SEPARATING MANY LOCALISATION CARDINALS ON THE GENERALISED BAIRE SPACE

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Abstract. Given a cofinal cardinal function $h \in {}^\kappa\kappa$ for κ inaccessible, we consider the dominating h-localisation number, that is, the least cardinality of a dominating set of h-slaloms such that every κ-real is localised by a slalom in the dominating set. It was proved in [3] that the dominating localisation numbers can be consistently different for two functions h (the identity function and the power function). We will construct a κ^+-sized family of functions h and their corresponding localisation numbers, and use a $\leq\kappa$-supported product of a cofinality-preserving forcing to prove that any simultaneous assignment of these localisation numbers to cardinals above κ is consistent. This answers an open question from [3].

In an effort to generalise the cardinal characteristics related to the null ideal from the context of the continuum ${}^\omega\omega$ to the generalised Baire space ${}^\kappa\kappa$, the authors of [3] considered localisation cardinals. These cardinals were first described in the context of ${}^\omega\omega$ by Bartoszyński [1] and are defined using the concept of slaloms.

DEFINITION 0.1. *Let κ be a regular strong limit cardinal (hence κ is inaccessible or equal to ω) and let $h \in {}^\kappa\kappa$ be increasing, then an h-slalom is a function $\varphi : \kappa \to [\kappa]^{<\kappa}$ such that $|\varphi(\alpha)| \leq |h(\alpha)|$ for all $\alpha \in \kappa$. A slalom φ localises a function $f \in {}^\kappa\kappa$, written symbolically as $f \in^* \varphi$, if there exists some $\xi \in \kappa$ such that $f(\alpha) \in \varphi(\alpha)$ for all $\alpha \in [\xi, \kappa)$. We denote the set of h-slaloms as* Loc_h.

We can define the following two cardinal characteristics, sometimes called *localisation cardinals*, having an increasing function $h \in {}^\kappa\kappa$ as parameter:

$$\mathfrak{b}^h_\kappa(\in^*) = \text{the least cardinality of a family } \mathcal{F} \subseteq {}^\kappa\kappa \text{ such that } \forall\varphi \in \mathrm{Loc}_h \exists f \in \mathcal{F}(f \notin^* \varphi),$$

$$\mathfrak{d}^h_\kappa(\in^*) = \text{the least cardinality of a family } \Phi \subseteq \mathrm{Loc}_h \text{ such that } \forall f \in {}^\kappa\kappa \exists\varphi \in \Phi(f \in^* \varphi).$$

In the case that $\kappa = \omega$ these cardinals give a combinatorial definition of two of the cardinal invariants of the Lebesgue null ideal $\mathcal{N}$:

$$\mathrm{add}(\mathcal{N}) = \text{the least cardinality of a family } \mathcal{A} \subseteq \mathcal{N} \text{ such that } \bigcup\mathcal{A} \notin \mathcal{N},$$

$$\mathrm{cof}(\mathcal{N}) = \text{the least cardinality of a family } \mathcal{C} \subseteq \mathcal{N} \text{ such that } \forall N \in \mathcal{N} \exists C \in \mathcal{C}(N \subseteq C).$$

Bartoszyński [1] introduced localisation cardinals to give the following characterisation.

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Fact 0.2. $\mathrm{add}(\mathcal{N}) = \mathfrak{b}^h_\omega(\in^*)$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}^h_\omega(\in^*)$.

It is clear from this fact that the choice of the parameter $h \in {}^\omega\omega$ is irrelevant, as it does not influence the cardinalities of $\mathfrak{b}^h_\omega(\in^*)$ and $\mathfrak{d}^h_\omega(\in^*)$. This turns out to be different in the generalised case. Recently, it was proved in [3] that $\mathfrak{d}^h_\kappa(\in^*)$ can consistently have different values for different $h \in {}^\kappa\kappa$. In particular, it was shown that $\mathfrak{d}^{\mathsf{pow}}_\kappa(\in^*) < \mathfrak{d}^{\mathsf{id}}_\kappa(\in^*)$ is consistent, where $\mathsf{pow} : \alpha \mapsto 2^{|\alpha|}$ and $\mathsf{id} : \alpha \mapsto |\alpha|$. This consistency was proved using a generalisation of Sacks forcing on ${}^\kappa 2$ that has the generalised pow-Sacks property, but not the generalised id-Sacks property.

Definition 0.3. *Let $h \in {}^\kappa\kappa$. A forcing $\mathbb{P}$ has the generalised h-Sacks property if for every $\mathbb{P}$-name $\dot f$ and $p \in \mathbb{P}$ such that $p \Vdash$ “ $\dot f : \check\kappa \to \check\kappa$ ” there exists an h-slalom φ and $q \leq p$ such that $q \Vdash$ “ $\dot f(\check\alpha) \in \check\varphi(\check\alpha)$ ” for all $\alpha \in \kappa$. We will from now on simply say h-Sacks property and omit “generalised”.*

Hence, if $\Phi \subseteq \mathrm{Loc}_h$ is a family as in the definition of $\mathfrak{d}^h_\kappa(\in^*)$ in the ground model, then Φ will still witness the size of $\mathfrak{d}^h_\kappa(\in^*)$ in extensions by forcings with the h-Sacks property. Specifically, the generalised Sacks forcing is unable to increase the size of $\mathfrak{d}^{\mathsf{pow}}_\kappa(\in^*)$. Meanwhile, it is possible to increase the size of $\mathfrak{d}^{\mathsf{id}}_\kappa(\in^*)$ by using an iteration or a product of the generalised Sacks forcing.

In this article we will answer an open question from [3] and prove that there exist functions $h_\xi \in {}^\kappa\kappa$ and cardinals λ_ξ with $\mathrm{cf}(\lambda_\xi) > \kappa$ for each $\xi \in \kappa^+$ such that it is simultaneously consistent that $\mathfrak{d}^{h_\xi}_\kappa(\in^*) = \lambda_\xi$ for all $\xi \in \kappa$. The strategy will be the same as the strategy used to separate $\mathfrak{d}^{\mathsf{id}}_\kappa(\in^*)$ from $\mathfrak{d}^{\mathsf{pow}}_\kappa(\in^*)$, in that we consider a product of Sacks-like forcings that have the h_ξ-Sacks property, but not the h_η-Sacks property for different ξ and η. In the first section, we introduce the Sacks-like forcing $\mathbb{S}^h_\kappa$. We will prove that it preserves cardinals and cofinalities and use fusion to show that it satisfies certain Sacks properties. In the second section we consider products of such forcings. We show that properties such as the preservation of cardinals and cofinalities and the relevant Sacks properties are preserved under $\leq\kappa$-support products and we use this to prove the consistency of κ many different cardinals. Finally, in the third section we will show that with a preparatory forcing, we can use our approach to prove the consistency of κ^+ many different cardinals.

For the sake of brevity, from now on we will assume that κ denotes a *strongly inaccessible cardinal*. We will also fix the convention that $h, H, F \in {}^\kappa\kappa$ denote *increasing cofinal cardinal functions* (i.e., $\mathrm{ran}(h)$ is cofinal in κ and $h(\alpha)$ is a cardinal for each $\alpha \in \kappa$). This convention extends to h_ξ, F_0, and other subscripts.

§1. The forcing. Let us establish some notation to discuss trees on ${}^\kappa\kappa$ before we define our forcing notion.

A subset $T \subseteq {}^{<\kappa}\kappa$ is called a *κ-tree* if for every $u \in T$ and $\beta \in \mathrm{dom}(u)$ we have $u \restriction \beta \in T$. A subset $C \subseteq T$ is a *chain* if for any $u, v \in C$ we have $u \subseteq v$ or $v \subseteq u$, and C is called *maximal* if there exists no chain $C' \subseteq T$ with $C \subsetneq C'$. A function $b : \alpha \to \kappa$ where $\alpha \leq \kappa$ is called a *branch* of T if there exists a maximal chain $C \subseteq T$ such that $b = \bigcup C$. The set of branches of T is denoted by $[T]$. We define the *subtree* of T generated by $u \in T$ as

$$(T)_u = \{v \in T \mid u \subseteq v \vee v \subseteq u\}.$$

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Given $u \in {}^{<\kappa}\kappa$ and $\beta \in \kappa$, we write $u^\frown\beta$ for the extension of the sequence u with the term β. If $u \in T$, let $v \in T$ be a *successor* of u if there exists $\beta \in \kappa$ such that $v = u^\frown\beta$. We denote the set of successors of u in T by $\mathrm{suc}(u, T)$.

We call u a λ-*splitting* node (of T), if $\lambda \leq |\mathrm{suc}(u, T)|$. We say u is a *splitting* node if it is a 2-splitting node, and a *non-splitting* node otherwise. If u is a λ-splitting node, but not a μ-splitting node for any cardinal μ with $\lambda < \mu$, then we say that u is a *sharp* λ-splitting node.

We let $\mathrm{Split}_\alpha(T)$ be the set of all $u \in T$ such that u splitting and

$$\mathrm{ot}(\{\beta \in \mathrm{ot}(u) \mid u \restriction \beta \text{ is splitting}\}) = \alpha,$$

where $\mathrm{ot}(X)$ denotes the order-type of X.

DEFINITION 1.1. *The conditions of the forcing $\mathbb{S}^h_\kappa$ are κ-trees $T \subseteq {}^{<\kappa}\kappa$ that satisfy the following properties*:

(i) *For any $u \in T$ there exists $v \in T$ such that $u \subseteq v$ and v is splitting.*
(ii) *If $u \in \mathrm{Split}_\alpha(T)$, then u is an $h(\alpha)$-splitting node in T.*
(iii) *If $C \subseteq T$ is a chain of splitting nodes with $|C| < \kappa$, then $\bigcup C$ is a splitting node in T.*

The order is defined as $T \leq_{\mathbb{S}^h_\kappa} S$ (here T provides more information than S) iff:

- *$T \subseteq S$ and*
- *for every $u \in T$, if $\mathrm{suc}(u, T) \neq \mathrm{suc}(u, S)$, then $|\mathrm{suc}(u, T)| < |\mathrm{suc}(u, S)|$.*

If the forcing notion is clear from context, we will write $T \leq S$ in place of $T \leq_{\mathbb{S}^h_\kappa} S$.

The generalised Sacks forcing from [6], which was used in [3] to separate $\mathfrak{d}^{\mathrm{id}}_\kappa(\in^*)$ from $\mathfrak{d}^{\mathrm{pow}}_\kappa(\in^*)$, is a special case of the forcing $\mathbb{S}^h_\kappa$. Indeed, the only difference is that splitting nodes split into $h(\alpha)$ successors in our forcing, instead of only two successors in the Sacks forcing. That is, if $f : \kappa \to \{2\}$ is the constant function that sends every $\alpha \in \kappa$ to 2, then the generalised Sacks forcing is equivalent to $\mathbb{S}^f_\kappa$.

However, contrary to the constant function f, we will assume that h is an increasing *cofinal* cardinal function, and thus (as a consequence of the next lemma) any node can be extended to a λ-splitting node for arbitrarily large $\lambda < \kappa$. In this sense we can also view $\mathbb{S}^h_\kappa$ as a bounded version of Miller forcing, where the set of successors of a node is $< \kappa$ but eventually arbitrarily large. On ${}^\omega\omega$, such a forcing has first been studied in [4] as *Miller Lite forcing*.

