal{T}_\lambda(b),\ z \in \mathbb{D} \right\},$$

and to this end we also define

$$\Omega_{\lambda,c} := \left\{ \frac{1+\lambda z}{1+\lambda c w} : z, w \in \mathbb{D} \right\}.$$

Note that $\Omega_{\lambda,c}$, as the union of circular discs (with w as parameter) containing the point 1, is a domain, starlike with respect to 1.

THEOREM 1.2. (i) *Let* $0 \not\equiv b \in \mathcal{B}_0$ *and* $c := \sup_{z\in\mathbb{D}} \int_0^1 |b(tz)|\,dt$. *Then*

$$\Gamma_\lambda(b) \subseteq \Omega_{\lambda,c}, \quad 0 < \lambda \leq 1.$$

(ii) *Furthermore, for each* $f \in \mathcal{T}_\lambda(b)$ *we have*

$$\overline{\left\{ \frac{zf'(z)}{f(z)} : z \in \mathbb{D} \right\}} \subset \Omega_{\lambda,c}.$$

(iii) *If, in addition, b satisfies the condition in* (1.2), *then*

$$\overline{\Gamma_\lambda(b)} = \overline{\Omega_{\lambda,c}}, \quad 0 < \lambda \leq 1. \tag{1.5}$$

The condition (1.2) holds for $b(z) = z$, so that (1.5) applies to $\mathcal{T}_\lambda$. In Theorem 2.1 below we shall describe the boundary of $\Omega_{\lambda,c}$ in parametric form, and this makes it possible to derive various sharp starlikeness statements. We give two examples. A function $f \in \mathscr{A}$ is said to be *strongly starlike of order* α, $0 < \alpha \leq 1$, if and only if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leq \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}.$$

Note that the case $\alpha = 1$ of the following corollary is Theorem 1.1.

COROLLARY 1.3. *Let $b \in \mathscr{B}_0$ and c be as before. For $0 < \alpha \leq 1$ let*

$$\mu(\alpha) := \frac{\sin(\alpha\pi/2)}{\sqrt{1 + 2c\cos(\alpha\pi/2) + c^2}}.$$

Then $f \in \mathscr{T}_{\mu(\alpha)}(b)$ implies that f is strongly starlike of order α. If b satisfies (1.2), *then the constant $\mu(\alpha)$ cannot be replaced by any larger number without violating the conclusion, although every single function $f \in \mathscr{T}_{\mu(\alpha)}(b)$ is strongly starlike of order $\alpha_f < \alpha$.*

The result in Corollary 1.3, without the sharpness part, was previously obtained by Ponnusamy and Singh [5].

A function $g \in \mathscr{A}$ is called *uniformly convex* if it maps every circular arc in $\mathbb{D}$, with center also in $\mathbb{D}$, univalently onto a convex arc. It is known [7] that $g \in \mathscr{A}$ has this property if and only if $f = zg'$ satisfies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D}. \tag{1.6}$$

Note that (1.6) means that $zf'(z)/f(z)$ is contained in a parabola.

COROLLARY 1.4. *Let $b \in \mathscr{B}_0$ and c be as before. If $f \in \mathscr{T}_{1/(c+2)}(b)$ and $f = zg'$ then g is uniformly convex. If b satisfies* (1.2), *then the constant* $1/(c+2)$ *cannot be increased without violating the conclusion.*

Taking $b(z) = z^n$ in Corollary 1.4 we find that $g \in \mathscr{A}$ is uniformly convex if $f = zg'$ satisfies

$$|f'(z) - 1| \leq \frac{n+1}{2n+3}, \quad z \in \mathbb{D},$$

and the bound is best possible. Non-sharp results for this case can also be found in [5].

A different situation where our method can be applied as well is the case of uniformly starlike functions $f \in \mathscr{A}$. A function f is called *uniformly starlike* if it maps each circular arc in $\mathbb{D}$ with center ζ, also in $\mathbb{D}$, univalently onto an arc starlike with respect to $f(\zeta)$.