LEMMA 1.2. *If $T \in \mathbb{S}^h_\kappa$ and $b \in [T]$ is a branch of T, then $\mathrm{dom}(b) = \kappa$. In particular, if $\alpha \in \kappa$ is limit and $u \in {}^\alpha\kappa$ is such that $u \restriction \beta \in T$ for all $\beta \in \alpha$, then $u \in T$.*

PROOF. Suppose that $C \subseteq T$ is a chain such that $u = \bigcup C$ and $\mathrm{dom}(u) \in \kappa$. If we show that $u \in T$, then $\mathrm{suc}(u, T) \neq \varnothing$ by (i) of Definition 1.1, thus it follows that C is not a branch. Let

$$C' = \{v \subseteq u \mid v \text{ is splitting in } T\}.$$

From (i) of Definition 1.1 we can conclude that either $\bigcup C' = u$ or there is some splitting node $v \in T$ with $w \subseteq v$ for all $w \in C$. In the first case it follows that $u \in T$ by (iii) of Definition 1.1, and in the second case $v \restriction \mathrm{dom}(u) = u \in T$ since T is a κ-tree. ⊣

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We naturally want $\mathbb{S}^h_\kappa$ to preserve cardinalities. If we assume that $\mathbf{V} \vDash$ " $2^\kappa = \kappa^+$ ", then it is clear that $\mathbb{S}^h_\kappa$ has the $<\kappa^{++}$-chain condition, since $|\mathbb{S}^h_\kappa| \leq |\mathcal{P}({}^{<\kappa}\kappa)| = 2^\kappa = \kappa^+$, where the former equality is implied by $\kappa^{<\kappa} = \kappa$, which in turn follows from κ being inaccessible. Therefore cardinalities above κ^+ are preserved under assumption of $\mathbf{V} \vDash$ " $2^\kappa = \kappa^+$ ".

To preserve cardinalities less than or equal to κ, we show that $\mathbb{S}^h_\kappa$ is $<\kappa$-closed.

LEMMA 1.3. *$\mathbb{S}^h_\kappa$ is $<\kappa$-closed. That is, for any $\lambda < \kappa$, if $\langle T_\xi \mid \xi \in \lambda \rangle$ is a descending chain of conditions, then there exists a condition T such that $T \leq T_\xi$ for all $\xi \in \lambda$.*

The condition T that is below all T_ξ will simply be $T = \bigcap_{\xi \in \lambda} T_\xi$, which can easily be seen to be a κ-tree as well. We will make use of the following claim to prove the lemma.

CLAIM 1.4. *If $u \in T = \bigcap_{\xi \in \lambda} T_\xi$, then there is $\eta \in \lambda$ such that* $\mathrm{suc}(u, T) = \mathrm{suc}(u, T_\eta)$.

PROOF. Suppose that $u \in T$, and let $\lambda_\xi = |\mathrm{suc}(u, T_\xi)|$, then the ordering on $\mathbb{S}^h_\kappa$ dictates that $\langle \lambda_\xi \mid \xi \in \lambda \rangle$ is a descending sequence of cardinals, hence there is $\eta \in \lambda$ such that $\lambda_\xi = \lambda_\eta$ for all $\xi \in [\eta, \lambda)$. But then $\mathrm{suc}(u, T_\xi) = \mathrm{suc}(u, T_\eta)$ for all $\xi \in [\eta, \lambda)$ by the ordering of $\mathbb{S}^h_\kappa$. ⊣

PROOF OF LEMMA 1.3. We show that $T = \bigcap_{\xi \in \lambda} T_\xi$ satisfies the lemma by verifying points (i)–(iii) from Definition 1.1 and showing that $T \leq T_\xi$ for all $\xi \in \lambda$.

(i) Let $u \in T$, and let $f \in [T]$ be a branch for which $u \subseteq f$. If $\mathrm{ot}(f) < \kappa$, then $f \in T_\xi$ for each ξ, thus by Claim 1.4 there is some $\eta \in \lambda$ for which $\mathrm{suc}(f, T_\eta) = \mathrm{suc}(f, T) = \varnothing$. Then clearly $T_\eta \notin \mathbb{S}^h_\kappa$, which is a contradiction, hence $\mathrm{ot}(f) = \kappa$.

Let $C_\xi = \{\alpha \in [\mathrm{ot}(u), \kappa) \mid f \restriction \alpha \text{ is splitting in } T_\xi\}$, then since $T_\xi \in \mathbb{S}^h_\kappa$ satisfies (i) and (iii), we see that C_ξ is a club set. But then $\bigcap_{\xi \in \lambda} C_\xi$ is club. Any $v \in \bigcap_{\xi \in \lambda} C_\xi$ is splitting in all T_ξ, thus by Claim 1.4 it is splitting in T, and by definition of C_ξ it follows that $u \subseteq v$.

(ii) If $u \in \mathrm{Split}_\alpha(T)$, then by Claim 1.4 there is $\eta \in \lambda$ such that $\mathrm{suc}(u, T_\eta) = \mathrm{suc}(u, T)$. Therefore $u \in \mathrm{Split}_\beta(T_\eta)$ for some $\beta \geq \alpha$, hence u is a $h(\beta)$-splitting in T. Remember that we assume h is increasing, so u is also $h(\alpha)$-splitting in T.

(iii) Let $C \subseteq T$ be a chain of splitting nodes, then for every $\xi \in \lambda$ we also see that C is a chain of splitting nodes in T_ξ, and thus $\bigcup C$ is a splitting node in all T_ξ, hence by Claim 1.4, $\bigcup C$ is splitting in T.

($\leq$) Clearly $T \subseteq T_\xi$ for each $\xi \in \lambda$, and if $u \in T$ and $\mathrm{suc}(u, T) \neq \mathrm{suc}(u, T_\xi)$, then by Claim 1.4 there exists $\eta \in \lambda$ such that $\mathrm{suc}(u, T) = \mathrm{suc}(u, T_\eta)$, and clearly $\xi < \eta$. Since $T_\xi \leq T_\eta$ by assumption, then $|\mathrm{suc}(u, T)| = |\mathrm{suc}(u, T_\eta)| < |\mathrm{suc}(u, T_\xi)|$. Hence $T \leq T_\xi$. ⊣

COROLLARY 1.5. *$\mathbb{S}^h_\kappa$ preserves all cardinalities and cofinalities $\leq \kappa$.*

What is left, is to show that κ^+ is also preserved. This will be a consequence of the proof that $\mathbb{S}^h_\kappa$ has the F-Sacks property for some suitably large $F \in {}^\kappa\kappa$, so we will prove this first. But before that, we will need to show that $\mathbb{S}^h_\kappa$ is closed under fusion. It will be helpful to establish the notion of sharp trees.

Let a κ-tree $T \in \mathbb{S}^h_\kappa$ be called *sharp* if every $u \in \mathrm{Split}_\alpha(T)$ is a sharp $h(\alpha)$-splitting node. It is clear that by pruning we may find a sharp T^* below any condition $T \in \mathbb{S}^h_\kappa$

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such that $\mathrm{Split}_\alpha(T^*) \subseteq \mathrm{Split}_\alpha(T)$ for every $\alpha \in \kappa$. We may assume that we can canonically do so, thus we will hereby fix the notation T^* to denote a canonical sharp κ-tree below condition T. We will write $(\mathbb{S}^h_\kappa)^* = \{T \in \mathbb{S}^h_\kappa \mid T \text{ is sharp}\}$, which embeds densely into $\mathbb{S}^h_\kappa$.

Definition 1.6. *For $T, S \in \mathbb{S}^h_\kappa$, we let $T \leq_\alpha S$ iff $T \leq S$ and $\mathrm{Split}_\alpha(T) = \mathrm{Split}_\alpha(S)$. A fusion sequence is a sequence $\langle T_\alpha \in \mathbb{S}^h_\kappa \mid \alpha \in \kappa \rangle$ such that $T_\beta \leq_\alpha T_\alpha$ for all $\alpha \leq \beta \in \kappa$.*

Lemma 1.7. *If $\langle T_\alpha \in \mathbb{S}^h_\kappa \mid \alpha \in \kappa \rangle$ is a fusion sequence, then $T = \bigcap_{\alpha \in \kappa} T_\alpha \in \mathbb{S}^h_\kappa$.*

Proof. Clearly T is a κ-tree on ${}^{<\kappa}\kappa$. We check conditions (i)–(iii) of Definition 1.1.

(i) Let $u \in T$ and $\alpha = \mathrm{ot}(u)$, then for any $\beta \geq \alpha$ we see that $\mathrm{Split}_\alpha(T_\beta) = \mathrm{Split}_\alpha(T_\alpha)$. Since $\alpha = \mathrm{ot}(u)$, necessarily there exists some $v \in \mathrm{Split}_\alpha(T_\alpha)$ such that $u \subseteq v$, and since $v \in \mathrm{Split}_\alpha(T_\beta)$ for all $\beta > \alpha$ we see that $v \in \mathrm{Split}_\alpha(T)$.

(ii) Let $u \in \mathrm{Split}_\alpha(T)$, then u is $h(\alpha)$-splitting in $T_{\alpha+1}$. Let $\lambda_u = |\mathrm{suc}(u, T_{\alpha+1})| \geq h(\alpha)$ and let $\langle v_\xi \mid \xi \in \lambda_u \rangle$ enumerate those $v \supseteq u$ such that $v \in \mathrm{Split}_{\alpha+1}(T_{\alpha+1})$. For all $\beta \geq \alpha + 1$ we have $\mathrm{Split}_{\alpha+1}(T_\beta) = \mathrm{Split}_{\alpha+1}(T_{\alpha+1})$, therefore for each $\xi \in \lambda_u$ we see that $v_\xi \in T_\beta$ for all $\beta > \alpha$, thus $v_\xi \in T$. Therefore u is $h(\alpha)$-splitting in T.

(iii) Let $C \subseteq T$ be a chain of splitting nodes with $|C| < \kappa$ and let $\gamma \in \kappa$ be large enough such that $C \subseteq \bigcup_{\alpha<\gamma} \mathrm{Split}_\alpha(T)$. It follows that $C \subseteq \bigcup_{\alpha<\gamma} \mathrm{Split}_\alpha(T_\gamma)$, and thus $\bigcup C \in \mathrm{Split}_\beta(T_\gamma)$ for some $\beta \leq \gamma$. Then also $\bigcup C \in \mathrm{Split}_\beta(T_{\gamma'})$ for all $\gamma' > \gamma$, hence $\bigcup C \in \mathrm{Split}_\beta(T)$.

($\leq$) Clearly $T \subseteq T_\alpha$ for all $\alpha \in \kappa$. Given $u \in T$ and $\alpha \in \kappa$ such that $\mathrm{suc}(u, T) \neq \mathrm{suc}(u, T_\alpha)$, we will show that $|\mathrm{suc}(u, T)| < |\mathrm{suc}(u, T_\alpha)|$. We may assume without loss of generality that u is splitting in T, so let $\beta \in \kappa$ be such that $u \in \mathrm{Split}_\beta(T)$. Since $\mathrm{Split}_\gamma(T) = \mathrm{Split}_\gamma(T_\alpha)$ for all $\gamma \leq \alpha$, we see that $\beta \geq \alpha$. We have $\mathrm{Split}_{\beta+1}(T_{\beta+1}) = \mathrm{Split}_{\beta+1}(T)$, and thus $\mathrm{suc}(u, T_{\beta+1}) = \mathrm{suc}(u, T)$. Finally $T_{\beta+1} \leq T_\alpha$ gives us $|\mathrm{suc}(u, T)| = |\mathrm{suc}(u, T_{\beta+1})| < |\mathrm{suc}(u, T_\alpha)|$. ⊣

We are now ready to prove the two main ingredients necessary for separating the localisation cardinals. We will show that for any h there is some faster growing F such that $\mathbb{S}^h_\kappa$ has the F-Sacks property, and reversely that for any F there exists some faster growing h such that $\mathbb{S}^h_\kappa$ does not have the F-Sacks property. In other words, for any F_0 we may find h and F_1 such that $\mathbb{S}^h_\kappa$ does not have the F_0-Sacks property, but does have the F_1-Sacks property.

If T is a κ-tree and $u \in T$, let $(T)_u = \{v \in T \mid u \subseteq v \vee v \subseteq u\}$. It is clear that $(T)_u \leq T$.

Theorem 1.8. *For any h there exists F such that $h \leq^* F$ and $\mathbb{S}^h_\kappa$ has the F-Sacks property.*

Proof. We will let $F : \alpha \mapsto h(\alpha)^{|\alpha|}$ and show that $\mathbb{S}^h_\kappa$ has the F-Sacks property.

Let $T_0 \in \mathbb{S}^h_\kappa$ and let $\dot{f}$ be a name such that $T_0 \Vdash$ "$\dot{f} : \check{\kappa} \to \check{\kappa}$". If $T_0 \Vdash$ "$\dot{f} \in \check{\mathbf{V}}$", then the existence of an appropriate F-slalom is obvious, so we assume that $T_0 \Vdash$ "$\dot{f} \notin \check{\mathbf{V}}$". We will construct a fusion sequence $\langle T_\xi \mid \xi \in \kappa \rangle$ and sets $\{D_\xi \subseteq \kappa \mid \xi \in \kappa\}$ with $|D_\xi| \leq F(\xi)$ such that

$$\bigcap_{\xi \in \kappa} T_\xi = T \Vdash \text{“} \dot{f}(\check{\xi}) \in \check{D}_\xi \text{”}$$

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for each $\xi \in \kappa$. Consequently, we can define the F-slalom $\varphi : \xi \mapsto D_\xi$ in the ground model, then it follows that $T \Vdash$ "$\dot{f}(\check{\xi}) \in \check{\varphi}(\check{\xi})$" for all $\xi \in \kappa$.

In general, we will assume each T_ξ has the following property:

(∗) For every $u \in \mathrm{Split}_\alpha(T_\xi)$ with $\alpha < \xi$ we have $|\mathrm{suc}(u, T_\xi)| = h(\alpha)$.

This is vacuously true for T_0, and by using sharp κ-trees at successor stages of our construction, $(*)$ will follow by induction. If γ is limit, we will let $T_\gamma = \bigcap_{\xi\in\gamma} T_\xi$, which will be a condition by the proof of Lemma 1.3. T_γ need not necessarily be a sharp κ-tree, but it is at least sharp for all splitting levels less than γ, which is enough for $(*)$.

Suppose T_ξ has been defined, then we will define $T_{\xi+1}$ such that it limits the possible values of $\dot{f}(\xi)$ and such that $T_{\xi+1} \leq_\xi T_\xi$. First note that if T_ξ has property $(*)$, then $T^*_\xi \leq_\xi T_\xi$: If u is splitting in T_ξ and $u \notin T^*_\xi$, then u was removed because there is some $v \subseteq u$ such that $\mathrm{suc}(v, T_\xi)$ is too large for sharpness. But then by $(*)$ it follows that $v \in \mathrm{Split}_\alpha(T_\xi)$ for some $\alpha \geq \xi$, hence $u \in \mathrm{Split}_\beta(T_\xi)$ for some $\beta > \xi$.

We define a set V_ξ of successor nodes of the ξ-th splitting level, that is,

$$V_\xi = \bigcup\{\mathrm{suc}(u, T^*_\xi) \mid u \in \mathrm{Split}_\xi(T^*_\xi)\}.$$

Our goal is to find a stronger condition below each subtree $(T^*_\xi)_v$ with $v \in V_\xi$ that decides $\dot{f}(\check{\xi})$, and glue these conditions back together to get a condition stronger than T^*_ξ. Since the size of V_ξ is limited, this limits the possible values of $\dot{f}(\check{\xi})$ to a small set.

For each $v \in V_\xi$ find a condition $T^v \leq (T^*_\xi)_v$ such that $T^v \Vdash$ "$\dot{f}(\check{\xi}) = \check{\beta}^v_\xi$" for some $\beta^v_\xi \in \kappa$. Choose some arbitrary $u \in \mathrm{Split}_\xi(T^v)$ and $w \in \mathrm{suc}(u, T^v)$, and consider the subtree $(T^v)_w$ of T^v generated by the initial segment w. We let $G_\xi : V_\xi \to \mathcal{P}(T_\xi)$ send $v \mapsto (T^v)_w$. Note that the α-th splitting level of $G_\xi(v) = (T^v)_w$ corresponds to the $(\xi + 1 + \alpha)$-th splitting level of T^v.

Now we define

$$\begin{aligned} T_{\xi+1} &= \bigcup G_\xi[V_\xi] = \bigcup\{G_\xi(v) \mid v \in V_\xi\}, \\ D_\xi &= \left\{\beta^v_\xi \mid v \in V_\xi\right\}. \end{aligned}$$

For each $v \in V_\xi$ we have $v \in G_\xi(v)$, thus each successor of a splitting node in $\mathrm{Split}_\xi(T_\xi)$ is in $T_{\xi+1}$. Therefore we see that $\mathrm{Split}_\xi(T_{\xi+1}) = \mathrm{Split}_\xi(T_\xi)$. If $u \in \mathrm{Split}_{\xi+1+\alpha}(T_{\xi+1})$ for some $\alpha \in \kappa$, then $u \in \mathrm{Split}_\alpha(G_\xi(v))$, thus $u \in \mathrm{Split}_{\xi+1+\alpha}(T^v)$, and since $T^v \in \mathbb{S}^h_\kappa$, we see that u is $h(\xi + 1 + \alpha)$-splitting. Therefore $T_{\xi+1}$ satisfies (ii) of Definition 1.1. It is easy to check (i) and (iii), thus we can conclude that $T_{\xi+1} \in \mathbb{S}^h_\kappa$ and that $T_{\xi+1} \leq_\xi T_\xi$.

Note that the set D_ξ is indeed small enough:

$$|D_\xi| \leq |V_\xi| = |\mathrm{Split}_\xi(T^*_\xi)| \cdot h(\xi) \leq h(\xi)^{|\xi|} = F(\xi).$$

For each $v \in V_\xi$ we have $T^v \Vdash$ "$\dot{f}(\check{\xi}) \in \check{D}_\xi$", and $\{T^v \mid v \in V_\xi\}$ is predense below $T_{\xi+1}$; thus,

$$T_{\xi+1} \Vdash \text{“} \dot{f}(\check{\xi}) \in \check{D}_\xi \text{”}.$$

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Let $T = \bigcap_{\xi\in\kappa} T_\xi$, then by the fusion lemma $T \in \mathbb{S}^h_\kappa$, and $T \Vdash$ " $\dot{f}(\check{\xi}) \in \check{D}_\xi$ " for all $\xi \in \kappa$. ⊣

As a corollary of $\mathbb{S}^h_\kappa$ having the F-Sacks property, we immediately get that κ^+ is preserved.

COROLLARY 1.9. *$\mathbb{S}^h_\kappa$ preserves κ^+.*

PROOF. Given an $\mathbb{S}^h_\kappa$-name $\dot{f}$ and $T \in \mathbb{S}^h_\kappa$ such that $T \Vdash$ " $\dot{f} : \check{\kappa} \to \check{\kappa}^+$ ", then using (the proof of) the F-Sacks property we may produce sets D_ξ with $|D_\xi| = F(\xi) < \kappa$ for each $\xi \in \kappa$ such that $T' \Vdash$ " $\dot{f}(\check{\xi}) \in \check{D}_\xi$ " for some stronger $T' \leq T$, and thus $\dot{f}$ is forced to have a range contained in $\bigcup_{\xi\in\kappa} D_\xi$ and cannot be cofinal in κ^+. ⊣

The second ingredient is to find a suitably fast growing h for a given function F such that $\mathbb{S}^h_\kappa$ does not have the F-Sacks property. We will need the following lemma.

LEMMA 1.10. *Let $T \in \mathbb{S}^h_\kappa$ and let $C_T = \{\alpha \in \kappa \mid \mathrm{Split}_\alpha(T) = T \cap {}^\alpha\kappa\}$, then C_T is a club set.*

PROOF. For $\alpha_0 \in \kappa$ we can recursively define α_{n+1} large enough such that $\mathrm{Split}_{\alpha_n}(T) \subseteq {}^{\leq\alpha_{n+1}}\kappa$ for each $n \in \omega$. Let $\alpha = \bigcup_{n\in\omega} \alpha_n$, then $\alpha \in C_T$, hence C_T is unbounded. It is easy to see that C_T is continuous. ⊣

THEOREM 1.11. *Let $F \in {}^\kappa\kappa$, then there exists h such that $\mathbb{S}^h_\kappa$ does not have the F-Sacks property.*

PROOF. Let h be such that $F(\alpha) < h(\alpha)$ for all $\alpha \in S$, where S is a stationary subset of κ. We will show that $\mathbb{S}^h_\kappa$ does not have the F-Sacks property.

Let $\dot{f}$ be a name for the generic $\mathbb{S}^h_\kappa$-real in ${}^\kappa\kappa$, let φ be an F-slalom, let $T \in \mathbb{S}^h_\kappa$ and let $\alpha_0 \in \kappa$. We want to find some $\alpha \geq \alpha_0$ and $S \leq T$ such that $S \Vdash$ " $\dot{f}(\check{\alpha}) \notin \check{\varphi}(\check{\alpha})$ ". If we can find $u \in T \cap {}^{\alpha+1}\kappa$ such that $u(\alpha) \notin \varphi(\alpha)$, then $(T)_u$ will be sufficient.

Let C_T be as defined in Lemma 1.10 and $\alpha \in C_T \cap S$ such that $\alpha_0 \leq \alpha$, then $\mathrm{Split}_\alpha(T) = T \cap {}^\alpha\kappa$, thus each $t \in T \cap {}^\alpha\kappa$ is an $h(\alpha)$-splitting node. Hence, there is a set $X \subseteq \kappa$ with $|X| = h(\alpha)$ such that $t^\frown\gamma \in T$ for all $\gamma \in X$. Since $|\varphi(\alpha)| = F(\alpha) < h(\alpha)$, there is some $\gamma \in X$ such that $\gamma \notin \varphi(\alpha)$, and thus $u = t^\frown\gamma$ is as desired. ⊣

As a final part of this section, we will discuss the relation between parameters that are almost equal. For functions $f, g \in {}^\kappa\kappa$, we say that f and g are *almost equal*, written as $f =^* g$, if there exists $\xi \in \kappa$ such that $f(\alpha) = g(\alpha)$ for all $\alpha \in [\xi, \kappa)$. A related notion is that g *dominates* f, written as $f \leq^* g$, if there exists $\xi \in \kappa$ such that $f(\alpha) \leq g(\alpha)$ for all $\alpha \in [\xi, \kappa)$.

FACT 1.12. *If $h =^* h'$, then $\mathbb{S}^h_\kappa$ and $\mathbb{S}^{h'}_\kappa$ are forcing equivalent.*

PROOF. Since $\mathbb{S}^h_\kappa \cap \mathbb{S}^{h'}_\kappa$ is dense in both $\mathbb{S}^h_\kappa$ and $\mathbb{S}^{h'}_\kappa$. ⊣

§2. Products. We see that for any F_0, we can find a faster growing F_1 and some suitable h such that the forcing $\mathbb{S}^h_\kappa$ has the F_1-Sacks property and not the F_0-Sacks property, thus forcing with $\mathbb{S}^h_\kappa$ will not increase $\mathfrak{d}^{F_1}_\kappa(\in^*)$, but has the potential to increase $\mathfrak{d}^{F_0}_\kappa(\in^*)$.

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In order to increase $\mathfrak{d}_\kappa^{F_0}(\in^*)$ we will need to add many $\mathbb{S}_\kappa^h$-generic κ-reals to the ground model. This can be either done with an iteration, or with a product. Iteration has the drawback that once we have forced 2^κ to be of size κ^{++}, the forcing $\mathbb{S}_\kappa^h$ no longer has the $<\kappa^{++}$-c.c., and thus we cannot sufficiently control the iteration past this point. While this does not form a problem to prove the consistency of $\kappa^+ = \mathfrak{d}_\kappa^{F_1}(\in^*) < \mathfrak{d}_\kappa^{F_0}(\in^*) = \kappa^{++}$, iteration proves to be an obstacle when we wish to force localisation cardinals to be larger than κ^{++}. In particular, our goal to simultaneously assign multiple localisation cardinals to different cardinalities requires a product.