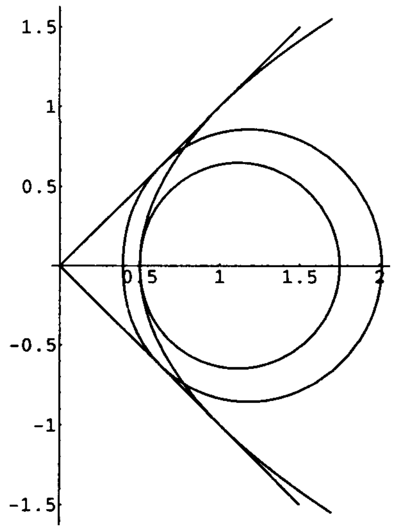

FIGURE 2. The situation for a strongly starlike function of order $1/2$ and uniformly convex with $b(z) = z$ (that is, $c = 1/2$).

THEOREM 1.5. *Let $b \in \mathscr{B}_0$ and $\mu := 1/\sqrt{2}$. Then each $f \in \mathscr{T}_\mu(b)$ is uniformly starlike, and, for every single $b \in \mathscr{B}_0$, the number μ is the largest one with this property. However, if with $b \in \mathscr{B}$*

$$f(z) = z + \mu \int_0^z b(t)w(t)\,dt \in \mathscr{T}_\mu(b)$$

satisfies $f' \in C^0(\overline{\mathbb{D}})$, then there exists $\mu_f > \mu$ such that $z + \mu_f \int_0^z b(t)w(t)\,dt$ is uniformly starlike as well.

This result is somewhat surprising, as it implies that, for instance, a restriction to functions of the form (1.3), however large n may be, does not increase the optimal value of μ. One of the first examples of a uniformly starlike function, given by Goodman in [3], was the second degree polynomial $f(z) = z + z^2/2\sqrt{2}$. Goodman showed that the coefficient of z^2 could be increased somewhat, and the polynomial would still be uniformly starlike. We see that this function satisfies the assumption of Theorem 1.5 (with $b(z) = z$) and that his observation reflects a more general fact.

Our final result deals with functions $f \in \mathscr{A}$ *starlike with respect to symmetrical points*, which are defined by the condition

$$\operatorname{Re} \frac{2zf'(z)}{f(z) - f(-z)} \geq 0, \quad z \in \mathbb{D}.$$

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Note that they form a superset of the uniformly starlike functions, but not a subset of $\mathscr{S}^*$. They are, however, close-to-convex univalent.

THEOREM 1.6. *Let* $b \in \mathscr{B}_0$ *and* $c := (1/2)\sup_{z\in\mathbb{D}} \int_{-1}^{1} |b(tz)|\,dt$, $\mu(c) := 1/\sqrt{1+c^2}$. *Then* $f \in \mathscr{T}_{\mu(c)}(b)$ *implies that* f *is starlike with respect to symmetrical points. If* b *is an even function satisfying* (1.2), *then the constants* $\mu(c)$ *are best possible.*

This result says, for instance, that $f \in \mathscr{T}_{2/\sqrt{5}}$ is starlike with respect to symmetrical points, but this is not known to be best possible ($b(z) = z$ is not even). However, if $f \in \mathscr{T}_{3/\sqrt{10}}$ is of the form $f(z) = z + a_3z^3 + a_4z^4 + \cdots$, then f is starlike with respect to symmetrical points, and the bound $3/\sqrt{10}$ is best possible (as we are dealing with the case $b(z) = z^2$).

As indicated before, the containment properties of function classes described in these theorems and corollaries are not too surprising, and some of them have been known before. What is striking here are the sharpness conclusions, and they all rely on the following single observation, which slightly extends previous results in [1] and in [8].

THEOREM 1.7. *Let* $x, y \in \overline{\mathbb{D}}$, $z_1, z_2 \in \partial\mathbb{D}$ *and* $b \in \mathscr{B}$. *Then there exists a sequence of functions* $V_k \in \mathscr{B} \cap \mathscr{H}(\overline{\mathbb{D}})$ *such that* $V_k(1) = z_1$ *and*

$$\lim_{k\to\infty} \int_x^y b(t)\,V_k(t)\,dt = z_2 \int_x^y b(t)\,dt. \tag{1.7}$$

The functions V_k *can be chosen independently of* x, y.

The proof of Theorem 1.7 will be given in the Appendix.

2. Proof of Theorem 1.2 and its corollaries

PROOF OF THEOREM 1.2. (i) Fix $a \in \Gamma_\lambda(b)$, so that there exists $f \in \mathscr{T}_\lambda(b)$, $z_0 \in \mathbb{D}$, with $a = z_0 f'(z_0)/f(z_0)$. We write

$$f(z) = z + \lambda \int_0^z b(t)w(t)\,dt = z + \lambda F(z).$$

Then

$$a = \frac{z_0 f'(z_0)}{f(z_0)} = \frac{1 + \lambda F'(z_0)}{1 + \lambda F(z_0)/z_0} = \frac{1 + \lambda\tilde{z}}{1 + \lambda c\tilde{w}} \in \Omega_{\lambda,c},$$

since

$$|\tilde{z}| = |F'(z_0)| < 1, \quad |\tilde{w}| = \left|\frac{1}{cz_0}\int_0^{z_0} b(t)F(t)\,dt\right| \le \frac{1}{c}\int_0^1 |b(tz_0)|\,dt < 1.$$

We remark that a simple calculation shows

$$\left\{ \frac{1+\lambda z}{1+\lambda c z} : z \in \overline{\mathbb{D}} \right\} \subset \Omega_{\lambda,c}. \tag{2.1}$$

(ii) Assume that for some $f \in \mathscr{T}_\lambda(b)$ there exists a sequence $v_k \in \mathbb{D}$, $v_k \to 1$, such that

$$\lim_{k\to\infty} \frac{v_k f'(v_k)}{f(v_k)} = \omega \in \partial\Omega_{\lambda,c}.$$

Then

$$\lim_{k\to\infty} f(v_k) = 1 + \lambda \int_0^1 b(t)w(t)\,dt = 1 + \lambda c z_2, \tag{2.2}$$

$$\lim_{k\to\infty} v_k f'(v_k) = 1 + \lambda \lim_{k\to\infty} b(v_k)w(v_k) = 1 + \lambda z_1, \tag{2.3}$$

with $z_1, z_2 \in \overline{\mathbb{D}}$. In fact, we must have $z_1, z_2 \in \partial\mathbb{D}$, since otherwise

$$\omega = \frac{1+\lambda z_1}{1+\lambda c z_2} \notin \partial\Omega_{\lambda,c}.$$

This and the assumption concerning c imply that

$$c = \left| \int_0^1 b(t)w(t)\,dt \right| \le \int_0^1 |b(t)|\,dt \le c,$$

so that $|w(t)| = 1$ and $\arg b(t) =$ constant. Without loss of generality we may assume $w \equiv 1$, and then (2.2) and (2.3) lead to

$$\arg z_1 = \arg b(1) = \arg \int_0^1 b(t)\,dt = \arg z_2,$$

or $z_1 = z_2$. Then (2.1) gives $\omega \notin \partial\Omega_{\lambda,c}$, a contradiction.