Definition 2.1. *Let A be a set of ordinals and $\mathbb{P}_\xi$ a forcing notion for each $\xi \in A$. Let $\mathcal{C}$ be the set of functions p with* $\mathrm{dom}(p) = A$ *such that $p(\xi) \in \mathbb{P}_\xi$ for each $\xi \in A$. For any $p \in \mathcal{C}$, the support of p is defined as*

$$\mathrm{supp}(p) = \left\{\xi \in A \;\middle|\; p(\xi) \neq \mathbb{1}_{\mathbb{P}_\xi}\right\}.$$

We define the $<\lambda$-support product as follows:

$$\overline{\mathbb{P}} = \textstyle\prod_{\xi \in A} \mathbb{P}_\xi = \{p \in \mathcal{C} \mid |\mathrm{supp}(p)| < \lambda\}.$$

This is a forcing poset under the ordering $q \leq_{\overline{\mathbb{P}}} p$ iff $q \leq_{\mathbb{P}_\xi} p$ for all $\xi \in A$. Generally we will say "$\leq\kappa$-support" instead of "$<\kappa^+$-support".

Let us fix a set of ordinals A, parameters $\langle h_\xi \mid \xi \in A\rangle$ for the forcings $\mathbb{S}_\kappa^{h_\xi}$ and the set $\mathcal{C}$ of functions p such that $p(\xi) \in \mathbb{S}_\kappa^{h_\xi}$ for each $\xi \in A$. For the remainder of this section we will also fix the shorthand $\overline{\mathbb{S}} = \prod_{\xi \in A} \mathbb{S}_\kappa^{h_\xi}$. If $p, q \in \overline{\mathbb{S}}$, we will often write $q \leq p$ instead of $q \leq_{\overline{\mathbb{S}}} p$ when the forcing $\overline{\mathbb{S}}$ is clear from context.

If $\langle p_\alpha \mid \alpha \in \gamma\rangle$ is a sequence of conditions in $\overline{\mathbb{S}}$ such that $p_{\alpha'} \leq_{\overline{\mathbb{S}}} p_\alpha$ for all $\alpha \leq \alpha'$, define

$$\textstyle\bigwedge_{\alpha \in \gamma} p_\alpha : \xi \mapsto \bigcap_{\alpha \in \gamma} p_\alpha(\xi) \quad \text{for } \xi \in A.$$

Lemma 2.2. *$\overline{\mathbb{S}}$ is $<\kappa$-closed.*

Proof. If $\langle p_\alpha \mid \alpha \in \gamma\rangle$ is a descending sequence in $\overline{\mathbb{S}}$ and $\gamma \in \kappa$, then $\bigwedge_{\alpha \in \gamma} p_\alpha$ is a condition below each p_α, since each $\mathbb{S}_\kappa^{h_\xi}$ is $<\kappa$-closed. ⊣

We will also need a generalisation of the fusion lemma to work on product forcings. The generalisation of fusion described here is analogous to what is described in [3] or [6].

Definition 2.3. *Given $p, q \in \overline{\mathbb{S}}$, $\alpha \in \kappa$, and $Z \subseteq A$ with $|Z| < \kappa$, let $q \leq_{Z,\alpha} p$ iff $q \leq p$ and for each $\xi \in Z$ we have $q(\xi) \leq_\alpha p(\xi)$.*

A generalised fusion sequence is a sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa\rangle$ such that:

(1) *$p_\alpha \in \overline{\mathbb{S}}$ and $Z_\alpha \in [A]^{<\kappa}$ for each $\alpha \in \kappa$.*
(2) *$p_\beta \leq_{Z_\alpha,\alpha} p_\alpha$ and $Z_\alpha \subseteq Z_\beta$ for all $\alpha \leq \beta \in \kappa$.*
(3) *For limit δ we have $Z_\delta = \bigcup_{\alpha \in \delta} Z_\alpha$.*
(4) *$\bigcup_{\alpha \in \kappa} Z_\alpha = \bigcup_{\alpha \in \kappa} \mathrm{supp}(p_\alpha)$.*

Lemma 2.4. *If $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa\rangle$ is a generalised fusion sequence, then $\bigwedge_{\alpha \in \kappa} p_\alpha \in \overline{\mathbb{S}}$.*

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PROOF. Suppose that $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa \rangle$ is a generalised fusion sequence, and let $p = \bigwedge_{\alpha\in\kappa} p_\alpha$. Point (4) of Definition 2.3 implies that every $\xi \in \operatorname{supp}(p)$ is an element of Z_{η_ξ} for some $\eta_\xi \in \kappa$. This means that if $\beta \geq \alpha \geq \eta_\xi$, then $p_\beta(\xi) \leq_\alpha p_\alpha(\xi)$, and thus $\langle p_\alpha(\xi) \mid \alpha > \eta_\xi \rangle$ is a fusion sequence in $\mathbb{S}_\kappa^{h_\xi}$. Since $\mathbb{S}_\kappa^{h_\xi}$ is closed under fusion sequences (Lemma 1.7), we can conclude that

$$p(\xi) = \bigcap\nolimits_{\alpha\in\kappa} p_\alpha(\xi) \in \mathbb{S}_\kappa^{h_\xi}.$$

Since $\operatorname{supp}(p) = \bigcup_{\alpha\in\kappa} Z_\alpha$, we see that $|\operatorname{supp}(p)| \leq \kappa$, thus we can conclude that $p \in \overline{\mathbb{S}}$. ⊣

By Lemma 2.2, $\overline{\mathbb{S}}$ preserves all cardinalities up to and including κ. Suppose that each $\mathbb{S}_\kappa^{h_\xi}$ has the F-Sacks property for some suitably large F. We will show in the next lemma that this implies that $\overline{\mathbb{S}}$ has the F-Sacks property and therefore preserves κ^+. Finally, if we assume that $\mathbf{V} \vDash$ "$2^\kappa = \kappa^+$", then a standard Δ-system argument (see, e.g., [5, Lemma 15.4]) shows that $\overline{\mathbb{S}}$ is $<\kappa^{++}$-c.c. as well. Thus, $\overline{\mathbb{S}}$ preserves all cardinals and cofinalities assuming that there exists some fixed $F \in {}^\kappa\kappa$ such that each $\mathbb{S}_\kappa^{h_\xi}$ has the F-Sacks property.

Before we prove the lemma, let us introduce some notation related to the product of forcings. Suppose $\overline{\mathbb{P}} = \prod_{\xi\in A} \mathbb{P}_\xi$ is a product with $\leq\kappa$-support, $X \subseteq \overline{\mathbb{P}}$ and $B \subseteq A$, we define

$$X \restriction B = \{p \restriction B \mid p \in X\}.$$

Let $B^c = A \setminus B$ and $G \subseteq \overline{\mathbb{P}}$ be $\overline{\mathbb{P}}$-generic over $\mathbf{V}$, then clearly $\overline{\mathbb{P}}$ and $(\overline{\mathbb{P}} \restriction B) \times (\overline{\mathbb{P}} \restriction B^c)$ are forcing equivalent, $(G \restriction B) \times (G \restriction B^c)$ is $(\overline{\mathbb{P}} \restriction B) \times (\overline{\mathbb{P}} \restriction B^c)$-generic and

$$\mathbf{V}[G] = \mathbf{V}[(G \restriction B) \times (G \restriction B^c)] = \mathbf{V}[G \restriction B][G \restriction B^c].$$

LEMMA 2.5. *Let $B \subseteq A$ be sets of ordinals and $B^c = A \setminus B$, and consider a sequence of functions $\langle h_\xi \mid \xi \in A \rangle$. We define the $\leq\kappa$-support product $\overline{\mathbb{S}} = \prod_{\xi\in A} \mathbb{S}_\kappa^{h_\xi}$, we assume G is an $\overline{\mathbb{S}}$-generic filter. If there exists $F \in {}^\kappa\kappa$ such that $F(\alpha)^{|\alpha|} = F(\alpha)$ for all $\alpha \in \kappa$ and $h_\xi \leq^* F$ for all $\xi \in B^c$, then for each $f \in ({}^\kappa\kappa)^{\mathbf{V}[G]}$ there is $\varphi \in (\mathrm{Loc}_F)^{\mathbf{V}[G\restriction B]}$ such that $f \in^* \varphi$.*

PROOF. Note that Fact 1.12 implies that we can assume without loss of generality that $h_\xi \leq F$ for each $\xi \in A$. Let $p \in \overline{\mathbb{S}}$ and $\dot{f}$ be a name such that $p \Vdash_{\overline{\mathbb{S}}}$ "$\dot{f} : \check{\kappa} \to \check{\kappa}$", then we will construct a name $\dot{\varphi}$ and a condition $p' \leq p$ such that $p' \Vdash$ "$\dot{\varphi} \in (\mathrm{Loc}_{\check{F}})^{\mathbf{V}[\dot{G}\restriction\check{B}]}$".

The proof is essentially the same as the proof of Theorem 1.8, except that we work with generalised fusion sequences and have to construct a name $\dot{\varphi}$ for the appropriate F-slalom in $\mathbf{V}[G \restriction B]$, since such a slalom is not generally present in the ground model. That is, we will construct a sequence $\langle (p_\xi, Z_\xi) \mid \xi \in \kappa \rangle$ with each $p_\xi \in \overline{\mathbb{S}}$ that is a generalised fusion sequence in $\overline{\mathbb{S}}$ and names $\dot{D}_\xi$ for sets of ordinals $D_\xi \in \mathbf{V}[G \restriction B]$ with $|D_\xi| \leq F(\xi)$, such that $p_{\xi+1} \Vdash$ "$\dot{f}(\check{\xi}) \in \dot{D}_\xi$".

For each $\xi \in \kappa$ and $\beta \in Z_\xi$ we will make sure that $p_\xi(\beta) \in (\mathbb{S}_\kappa^{h_\beta})^*$ is sharp. To start, we let $p_0 = p$ and we let $Z_0 = \varnothing$. At limit stages δ we can define $p'_\delta = \bigwedge_{\xi\in\delta} p_\xi$

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and let $p_\delta \leq p'_\delta$ be defined elementwise such that $p_\delta(\beta) = (p'_\delta(\beta))^*$ is sharp for each $\beta \in Z_\delta$.

Suppose we have defined $p_\xi \in \overline{\mathbb{S}}$ and Z_ξ and that $|Z_\xi| \leq |\xi|$. As in the proof of Theorem 1.8, we will consider the successor nodes of the ξ-th splitting level, find subtrees that decide the value of $\dot{f}(\check{\xi})$, and glue the subtrees together. However, in this situation we have to deal with multiple trees at once, namely with each $p_\xi(\beta)$ such that $\beta \in Z_\xi$. For each $\beta \in Z_\xi$ we define the set of successor nodes of the ξ-th splitting level of $p_\xi(\beta)$:

$$V_\xi^\beta = \bigcup \left\{ \mathrm{suc}(u, p_\xi(\beta)) \;\middle|\; u \in \mathrm{Split}_\xi(p_\xi(\beta)) \right\}.$$

To deal with $p_\xi(\beta)$ for all $\beta \in Z_\xi$ simultaneously, we have to consider combinations of elements of V_ξ^β for $\beta \in Z_\xi$, and for each combination we will define a condition that decides $\dot{f}(\check{\xi})$. These combinations are given by functions $g : Z_\xi \to \bigcup_{\beta \in Z_\xi} V_\xi^\beta$ with the property that $g(\beta) \in V_\xi^\beta$. We will refer to such g as *choice functions*, since g chooses an element of V_ξ^β for each $\beta \in Z_\xi$.

Let $\mathcal{V}_\xi$ be the set of choice functions on $\{V_\xi^\beta \mid \beta \in Z_\xi\}$ and $\mathcal{V}'_\xi$ the set of choice functions on $\{V_\xi^\beta \mid \beta \in Z_\xi \setminus B\}$, that is, $\mathcal{V}'_\xi$ is the set of $g \upharpoonright (Z_\xi \setminus B)$ with $g \in \mathcal{V}_\xi$.