(iii) If $\omega \in \overline{\Omega_{\lambda,c}}$, then there are $z, w \in \overline{\mathbb{D}}$ such that $\omega = (1+\lambda z)/(1+\lambda c w)$. A simple homotopy argument can be used to show that we can assume that $|z| = |w| = r \le 1$, and re-writing the above representation we get

$$\omega = \frac{1+\lambda r z_1}{1+\lambda r c z_2}, \quad z_1, z_2 \in \partial\mathbb{D}.$$

In Theorem 1.7 let $x = 0$, $y = 1$ and choose the functions V_k accordingly. Then

$$f_k(z) := z + \lambda r \int_0^z b(t) V_k(t)\,dt \in \mathscr{T}_{\lambda r}(b) \subset \mathscr{T}_\lambda(b),$$

and we have

$$\lim_{k\to\infty} \frac{f_k'(1)}{f_k(1)} = \frac{1+\lambda r V_k(1)}{1+\lambda r \int_0^1 b(t) V_k(t)\,dt} = \frac{1+\lambda r z_1}{1+\lambda r z_2 \int_0^1 b(t)\,dt} = \omega,$$

which clearly implies that $\omega \in \overline{\Gamma_\lambda(b)}$. $\square$

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To exploit the properties described in Theorem 1.2 we need a more explicit description of the boundary of $\Omega_{\lambda,c}$.

THEOREM 2.1. *For* $0 < c \le 1$ *and* $0 < \lambda \le 1$, $c\lambda \neq 1$, *the boundary of* $\Omega_{\lambda,c}$ *is described by* $(x(\theta), y(\theta))$, $\theta \in [0, 2\pi)$, *where*

$$\begin{aligned} x(\theta) &= \frac{1+\lambda\cos\theta - c^2\lambda^2\sin^2\theta + c\lambda\cos\theta\sqrt{1+\lambda^2+2\lambda\cos\theta - c^2\lambda^2\sin^2\theta}}{1-c^2\lambda^2}, \\ y(\theta) &= (x(\theta)-1)\tan\theta. \end{aligned} \tag{2.4}$$

PROOF. Let $G(\theta,\varphi) = (1+\lambda e^{i\theta})/(1+c\lambda e^{i\varphi})$. For fixed θ the function $G(\theta,\varphi)$ describes the circle

$$\left(x - \frac{1+\lambda\cos\theta}{1-c^2\lambda^2}\right)^2 + \left(y - \frac{\lambda\sin\theta}{1-c^2\lambda^2}\right)^2 = \frac{c^2\lambda^2(1+\lambda^2+2\lambda\cos\theta)}{(1-c^2\lambda^2)^2}. \tag{2.5}$$

We need to determine the outer envelope of the circles (2.5) when θ varies from 0 to 2π. Hence we have to solve the system consisting of (2.5) and the equation that we get by differentiating (2.5) with respect to θ, namely

$$x\sin\theta - y\cos\theta = \sin\theta. \tag{2.6}$$

A combination of (2.5) and (2.6) yields the parametrization shown in Theorem 2.1. □

THEOREM 2.2. (i) *For* $c > 0$ *and* $0 < \alpha \le 1$ *let*

$$\lambda(\alpha) := \frac{\sin(\alpha\pi/2)}{\sqrt{1+2c\cos(\alpha\pi/2)+c^2}}. \tag{2.7}$$

Then $|\arg\omega| \le \alpha\pi/2$ *holds for every* $\omega \in \Omega_{\lambda(\alpha),c}$.

(ii) *For* $z_0 = e^{i\varphi_0}$, $w_0 = e^{i\psi_0}$, *with* $\varphi_0 = \sin^{-1}(\sqrt{1-\lambda(\alpha)^2}) \in (\pi/2, \pi)$ *and* $\psi_0 = \varphi_0 - (1+\alpha/2)\pi$, *we have*

$$\left|\arg\frac{1+x\lambda(\alpha)z_0}{1+xc\lambda(\alpha)w_0}\right| > \alpha\frac{\pi}{2}, \quad x > 1.$$

In particular, $\lambda(\alpha)$ *is the largest constant with the property described in* (i).

This result can be derived from Theorem 2.1. Since a more elegant direct proof is available, we prefer to include this proof here.

PROOF. Let z and w be points in the unit disk. Then

$$\left|\arg\frac{1+\lambda z}{1+c\lambda w}\right| \le \sin^{-1}\lambda + \sin^{-1}(c\lambda).$$

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With $\kappa = \sin(\alpha\pi/2)$ the condition to check is

$$\sin\left[\sin^{-1}\lambda + \sin^{-1}(c\lambda)\right] \le \kappa$$

which can be written as

$$\lambda\sqrt{1-c^2\lambda^2} + c\lambda\sqrt{1-\lambda^2} \le \kappa.$$

After squaring and simplifying this becomes

$$\left(\left(1-c^2\right)^2 + 4\kappa^2c^2\right)\lambda^4 - 2\left(1+c^2\right)\kappa^2\lambda^2 + \kappa^4 \ge 0.$$

The smallest positive root in λ^2 of this bi-quadratic equation is

$$\lambda^2 = \frac{(1+c^2)\kappa^2 - 2c\kappa^2\sqrt{1-\kappa^2}}{(1-c^2)^2 + 4\kappa^2c^2},$$

which translates into (2.7).

To find the extremal values of φ and ψ we draw the circles $1+\lambda e^{i\varphi}$ and $1+c\lambda e^{i\psi}$, and the tangent to these two circles from the origin. A simple geometric argument then gives the values φ_0 and ψ_0 as stated. □

PROOF OF COROLLARY 1.3. That $f \in \mathscr{T}_{\mu(\alpha)}(b)$ implies that f is strongly starlike of order α follows immediately from Theorem 2.2, while the sharpness statement is a consequence of Theorem 1.2 (ii). □

PROOF OF COROLLARY 1.4. The parabola $|w-1| \le \operatorname{Re} w$ intersects the real axis in $x = 1/2$. The curve $\gamma_{c,\lambda}$, given by (2.4), intersects the real axis in the points $x(0)$ and $x(\pi)$ with $x(\pi) < x(0)$. For the functions in $\mathscr{T}_\lambda(b)$ to satisfy (1.6) it is therefore necessary that λ is so small that $x(\pi) \ge 1/2$. This is sharp for $\lambda = 1/(c+2)$, and we will prove that indeed the curve $\gamma_{c,1/(c+2)}$ lies inside the parabolic domain. The upper half of the parabola is given by $y = \sqrt{2x-1}$, $x \ge 1/2$, so it is enough to prove that

$$y(\theta) \le \sqrt{2x(\theta)-1}, \quad 0 \le \theta \le \pi.$$

For $c > 0$ and $\lambda = 1/(c+2)$ we get

$$2x(\theta) - 1 = \frac{c^2 + 2c + 2 + (c+2)\cos\theta - c^2\sin^2\theta + cB(\cos\theta)\cos\theta}{2c+2},$$

$$y(\theta) = \frac{c^2\sin\theta\cos\theta + (c+2)\sin\theta + cB(\cos\theta)\sin\theta}{4c+4},$$

where $B(u) := \sqrt{4c+5+2(c+2)u+c^2u^2}$.

Writing $u := \cos\theta$ we have to show $h(u) := 2x(\theta) - 1 - y(\theta)^2 > 0, -1 \le u \le 1$, and

$$h(u) = \frac{1}{(4c+4)^2}\big[12 + 28c + 10c^2 - 4c^3 + (16 + 24c + 4c^3)u \\ + (4 + 4c + 14c^2 + 12c^3 - 2c^4)u^2 + (8c^2 + 4c^3)u^3 + 2c^4u^4 \\ +(2c^3u^3 + (4c + 2c^2)u^2 + (8c + 8c^2 - 2c^3)u - 4c - 2c^2)B(u)\big].$$

Separating the term containing $B(u)$, squaring both sides and simplifying we see that $h(u) > 0$ holds if

$$16(1+c)^2(u+1)((1+c^2)u+1+2c-c^2)((1+c^2)u^2+(6+2c)u+9+6c-c^2)>0.$$

Here all factors are positive for $u > -1$. The sharpness statement follows from Theorem 1.2. □