By induction hypothesis $p_\xi(\beta) \in (\mathbb{S}_\kappa^{h_\beta})^*$ for each $\beta \in Z_\xi \setminus B$, hence we know that

$$|\mathrm{Split}_\xi(p_\xi(\beta))| \leq h_\beta(\xi)^{|\xi|} \leq F(\xi),$$

and thus $|V_\xi^\beta| \leq F(\xi)$ for all $\beta \in Z_\xi \setminus B$. Since we assume that $|Z_\xi| \leq |\xi|$, we therefore have $|\mathcal{V}'_\xi| \leq F(\xi)^{|\xi|} = F(\xi)$ (assuming without loss of generality that $F(\xi)$ is infinite). Hence, if we restrict our attention to $Z_\xi \setminus B$, we have a small number of choice functions. Consequently, we can describe a name $\dot{D}_\xi$ depending only on the support in B, i.e., $\dot{D}_\xi$ names a set in $\mathbf{V}[G \upharpoonright B]$, such that $\dot{D}_\xi$ is bounded in cardinality by $F(\xi)$.

For any choice function $g \in \mathcal{V}_\xi$, let $(p_\xi)_g$ be the condition defined by

$$(p_\xi)_g(\beta) = \begin{cases} p_\xi(\beta), & \text{if } \beta \notin Z_\xi, \\ (p_\xi(\beta))_{g(\beta)}, & \text{if } \beta \in Z_\xi. \end{cases}$$

Here $(p_\xi(\beta))_{g(\beta)}$ is the subtree of $p_\xi(\beta)$ generated by the initial segment $g(\beta) \in V_\xi^\beta$.

Let $\zeta = |\mathcal{V}_\xi|$ then $\zeta < \kappa$ by inaccessibility of κ. Fix some enumeration $\langle g_\eta \mid \eta \in \zeta \rangle$ of $\mathcal{V}_\xi$, which we will use to recursively define a decreasing sequence of conditions r_η with $r_\eta \leq_{Z_\xi,\xi} p_\xi$ for each $\eta \in \zeta$. Essentially, our recursive construction will result in $r_{\eta+1}$ being like r_η, except that $(r_\eta)_{g_\eta}$ is replaced by a stronger condition that decides $\dot{f}(\check{\xi})$. At the end of the recursion, we will be left with a condition r_ζ such that $(r_\zeta)_g$ decides $\dot{f}(\check{\xi})$ for every $g \in \mathcal{V}_\xi$. We then gather the possible values of $\dot{f}(\check{\xi})$ to construct the name $\dot{D}_\xi$.

Let $r_0 = p_\xi$. For limit $\delta \in \zeta$ let $r_\delta = \bigwedge_{\eta \in \delta} r_\eta$, which is a condition by $<\kappa$-closure (Lemma 2.2). Assuming that $r_\eta \leq_{Z_\xi,\xi} p_\xi$ for each $\eta \in \delta$, it is easy to see that $r_\delta \leq_{Z_\xi,\xi} p_\xi$ as well.

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Suppose r_η is defined and $r_\eta \leq_{Z_\xi,\xi} p_\xi$, then in particular $r_\eta(\beta) \leq_\xi p_\xi(\beta)$ for all $\beta \in Z_\xi$, and thus $\mathrm{Split}_\xi(r_\eta(\beta)) = \mathrm{Split}_\xi(p_\xi(\beta))$ for all $\beta \in Z_\xi$. Therefore by definition of the ordering on $\mathbb{S}_\kappa^{h_\beta}$ and the fact that $p_\xi(\beta)$ is sharp, we see that V_ξ^β is exactly the set of successors of nodes at the ξ-th splitting level of $r_\eta(\beta)$. Take the η-th choice function $g_\eta \in \mathcal{V}_\xi$, and let $r'_\eta \leq (r_\eta)_{g_\eta}$ be such that $r'_\eta \Vdash$ " $\dot{f}(\check{\xi}) = \check{\beta}_\xi^\eta$ " for some ordinal β_ξ^η. We define $r_{\eta+1}$ elementwise.

If $\beta \notin Z_\xi$, then we simply take $r_{\eta+1}(\beta) = r'_\eta(\beta)$.

If $\beta \in Z_\xi$, fix some $w \in \mathrm{suc}(u, r'_\eta(\beta))$ for some $u \in \mathrm{Split}_\xi(r'_\eta(\beta))$ and consider the subtree $(r'_\eta(\beta))_w$ generated by the initial segment w. Now we are ready to define $r_{\eta+1}(\beta)$ as

$$r_{\eta+1}(\beta) = (r'_\eta(\beta))_w \;\cup\; \left\{u \in r_\eta(\beta) \;\middle|\; \exists v \in V_\xi^\beta \setminus \{g_\eta(\beta)\}\ (u \subseteq v \text{ or } v \subseteq u)\right\}.$$

In words, $r_{\eta+1}(\beta)$ is the result of replacing the extensions of $g_\eta(\beta) \in r_\eta(\beta)$ by $(r'_\eta(\beta))_w$ that decides $\dot{f}(\check{\xi})$, where we use the subtree $(r'_\eta(\beta))_w$ instead of $r'_\eta(\beta)$ to make sure that $r_{\eta+1}(\beta)$ has enough successors at each splitting level to be in $\mathbb{S}_\kappa^{h_\beta}$ (compare this to the role of $(T^v)_w$ instead of T^v in the proof of Theorem 1.8).

To finish the construction of the next condition in the fusion sequence, we use $<\kappa$-closure to define $p'_{\xi+1} = \bigwedge_{\eta\in\zeta} r_\eta$ and let $p_{\xi+1} = (p'_{\xi+1})^*$ be sharp. To see that $p_{\xi+1} \leq_{Z_\xi,\xi} p_\xi$, note that for every $\beta \in Z_\xi$ and $v \in V_\xi^\beta$ we have $v \in r_\eta(\beta)$ for all $\eta \in \zeta$, hence $v \in p_{\xi+1}(\beta)$. This implies by definition of V_ξ^β that $p_{\xi+1}(\beta) \leq_\xi p_\xi(\beta)$ for all $\beta \in Z_\xi$. Finally, we can let $Z_{\xi+1} = Z_\xi \cup \{\delta\}$ for some ordinal δ, using bookkeeping to make sure that $\bigcup_{\xi\in\kappa} Z_\xi = \bigcup_{\xi\in\kappa} \mathrm{supp}(p_\xi)$.

Note that the set of conditions $r \leq p_{\xi+1}$ with $|r(\beta) \cap V_\xi^\beta| = 1$ for all $\beta \in Z_\xi$, is dense below $p_{\xi+1}$. For any such r, let g map β to the unique element of $r(\beta) \cap V_\xi^\beta$ for each $\beta \in Z_\xi$, then $g \in \mathcal{V}_\xi$ is a choice function, so we see that there exists $\eta \in \zeta$ such that $g = g_\eta$. We will show that $r \leq r'_\eta$, which implies that $r \Vdash$ " $\dot{f}(\check{\xi}) = \check{\beta}_\xi^\eta$ ".

For any β we have $r(\beta) \leq p_{\xi+1}(\beta) \leq r_{\eta+1}(\beta)$. If $\beta \notin Z_\xi$, then we simply have $r_{\eta+1}(\beta) = r'_\eta(\beta)$, thus we are done. Otherwise $\beta \in Z_\xi$, and we know that $g(\beta)$ is an initial segment of the stem of $r(\beta)$, hence

$$r(\beta) = (r(\beta))_{g(\beta)} \subseteq (r_{\eta+1}(\beta))_{g(\beta)} = (r'_\eta(\beta))_w,$$

where w is as in the definition of $r_{\eta+1}(\beta)$ above. Since $r(\beta) \leq r_{\eta+1}(\beta)$, we also have

$$r(\beta) = (r(\beta))_w \leq (r_{\eta+1}(\beta))_w = (r'_\eta(\beta))_w \leq r'_\eta(\beta),$$

and thus $r(\beta) \leq r'_\eta(\beta)$.

We are now ready to construct the names $\dot{D}_\xi$ such that

$$p_{\xi+1} \Vdash \text{“}\, \dot{f}(\check{\xi}) \in \dot{D}_\xi \text{ and } \dot{D}_\xi \in \mathbf{V}[\dot{G} \restriction \check{B}] \text{ and } |\dot{D}_\xi| \leq \check{F}(\check{\xi}) \,\text{”}.$$

For any $g \in \mathcal{V}_\xi$, we define

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$$g'' = g \restriction (Z_\xi \cap B),$$
$$E_g = \{\eta \in \zeta \mid \exists g' \in \mathcal{V}'_\xi(g' \cup g'' = g_\eta)\},$$
$$D^g_\xi = \left\{\beta^\eta_\xi \mid \eta \in E_g\right\}.$$

Since $|\mathcal{V}'_\xi| \leq F(\xi)$, we see that $|E_g| \leq F(\xi)$, hence $|D^g_\xi| \leq F(\xi)$. Clearly, if $g, \tilde{g} \in \mathcal{V}_\xi$ and $g \restriction (Z_\xi \cap B) = \tilde{g} \restriction (Z_\xi \cap B)$, then $D^g_\xi = D^{\tilde{g}}_\xi$.

Let $\mathcal{A}_\xi$ be an antichain below $p_{\xi+1}$ such that $r \in \mathcal{A}_\xi$ implies $|r(\beta) \cap V^\beta_\xi| = 1$ for all $\beta \in Z_\xi$, and let $g_r \in \mathcal{V}_\xi$ be such that $g_r(\beta)$ is the single element of $r(\beta) \cap V^\beta_\xi$ for each $\beta \in Z_\xi$. We define

$$\dot{D}_\xi = \left\{(r, \check{D}^{g_r}_\xi) \mid r \in \mathcal{A}_\xi\right\}.$$

It is clear by the above that for each $r \in \mathcal{A}_\xi$ and η such that $g_r = g_\eta$ we have

$$r \Vdash \text{“} \dot{f}(\check{\xi}) = \check{\beta}^\eta_\xi \in \check{D}^{g_r}_\xi \text{ and } |\check{D}^{g_r}_\xi| \leq \check{F}(\check{\xi}) \text{”},$$

so by denseness

$$p_{\xi+1} \Vdash \text{“} \dot{f}(\check{\xi}) \in \dot{D}_\xi \text{ and } |\dot{D}_\xi| \leq \check{F}(\check{\xi}) \text{”}.$$

To see that $p_{\xi+1} \Vdash \text{“} \dot{D}_\xi \in \mathbf{V}[\dot{G} \restriction \check{B}] \text{”}$, we argue within $\mathbf{V}[G \restriction B]$. For every $r, \tilde{r} \in \mathcal{A}_\xi$ such that both $r \restriction B$ and $\tilde{r} \restriction B$ are elements of $G \restriction B$ we see that the corresponding g_r and $g_{\tilde{r}}$ have the property that $g_r \restriction (Z_\xi \cap B) = g_{\tilde{r}} \restriction (Z_\xi \cap B)$, and therefore $D^{g_r}_\xi = D^{g_{\tilde{r}}}_\xi$. Thus, we can fix any arbitrary such $r \in \mathcal{A}_\xi$ for which $r \restriction B \in G \restriction B$ holds, and see that

$$\mathbf{V}[G \restriction B] \vDash \text{“} p_{\xi+1} \restriction B^c \Vdash \dot{D}_\xi = \check{D}^{g_r}_\xi \text{”}.$$

Let $p' = \bigwedge_{\xi \in \kappa} p_\xi$ be the limit of the generalised fusion sequence, and let $\dot{\varphi}$ be a name such that $p' \Vdash \text{“} \dot{\varphi} : \check{\xi} \mapsto \dot{D}_\xi \text{”}$, then $\dot{\varphi}$ names an F-slalom in $\mathbf{V}[G \restriction B]$ and $p' \Vdash \text{“} \dot{f} \in^* \dot{\varphi} \text{”}$. ⊣

If we let $B = \varnothing$ in the definition of the lemma, then we can simplify this lemma to the following corollary, providing us with the preservation of the Sacks property.

Corollary 2.6. *If $\overline{\mathbb{S}} = \prod_{\xi \in A} \mathbb{S}^{h_\xi}_\kappa$ and each $h_\xi \leq^* h$ and $F : \alpha \mapsto h(\alpha)^{|\alpha|}$, then $\overline{\mathbb{S}}$ has the F-Sacks property.*

Finally the following lemma is based on Theorem 1.11 and shows how we can use products of forcings $\mathbb{S}^{h_\xi}_\kappa$ to increase the cardinality of $\mathfrak{d}^F_\kappa(\in^*)$.