3. Proof of Theorems 1.5 and 1.6

PROOF OF THEOREM 1.5. A function $f \in \mathscr{A}$ is uniformly starlike if and only if

$$\operatorname{Re} \frac{(1-x)zf'(z)}{f(z) - f(xz)} \ge 0, \quad z, x \in \mathbb{D},$$

(compare [6]). If

$$f(z) = z + \mu \int_0^z b(t)w(t)\,dt = z + \mu F(z) \in \mathscr{T}_\mu(b),$$

then we need to show that

$$\operatorname{Re} \frac{1 + \mu F'(z)}{1 + \mu(F(xz) - F(z))/(xz - z)} > 0, \quad x, z \in \mathbb{D}. \tag{3.1}$$

However,

$$|F'(z)| = |b(z)w(z)| < 1, \qquad \left|\frac{F(xz) - F(z)}{xz - z}\right| = \left|\frac{1}{z - xz}\int_{xz}^{z} b(t)w(t)\,dt\right| < 1,$$

so that Theorem 2.1 with $c = 1$, $\lambda = \mu = 1/\sqrt{2}$ establishes (3.1). For the proof that μ cannot be replaced by any larger number we assume without loss of generality that $b(1) = 1$, and note

$$\operatorname{Re} \frac{1 + x\mu z_1}{1 + x\mu z_2} < 0, \quad \text{for } x > 1,\ z_1 = \bar{z}_2 = \frac{-1+i}{\sqrt{2}}. \tag{3.2}$$

For this choice of z_1, z_2 we define V_k as in Theorem 1.7, so that $V_k(1) = z_1$, and for all $x \in \mathbb{D}$

$$\frac{1}{1-x}\int_x^1 b(t)V_k(t)\,dt \to z_2\frac{1}{1-x}\int_x^1 b(t)\,dt, \quad k \to \infty.$$

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It is now clear that we can choose a sequence $x_j \to 1$ and $k_j \to \infty$ for $j \to \infty$ such that

$$\frac{1}{1-x_j}\int_{x_j}^{1} b(t)\,V_{k_j}(t)\,dt \to z_2, \quad j \to \infty.$$

Hence, if $x > 1$ we have by (3.2)

$$\operatorname{Re}\frac{1+x\mu V_{k_j}(1)}{1+x\mu/(1-x_j)\int_{x_j}^{1} b(t)\,V_{k_j}(t)\,dt} < 0$$

for some j large enough. This implies that the corresponding function

$$f(z) := z + x\mu\int_0^z b(t)\,V_{k_j}(t)\,dt$$

cannot be uniformly starlike.

If $f' \in C^0(\overline{\mathbb{D}})$ holds for

$$f(z) = z + \mu\int_0^z b(t)w(t)\,dt \in \mathcal{T}_\mu(b),$$

then we can easily show that

$$\text{(3.3)} \qquad \inf_{z,x\in\overline{\mathbb{D}}} \operatorname{Re}\frac{1+\mu b(z)w(z)}{1+\mu/(z-xz)\int_{xz}^{z} b(t)w(t)\,dt} > 0.$$

If not, then we must have

$$1 = |b(z)w(z)| = \left|\frac{1}{z-xz}\int_{xz}^{z} b(t)w(t)\,dt\right|$$

which is only possible for the limiting case $x \to 1$ and some $z \in \partial\mathbb{D}$. Then, however,

$$b(z)w(z) = \lim_{x\to 1}\frac{1}{z-xz}\int_{xz}^{z} b(t)w(t)\,dt,$$

and we are in a situation of (2.1), which yields a contradiction. The final claim of Theorem 1.5 follows readily from (3.3). $\square$