Lemma 2.7. *Let $B \subseteq A$ be sets of ordinals, and consider a sequence of functions $\langle h_\xi \mid \xi \in A\rangle$. We define the $\leq\kappa$-support product $\overline{\mathbb{S}} = \prod_{\xi \in A} \mathbb{S}^{h_\xi}_\kappa$ and we assume G is an $\overline{\mathbb{S}}$-generic filter. Let $\langle S_\xi \mid \xi \in B\rangle$ be a sequence of stationary sets. If F is such that for each $\xi \in B$ we have $F(\alpha) < h_\xi(\alpha)$ for all $\alpha \in S_\xi$, then $\mathbf{V}[G] \vDash$ “$|B| \leq \mathfrak{d}^F_\kappa(\in^*)$”.*

Proof. The lemma is trivial if $|B| \leq \kappa^+$, so we will assume that $|B| \geq \kappa^{++}$.

We work in $\mathbf{V}[G]$. Let $\mu < |B|$ and let $\{\varphi_\xi \mid \xi \in \mu\}$ be a family of F-slaloms, then we want to describe some $f \in {}^\kappa\kappa$ such that $f \notin^* \varphi_\xi$ for each $\xi \in \mu$. Since

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$\overline{\mathbb{S}}$ is $<\kappa^{++}$-c.c., we could find $A_\xi \subseteq A$ with $|A_\xi| \leq \kappa^+$ for each $\xi \in \mu$ such that $\varphi_\xi \in \mathbf{V}[G \restriction A_\xi]$. Since $|B| > \mu \cdot \kappa^+$, we may fix some $\beta \in B \setminus \bigcup_{\xi \in \mu} A_\xi$ for the remainder of this proof. Let $f = \bigcap_{p \in G} p(\beta)$, then $f \in {}^\kappa\kappa$ is the generic κ-real added by the β-th term of the product $\overline{\mathbb{S}}$.

Continuing the proof in the ground model, let $\dot{f}$ be an $\overline{\mathbb{S}}$-name for f and $\dot{\varphi}_\xi$ be an $\overline{\mathbb{S}}$-name for φ_ξ, let $p \in \overline{\mathbb{S}}$ and $\alpha_0 \in \kappa$. We want to find some $\alpha \geq \alpha_0$ and $q \leq p$ such that $q \Vdash$ " $\dot{f}(\check{\alpha}) \notin \dot{\varphi}_\xi(\check{\alpha})$ ".

Let $C = \{\alpha \in \kappa \mid p(\beta) \cap {}^\alpha\kappa = \mathrm{Split}_\alpha(p(\beta))\}$, which is a club set by Lemma 1.10. Since S_β is stationary, there exists some $\alpha \geq \alpha_0$ such that $\alpha \in C \cap S_\beta$. Choose some $p_0 \leq p$ such that $p_0(\beta) = p(\beta)$ and such that there is a $Y \in [\kappa]^{\leq F(\alpha)}$ for which $p_0 \Vdash$ " $\dot{\varphi}_\xi(\check{\alpha}) = \check{Y}$ ". This is possible, since $\varphi_\xi \in \mathbf{V}[G \restriction A_\xi]$ and $\beta \notin A_\xi$, therefore we could find $p_0' \in \overline{\mathbb{S}} \restriction A_\xi$ with $p_0' \leq p \restriction A_\xi$ and Y with the aforementioned property, and then let $p_0(\eta) = p_0'(\eta)$ if $\eta \in A_\xi$ and $p_0(\eta) = p(\eta)$ otherwise.

Each $t \in p_0(\beta) \cap {}^\alpha\kappa$ is a $h_\beta(\alpha)$-splitting node, hence the set $X = \{\chi \in \kappa \mid t^\frown\chi \in p_0(\beta)\}$ has cardinality $|X| \geq h_\beta(\alpha)$. Because $\alpha \in S_\beta$ and $\beta \in B$, we have by our assumptions on F that $|Y| \leq F(\alpha) < h_\beta(\alpha) \leq |X|$. We can therefore find some $\chi \in X$ such that $\chi \notin Y$. Let $q \leq p_0$ be defined as

$$q(\eta) = \begin{cases} (p_0(\beta))_{t^\frown\chi}, & \text{if } \eta = \beta, \\ p_0(\eta), & \text{otherwise.} \end{cases}$$

Here $(p_0(\beta))_{t^\frown\chi}$ is the subtree of $p_0(\beta)$ generated by the initial segment $t^\frown\chi$. Then $q \leq p_0 \leq p$ and $q \Vdash$ " $\dot{f}(\check{\alpha}) \notin \check{Y} = \dot{\varphi}_\xi(\check{\alpha})$ ". ⊣

Lemma 2.8. *Let A be a set of ordinals such that $\kappa < \mathrm{cf}(|A|)$, let $\langle h_\xi \mid \xi \in A\rangle$ be a sequence of functions, let $\overline{\mathbb{S}} = \prod_{\xi \in A} \mathbb{S}_\kappa^{h_\xi}$ with $\overline{\mathbb{S}}$-generic G, and let $F \in {}^\kappa\kappa$. Assuming $\mathbf{V} \vDash$ " $2^\kappa = \kappa^+$ ", it follows that*

$$\mathbf{V}[G] \vDash \text{“ } 2^\kappa = |\mathrm{Loc}_F| = \kappa^+ \cdot |A| \text{ ”}.$$

Proof. This is a standard argument of counting names. ⊣

We are now ready to use our product forcing to separate κ many cardinals of the form $\mathfrak{d}_\kappa^h(\in^*)$.

Theorem 2.9. *There exists a family of functions $\{g_\eta \mid \eta \in \kappa\} \subseteq {}^\kappa\kappa$ such that for any $\gamma \in \kappa^+$ and any increasing sequence $\langle \lambda_\xi \mid \xi \in \gamma\rangle$ of cardinals with $\kappa < \mathrm{cf}(\lambda_\xi)$ for all $\xi \in \gamma$ and any $\sigma : \kappa \to \gamma$, there exists a forcing extension in which $\mathfrak{d}_\kappa^{g_\eta}(\in^*) = \lambda_{\sigma(\eta)}$ for all $\eta \in \kappa$.*

Proof. We assume that $\mathbf{V} \vDash$ " $2^\kappa = \kappa^+$ ", or otherwise we first use a forcing to collapse 2^κ to become κ^+. By a result of Solovay (Theorem 8.10 in [5]) there exists a family of κ many disjoint stationary subsets of κ, thus let $\{S_\eta \mid \eta \in \kappa\}$ be such a family. Let $\kappa \leq \gamma \in \kappa^+$ and $\sigma : \kappa \to \gamma$ be given. We will assume without loss of generality that σ is bijective, and hence that $\sigma^{-1} : \gamma \to \kappa$ is a well-defined bijection. Let $\langle \lambda_\xi \mid \xi \in \gamma\rangle$ be an increasing sequence of cardinals with $\mathrm{cf}(\lambda_\xi) > \kappa$ for all $\xi \in \gamma$.

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Fix some $F \in {}^\kappa\kappa$ such that $F(\alpha)^{|\alpha|} = F(\alpha)$ and $2^{F(\alpha)} \leq F(\beta)$ for any $\alpha < \beta$. For each $\eta \in \kappa$ we define a function g_η as follows:

$$g_\eta(\alpha) = \begin{cases} F(\alpha), & \text{if } \alpha \in S_\eta, \\ 2^{F(\alpha)}, & \text{otherwise.} \end{cases}$$

For each $\xi \in \gamma$ we define $H_\xi \in {}^\kappa\kappa$ as follows:

$$H_\xi(\alpha) = \begin{cases} F(\alpha), & \text{if } \alpha \in \bigcup_{\zeta \in \xi} S_{\sigma^{-1}(\zeta)}, \\ 2^{F(\alpha)}, & \text{otherwise.} \end{cases}$$

For each $\xi \in \gamma$ let A_ξ be a set of ordinals with $|A_\xi| = \lambda_\xi$, such that $\langle A_\xi \mid \xi \in \gamma \rangle$ is a sequence of mutually disjoint sets, and let $A = \bigcup_{\xi \in \gamma} A_\xi$. For each $\xi \in \gamma$ and $\beta \in A_\xi$, we define $h_\beta = H_\xi$.

We now consider the product forcing $\overline{\mathbb{S}} = \prod_{\beta \in A} \mathbb{S}_\kappa^{h_\beta}$ with $\leq\kappa$-support. Let G be $\overline{\mathbb{S}}$-generic. We will fix some $\eta \in \kappa$, and let $B = \bigcup_{\xi \in \sigma(\eta)+1} A_\xi$ and $B^c = A \setminus B$. By Lemma 2.8 we see that $(\mathrm{Loc}_{g_\eta})^{\mathbf{V}[G \restriction B]}$ has cardinality

$$\kappa^+ \cdot |B| = \kappa^+ \cdot \left|\sup_{\xi \leq \sigma(\eta)} A_\xi\right| = \kappa^+ \cdot \left|A_{\sigma(\eta)}\right| = \lambda_{\sigma(\eta)}.$$

To use Lemma 2.5, we need that $h_\beta \leq^* g_\eta$ for all $\beta \in B^c$, equivalently, that $H_\xi \leq^* g_\eta$ for all $\xi \in (\sigma(\eta), \gamma)$. But this is true for any $\xi \in (\sigma(\eta), \gamma)$, since $g_\eta = F(\alpha)$ iff $\alpha \in S_\eta = S_{\sigma^{-1}(\sigma(\eta))}$ and because $\sigma(\eta) \in \xi$ we see that $H_\xi(\alpha) = F(\alpha)$ as well. Meanwhile for all $\alpha \notin S_\eta$ we have $g_\eta(\alpha) = 2^{F(\alpha)} \geq H_\xi(\alpha)$. Therefore Lemma 2.5 shows that $(\mathrm{Loc}_{g_\eta})^{\mathbf{V}[G \restriction B]}$ is a family in $\mathbf{V}[G]$ of size $\lambda_{\sigma(\eta)}$ that forms a witness for

$$\mathbf{V}[G] \vDash \text{“}\, \mathfrak{d}_\kappa^{g_\eta}(\in^*) \leq \lambda_{\sigma(\eta)} \,\text{”}.$$

On the other hand, if $\beta \in A_{\sigma(\eta)}$, then $h_\beta = H_{\sigma(\eta)}$ and thus for any $\alpha \in S_\eta = S_{\sigma^{-1}(\sigma(\eta))}$ we see that $g_\eta(\alpha) < H_{\sigma(\eta)}(\alpha)$. Therefore by Lemma 2.7 we see that

$$\mathbf{V}[G] \vDash \text{“}\, \lambda_{\sigma(\eta)} = |A_{\sigma(\eta)}| \leq \mathfrak{d}_\kappa^{g_\eta}(\in^*) \,\text{”}.$$

In conclusion, we get for every $\eta \in \kappa$ that

$$\mathbf{V}[G] \vDash \text{“}\, \lambda_{\sigma(\eta)} = \mathfrak{d}_\kappa^{g_\eta}(\in^*) \,\text{”}. \qquad \dashv$$

COROLLARY 2.10. *There exists functions h_ξ for each $\xi \in \kappa$ such that for any cardinals $\lambda_\xi > \kappa$ with $\mathrm{cf}(\lambda_\xi) > \kappa$ it is consistent that simultaneously $\mathfrak{d}_\kappa^{h_\xi}(\in^*) = \lambda_\xi$ for all $\xi \in \kappa$.*

§3. Separating κ^+ many cardinals. We saw in the previous section that we can use a partition of κ into disjoint stationary sets $\{S_\eta \mid \eta \in \kappa\}$, and associate a function g_η with each stationary S_η such that the cardinals $\mathfrak{d}_\kappa^{g_\eta}(\in^*)$ can consistently be put in any arbitrary well-order.