PROOF OF THEOREM 1.6. Let

$$f(z) = z + \mu(c)\int_0^z b(t)w(t)\,dt \in \mathcal{T}_{\mu(c)}(b).$$

Then, by Theorem 2.2 with $\alpha = 1$,

$$\operatorname{Re}\frac{2zf'(z)}{f(z)-f(-z)} = \operatorname{Re}\frac{1+\mu(c)b(z)w(z)}{1+(\mu(c)/2)\int_{-1}^{1} b(tz)w(tz)\,dt} > 0,$$

since

$$|b(z)w(z)| < 1, \qquad \left|\frac{1}{2}\int_{-1}^{1} b(tz)w(tz)\,dt\right| < c.$$

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If b is even and satisfies (1.2), then $(1/2)\int_{-1}^{1} b(t)\,dt = c$, and an application of Theorem 1.7 with $x = -1$, $y = 1$ produces a sequence of functions $V_k \in \mathscr{B}$ such that for every $x > 1$

$$\lim_{k\to\infty} \operatorname{Re} \frac{1 + x\mu(c)b(z)V_k(z)}{1 + (x\mu(c)/2)\int_{-1}^{1} b(tz)V_k(tz)\,dt} < 0$$

for $z \in \mathbb{D}$ close enough to 1. We omit the details which are similar to those in previous proofs. □

4. Appendix

Proof of Theorem 1.7. We may assume that $z_1 = 1$. If $z_2 = 1$ as well, then $V_k \equiv 1, k \in \mathbb{N}$ will work. Hence we assume that $z_2 = e^{i\varphi}, \varphi \neq 0$. Let r_k be a sequence of numbers with $0 < r_k < 1$ and $r_k \to 1$ for $k \to \infty$. We define

$$V_k(z) := \frac{1 - r_k z_2}{1 - r_k \bar{z}_2} \frac{z + r_k(z_2 - r_k)/(1 - r_k z_2)}{1 + z r_k(\bar{z}_2 - r_k)/(1 - r_k \bar{z}_2)},$$

and note that $V_k \in \mathscr{B} \cap \mathscr{H}(\overline{\mathbb{D}})$, with $V_k(1) = 1, k \in \mathbb{N}$. Furthermore, introducing the number

$$A = \frac{1 - r_k}{|1 - z||1 - z_2|}$$

we see that

$$\begin{aligned}
|V_k(z) - z_2| &= (1 - r_k)\left|\frac{r_k - z_2 - r_k z_2 z + z}{1 - r_k\bar{z}_2 + r_k\bar{z}_2 z - r_k^2 z}\right| \\
&= \frac{1 - r_k}{|1 - z|}\left|\frac{1 - z_2}{1 - \bar{z}_2}\right|\left|\frac{1 + z + (1 - r_k)(z_2 z - 1)/(1 - z_2)}{1 - (1 - r_k)(\bar{z}_2(z - 1) - z(1 + r_k))/(1 - z)(1 - \bar{z}_2)}\right| \\
&\leq \frac{2(1 - r_k)}{|1 - z|}\frac{1 + A|1 - z|}{1 - 4A} \leq \frac{4(1 - r_k)}{|1 - z|}
\end{aligned}$$

if $A \leq 1/10$. It is easily seen that the condition

$$|1 - z| \geq \frac{10(1 - r_k)}{|\sin\varphi|}$$

is sufficient for $A \leq 1/10$. That means that $V_k \to z_2$ uniformly in all closed subsets of $\overline{\mathbb{D}}$ which do not contain $z = 1$. From here it is clear that (1.7) holds for all choices of x, y in which not at least one, say y, equals 1. Hence we assume $x \neq 1, y = 1$. A straightforward estimation of the quantity

$$q_k := \left|\frac{1}{1 - x}\int_x^1 b(t)[V_k(t) - z_2]\,dt\right| \leq \int_0^1 |b(\gamma(t))||V_k(\gamma(t)) - z_2|\,dt,$$

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with $\gamma(t) := x + t(1 - x)$ now shows that $q_k \to 0$ for $k \to \infty$, which implies (1.7). Note that the V_k's have been defined independently of x, y. $\square$

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