It is natural to ask if we can do better than this, and separate κ^+ many cardinalities. Clearly we cannot do this using a disjoint family of stationary sets, since no such family of size κ^+ exists. Fortunately we can work around this by using an *almost* disjoint family of stationary sets, that is, a family $\mathcal{S}$ of stationary subsets of κ, such that $|S \cap S'| < \kappa$ for any distinct $S, S' \in \mathcal{S}$. Let us refer to such families as *stationary almost disjoint families*, or *sad families*.

The existence of a sad family of size 2^κ is a consequence of $\diamondsuit_\kappa$. Let $\langle A_\alpha \mid \alpha \in \kappa\rangle$ be a $\diamondsuit_\kappa$-sequence, that is, a sequence such that for any $X \in \mathcal{P}(\kappa)$ the following set is stationary:

$$S_X = \{\alpha \in \kappa \mid X \cap \alpha = A_\alpha\}.$$

If $X, Y \in \mathcal{P}(\kappa)$ are distinct, and ξ is the least element of the symmetric difference $X \triangle Y$, then it is easy to see that $S_X \cap S_Y \subseteq \xi + 1$, thus $\{S_X \mid X \in \mathcal{P}(\kappa)\}$ is a sad family of size 2^κ.

However, generalising the proof of Theorem 2.9 to work with a sad family is not as straightforward as it seems. As we have seen in Lemmas 2.5 and 2.7, the forcing $\mathbb{S}^h_\kappa$ will not increase $\mathfrak{d}^g_\kappa(\in^*)$ if $\mathbb{S}^h_\kappa$ has the F-Sacks property for some $F \leq^* g$, but it will increase $\mathfrak{d}^g_\kappa(\in^*)$ if there exists a stationary set S such that $g(\alpha) < h(\alpha)$ for all $\alpha \in S$.

Let us assume $\mathbf{V} \vDash$ "$2^\kappa = \kappa^+$ and $\diamondsuit_\kappa$" and fix a sad family $\{S_\eta \mid \eta \in \kappa^+\}$. We assume that $F \in {}^\kappa\kappa$ is some arbitrary function such that $F(\alpha)^{|\alpha|} = F(\alpha)$ and $2^{F(\alpha)} \leq F(\beta)$ for all $\alpha < \beta$. For every $\eta \in \kappa^+$ we can define the functions g_η, forming the parameters of $\mathfrak{d}^{g_\eta}_\kappa(\in^*)$:

$$g_\eta(\alpha) = \begin{cases} F(\alpha), & \text{if } \alpha \in S_\eta. \\ 2^{F(\alpha)}, & \text{otherwise.} \end{cases}$$

In analogy with Theorem 2.9, we want to define functions H_ξ such that $\mathbb{S}^{H_\xi}_\kappa$ keeps $\mathfrak{d}^{g_\eta}_\kappa(\in^*)$ small when $\eta \in X$ and increases $\mathfrak{d}^{g_\eta}_\kappa(\in^*)$ when $\eta \in Y$, where $\{X, Y\}$ forms a partition of κ^+. Assuming $H_\xi(\alpha) = H_\xi(\alpha)^{|\alpha|}$ for all $\alpha \in \kappa$, this means that we want to define H_ξ such that:

$$\begin{aligned} &\text{when } \eta \in X: && H_\xi \leq^* g_\eta, \\ &\text{when } \eta \in Y: && g_\eta(\alpha) < H_\xi(\alpha) \text{ for all } \alpha \in S, \text{ where } S \text{ is stationary.} \end{aligned}$$

Note that $g_\eta(\alpha)$ can only have two possible values, either $F(\alpha)$ or $2^{F(\alpha)}$, regardless of $\eta \in \kappa^+$. We can therefore assume without loss of generality that the same holds for $H_\xi(\alpha)$. Let z be the set on which H_ξ is small:

$$z = \{\alpha \in \kappa \mid H_\xi(\alpha) = F(\alpha)\}.$$

If $\eta \in X$, then $H_\xi(\alpha) \leq g_\eta(\alpha)$ is true for all $\alpha \notin S_\eta$, but since $H_\xi(\alpha) \leq g_\eta(\alpha)$ has to hold for almost all $\alpha \in \kappa$, we also need $|\{\alpha \in S_\eta \mid \alpha \notin z\}| < \kappa$. Let us fix the notation that $a \subseteq^* b$ iff $a \setminus c \subseteq b$ for some c with $|c| < \kappa$, then our condition above states that $S_\eta \subseteq^* z$ should hold.

On the other hand, if $\eta \in Y$, then $g_\eta(\alpha) < H_\xi(\alpha)$ is possible if

$$g_\eta(\alpha) = F(\alpha) < 2^{F(\alpha)} = H_\xi(\alpha).$$

Thus $(\kappa \setminus z) \cap S_\eta$ needs to be stationary. The assumption that $|z \cap S_\eta| < \kappa$ is sufficient for this.

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Given our sad family $\mathcal{S} = \langle S_\eta \mid \eta \in \kappa^+ \rangle$, the existence of a set z such that $S_\eta \subseteq^* z$ for all $\eta \in X$ and $|z \cap S_\eta| < \kappa$ for all $\eta \in Y$, is not immediately clear, but we can add such z generically through forcing.

We define the forcing $\mathbb{W}_\kappa^{X,Y}$ (where $\mathbb{W}$ stands for *wedge*). If $s \in [\kappa]^{<\kappa}$, let σ_s be the least ordinal such that $s \subseteq \sigma_s$.

Definition 3.1. *Given a sequence $\mathcal{S} = \{S_\eta \mid \eta \in \kappa^+\}$ of almost disjoint subsets of κ and a partition $\{X, Y\}$ of κ^+, we define $\mathbb{W}_\kappa^{X,Y}$ to have tuples $p = (s_p, A_p, B_p)$ as conditions, where $s_p \in [\kappa]^{<\kappa}$ and $A_p \in [X]^{<\kappa}$ and $B_p \in [Y]^{<\kappa}$ are such that*

$$\bigcup_{\eta \in A_p} S_\eta \cap \bigcup_{\eta \in B_p} S_\eta \subseteq \sigma_{s_p}.$$

The ordering on $\mathbb{W}_\kappa^{X,Y}$ is given by $(s_q, A_q, B_q) \leq (s_p, A_p, B_p)$ if all of the following hold:

(i) $A_p \subseteq A_q$,
(ii) $B_p \subseteq B_q$,
(iii) $s_p = s_q \cap \sigma_{s_p}$,
(iv) $s_q \cap [\sigma_{s_p}, \sigma_{s_q}) \supseteq \bigcup_{\eta \in A_p} S_\eta \cap [\sigma_{s_p}, \sigma_{s_q})$,
(v) $s_q \cap \bigcup_{\eta \in B_p} S_\eta \subseteq \sigma_{s_p}$.

If $G \subseteq \mathbb{W}_\kappa^{X,Y}$ is a generic filter, then let $z_G = \bigcup_{p \in G} s_p$. It is not hard to see that z_G indeed has the desired properties.

Lemma 3.2. *If $\eta \in X$, then $S_\eta \subseteq^* z_G$. If $\eta \in Y$, then $|S_\eta \cap z_G| < \kappa$.*

Proof. Let $p \in \mathbb{W}_\kappa^{X,Y}$. It is clear from the way we have defined the forcing that for any $\eta \in A_p$ we have $p \Vdash$ " $\check{S}_\eta \subseteq^* \dot{z}_G$ " and for any $\eta \in B_p$ we have $p \Vdash$ " $\check{S}_\eta \cap \dot{z}_G \subseteq \check{\sigma}_p \in \kappa$ ". Therefore, we are done if we prove that:

(1) for every $\eta \in X$ there is $q \leq p$ such that $\eta \in A_q$, and
(2) for every $\eta \in Y$ there is $q \leq p$ such that $\eta \in B_q$.

Proving (1) and (2) happens in the same way, so we only prove (1) below.

Fix some $\eta \in X$. Since S_η is almost disjoint from S_ξ for all $\xi \in B_p$, we can define $\gamma_\xi \in \kappa$ such that $S_\xi \cap S_\eta \subseteq \gamma_\xi$ for each $\xi \in B_p$. Since $|B_p| < \kappa$ we see that $\bigcup_{\xi \in B_p} \gamma_\xi \in \kappa$. Pick some $\gamma \in \bigcup_{\xi \in A_p} S_\xi$ such that $\gamma \geq \sigma_{s_p} \cup \bigcup_{\xi \in B_p} \gamma_\xi$.

We define $q \leq p$ by

$$\begin{aligned} s_q &= s_p \cup \left(\bigcup_{\xi \in A_p} S_\xi \cap [\sigma_{s_p}, \gamma] \right), \\ A_q &= A_p \cup \{\eta\}, \\ B_q &= B_p. \end{aligned}$$

Note that $\gamma \in s_q$, thus $\sigma_{s_q} = \gamma + 1$. Furthermore, note that

$$\bigcup_{\xi \in A_p} S_\xi \cap \bigcup_{\xi \in B_p} S_\xi \subseteq \sigma_{s_p} \leq \gamma, \quad \text{and}$$
$$S_\eta \cap \bigcup_{\xi \in B_p} S_\xi \subseteq \bigcup_{\xi \in B_p} \gamma_\xi \leq \gamma.$$

Therefore, q is indeed a condition. ⊣

We need to show that our forcing has several nice properties to satisfy our needs. Firstly, it is essential that the sad family $\{S_\eta \mid \eta \in \kappa^+\}$ will remain a sad family, in

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particular, the forcing should not destroy any stationary sets. Secondly, our forcing needs to preserve cardinals. In particular, we may not collapse κ^+ to κ, since our goal of proving the consistency of κ^+ many distinct cardinal characteristics requires our sad family to have cardinality κ^+. Thirdly, our forcing should preserve $2^\kappa = \kappa^+$, which we need for the forcings of type $\mathbb{S}^h_\kappa$ afterwards.

All of these properties hold for $\mathbb{W}^{X,Y}_\kappa$ under the assumption that $|X| = \kappa$, since we can show that the forcing is $<\kappa$-closed and κ-centred in this case, and our forcing is small enough that it does not increase 2^κ.

LEMMA 3.3. *$\mathbb{W}^{X,Y}_\kappa$ is $<\kappa$-closed.*

PROOF. Let $\gamma \in \kappa$ be limit and let $\langle p_\eta \mid \eta < \gamma\rangle$ be a descending sequence of conditions. We will write $p_\eta = (s_\eta, A_\eta, B_\eta)$. Let $p = (s_p, A_p, B_p)$ be given by $s_p = \bigcup_{\eta\in\gamma} s_\eta$ and $A_p = \bigcup_{\eta\in\gamma} A_\eta$ and $B_p = \bigcup_{\eta\in\gamma} B_\eta$. That p is a condition and that $p \le p_\eta$ for each $\eta \in \gamma$ are easy to check. ⊣

COROLLARY 3.4. *$\mathbb{W}^{X,Y}_\kappa$ preserves stationary sets.*

PROOF. See, for example, Lemma 23.7 in [5]. ⊣

DEFINITION 3.5. *Let $\mathbb{P}$ be a forcing. We will call a set $P \subseteq \mathbb{P}$ centred if for every $Q \in [P]^{<\kappa}$ there exists $q \in \mathbb{P}$ such that $q \le p$ for all $p \in Q$. We say that $\mathbb{P}$ is κ-centred if $\mathbb{P} = \bigcup_{\alpha\in\kappa} P_\alpha$ where each P_α is centred.*

Note that our notion of a centred set is not the usual definition: more commonly P is called centred if every *finite* $Q \subseteq P$ has a common extension, but since we are working in the context of ${}^\kappa\kappa$ we have to replace "finite" by "$<\kappa$" to have a proper analogy.

Clearly if $A \subseteq \mathbb{P}$ is an antichain and $\mathbb{P} = \bigcup_{\alpha\in\kappa} P_\alpha$ is κ-centred, then $|A| \le \kappa$, since $|P_\alpha \cap A| \le 1$ for every $\alpha \in \kappa$. Therefore, if $\mathbb{P}$ is κ-centred, it is $<\kappa^+$-c.c. as well.

LEMMA 3.6. *If $|X| \le \kappa$, then $\mathbb{W}^{X,Y}_\kappa$ is κ-centred.*

PROOF. For any $s \in [\kappa]^{<\kappa}$ and $A \in [X]^{<\kappa}$ we define

$$W_{s,A} = \left\{p \in \mathbb{W}^{X,Y}_\kappa \mid s_p = s \wedge A_p = A\right\}.$$

Since $|X| \le \kappa$ implies that $|[\kappa]^{<\kappa} \times [X]^{<\kappa}| = \kappa$, we are done if we show that each $W_{s,A}$ is centred. Let $Q \in [W_{s,A}]^{<\kappa}$ and $B = \bigcup_{p\in Q} B_p$. We claim that $q = \langle s, A, B\rangle$ is a condition and that $q \le p$ for all $p \in Q$.

Suppose that $\alpha \in \bigcup_{\eta\in A} S_\eta \cap \bigcup_{\eta\in B} S_\eta$, then there is $p \in Q$ such that $\alpha \in \bigcup_{\eta\in A} S_\eta \cap \bigcup_{\eta\in B_p} S_\eta$, and since p is a condition it follows that $\alpha \in \sigma_{s_p} = \sigma_s$. Hence q is a condition. To check that $q \le p$ for each $p \in Q$, note that (i)–(iii) of Definition 3.1 are immediate, while (iv) and (v) hold vacuously by $s_p = s_q$. ⊣

COROLLARY 3.7. *If $|X| \le \kappa$, then $\mathbb{W}^{X,Y}_\kappa$ preserves all cardinalities.*

Finally, we have to look at adding multiple generics of forcings of the type $\mathbb{W}^{X,Y}_\kappa$. Our goal is to define functions H_ξ for each $\xi \in \kappa^+$. Fix some bijection $\sigma : \kappa^+ \to \kappa^+$, then we want to add a generic set z for the forcing $\mathbb{W}^{X_\xi,Y_\xi}_\kappa$ for each $\xi \in \kappa^+$, where

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$X_\xi = \sigma(\xi)$ and $Y_\xi = \kappa^+ \setminus X$. This means that we also need to guarantee that a $<\kappa$-support products of size κ^+ of forcings of the form $\mathbb{W}_\kappa^{X,Y}$ behaves nicely, in the sense that it preserves cardinals, stationary sets and $2^\kappa = \kappa^+$.

LEMMA 3.8. *Let $\langle \{X_\xi, Y_\xi\} \mid \xi \in \kappa^+\rangle$ be a sequence of partitions of κ^+ such that $|X_\xi| \le \kappa$ for each $\xi \in \kappa^+$ and let $\overline{\mathbb{W}} = \prod_{\xi\in\kappa^+} \mathbb{W}_\kappa^{X_\xi,Y_\xi}$ be the $<\kappa$-support product of the forcings $\mathbb{W}_\kappa^{X_\xi,Y_\xi}$. Then $\overline{\mathbb{W}}$ is $<\kappa$-closed, $<\kappa^+$-c.c. and if G is $\overline{\mathbb{W}}$-generic over $\mathbf{V}$ and $\mathbf{V} \vDash$ " $2^\kappa = \kappa^+$ ", then $\mathbf{V}[G] \vDash$ " $2^\kappa = \kappa^+$ ".*

PROOF. Note that each term $\mathbb{W}_\kappa^{X_\xi,Y_\xi}$ is $<\kappa$-closed and $<\kappa^+$-c.c. (the latter as a consequence of κ-centredness), thus a $<\kappa$-support product of length κ^+ is also $<\kappa$-closed, which is easily proved, and $<\kappa^+$-c.c., which is proved using a Δ-system argument. That $2^\kappa = \kappa^+$ will remain true, follows from an argument by counting names, using that $|\mathbb{W}_\kappa^{X_\xi,Y_\xi}| = \kappa^+$ for each $\xi \in \kappa^+$, and that the product has κ^+ many terms, thus $|\overline{\mathbb{W}}| = \kappa^+$. ⊣

COROLLARY 3.9. *$\overline{\mathbb{W}}$ preserves cardinals and stationary sets.*

Now we are finally ready to prove our last theorem, which is an extension of Theorem 2.9, and shows that there can be consistently κ^+ many distinct cardinal characteristics of the form $\mathfrak{d}_\kappa^h(\in^*)$.

THEOREM 3.10. *Assuming $2^\kappa = \kappa^+$ and $\diamondsuit_\kappa$, there exists a family of functions $\{g_\eta \mid \eta \in \kappa^+\} \subseteq {}^\kappa\kappa$ such that for any increasing sequence $\langle \lambda_\xi \mid \xi \in \kappa^+\rangle$ of cardinals with $\kappa < \mathrm{cf}(\lambda_\xi)$ and any function $\sigma : \kappa^+ \to \kappa^+$, there exists a forcing extension in which $\mathfrak{d}_\kappa^{g_\eta}(\in^*) = \lambda_{\sigma(\eta)}$ for all $\eta \in \kappa^+$.*

PROOF. We start with a model $\mathbf{V} \vDash$ " $2^\kappa = \kappa^+$ and $\diamondsuit_\kappa$ " containing a sad family $\mathcal{S} = \langle S_\eta \mid \eta \in \kappa^+\rangle$, and we will assume without loss of generality that $\sigma : \kappa^+ \to \kappa^+$ is a bijection. We define the functions g_η for each $\eta \in \kappa^+$ as

$$g_\eta(\alpha) = \begin{cases} F(\alpha), & \text{if } \alpha \in S_\eta, \\ 2^{F(\alpha)}, & \text{otherwise.} \end{cases}$$

For each $\eta \in \kappa^+$ we define the partition $\{X_\eta, Y_\eta\}$ of κ^+ by

$$X_\eta = \sigma^{-1}[\sigma(\eta)] = \{\zeta \in \kappa^+ \mid \sigma(\zeta) \in \sigma(\eta)\},$$
$$Y_\eta = \kappa^+ \setminus X_\eta.$$

We then force with a $<\kappa$-support product $\overline{\mathbb{W}} = \prod_{\eta\in\kappa^+} \mathbb{W}_\kappa^{X_\eta,Y_\eta}$. Note, in particular, that $|X_\eta| \le \kappa$. Let G be $\overline{\mathbb{W}}$-generic, then we will work in $\mathbf{V}[G]$. We define $z_{\sigma(\eta)} = \bigcup_{p\in G} s_{p(\eta)}$, then $z_{\sigma(\eta)}$ is $\mathbb{W}_\kappa^{X_\eta,Y_\eta}$-generic over $\mathbf{V}$.

By Lemma 3.8, we know that $\mathbf{V}[G] \vDash$ " $2^\kappa = \kappa^+ + \check{\mathcal{S}}$ is a sad family ". Therefore by Lemma 3.2, if $\eta \in \kappa^+$, then we have $S_\zeta \subseteq^* z_{\sigma(\eta)}$ for all $\zeta \in X_\eta$ and $|S_\zeta \cap z_{\sigma(\eta)}| < \kappa$ for all $\zeta \in Y_\eta$. Equivalently, using the definition of X_η and Y_η, if $\xi \in \kappa^+$, then we have $S_\zeta \subseteq^* z_\xi$ for all $\zeta \in \kappa^+$ such that $\sigma(\zeta) \in \xi$ and $|S_\zeta \cap z_\xi| < \kappa$ for all $\zeta \in \kappa^+$ such that $\sigma(\zeta) \in [\xi, \kappa^+)$.

For each $\xi \in \kappa^+$ we define $H_\xi \in {}^\kappa\kappa$ as follows:

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$$H_\xi(\alpha) = \begin{cases} F(\alpha), & \text{if } \alpha \in z_\xi, \\ 2^{F(\alpha)}, & \text{otherwise.} \end{cases}$$

The remainder of the proof mirrors the proof of Theorem 2.9 almost exactly.

For each $\xi \in \kappa^+$ let A_ξ be a set of ordinals with $|A_\xi| = \lambda_\xi$, such that $\langle A_\xi \mid \xi \in \kappa^+\rangle$ is a sequence of mutually disjoint sets, and let $A = \bigcup_{\xi\in\kappa^+} A_\xi$. For each $\xi \in \kappa^+$ and $\beta \in A_\xi$, we define $h_\beta = H_\xi$.

We now consider the product forcing $\overline{\mathbb{S}} = \prod_{\beta\in A} \mathbb{S}_\kappa^{h_\beta}$ with $\leq\kappa$-support. Let K be $\overline{\mathbb{S}}$-generic. We will fix some $\eta \in \kappa^+$, and let $B = \bigcup_{\xi\in\sigma(\eta)+1} A_\xi$ and $B^c = A \setminus B$. By Lemma 2.8 we see that $(\mathrm{Loc}_{g_\eta})^{\mathbf{V}[G][K\restriction B]}$ has cardinality

$$\kappa^+ \cdot |B| = \kappa^+ \cdot \left|\sup_{\xi\leq\sigma(\eta)} A_\xi\right| = \kappa^+ \cdot \left|A_{\sigma(\eta)}\right| = \lambda_{\sigma(\eta)}.$$

To use Lemma 2.5, we need that $h_\beta \leq^* g_\eta$ for all $\beta \in B^c$, equivalently, that $H_\xi \leq^* g_\eta$ for all $\xi \in (\sigma(\eta), \kappa^+)$. But this is true for any $\xi \in (\sigma(\eta), \kappa^+)$, since $g_\eta = F(\alpha)$ iff $\alpha \in S_\eta$, while $H_\xi(\alpha) = F(\alpha)$ iff $\alpha \in z_\xi$, and because $\sigma(\eta) \in \xi$ we have $S_\eta \subseteq^* z_\xi$. Therefore Lemma 2.5 shows that $(\mathrm{Loc}_{g_\eta})^{\mathbf{V}[G][K\restriction B]}$ is a family in $\mathbf{V}[G][K]$ of size $\lambda_{\sigma(\eta)}$ that forms a witness for

$$\mathbf{V}[G][K] \vDash \text{“}\,\mathfrak{d}_\kappa^{g_\eta}(\in^*) \leq \lambda_{\sigma(\eta)}\,\text{”}.$$

On the other hand, if $\beta \in A_{\sigma(\eta)}$, then $h_\beta = H_{\sigma(\eta)}$ and thus $\sigma(\eta) \in [\sigma(\eta), \kappa^+)$ implies that $|S_\eta \cap z_{\sigma(\eta)}| < \kappa$. In particular, $S_\eta \setminus z_{\sigma(\eta)}$ is stationary and if $\alpha \in S_\eta \setminus z_{\sigma(\eta)}$, then $g_\eta(\alpha) < H_{\sigma(\eta)}(\alpha)$. Hence by Lemma 2.7 we see that

$$\mathbf{V}[G][K] \vDash \text{“}\,\lambda_{\sigma(\eta)} = |A_{\sigma(\eta)}| \leq \mathfrak{d}_\kappa^{g_\eta}(\in^*)\,\text{”}.$$

In conclusion, we get for every $\eta \in \kappa$ that

$$\mathbf{V}[G][K] \vDash \text{“}\,\lambda_{\sigma(\eta)} = \mathfrak{d}_\kappa^{g_\eta}(\in^*)\,\text{”}. \qquad \dashv$$

§4. Concluding remarks. With Theorem 3.10 we improved the known consistency of $\mathfrak{d}_\kappa^{\mathsf{pow}}(\in^*) < \mathfrak{d}_\kappa^{\mathsf{id}}(\in^*)$ to a family of κ^+ many cardinal invariants that are mutually independent in the sense that any ordering of the cardinals with order-type κ^+ is consistent. This answers Questions 72 and 73 from [3] positively. Moreover, we have shown that there exist functions $h, h' \in {}^\kappa\kappa$ for which it is consistent that $\mathfrak{d}_\kappa^{h}(\in^*) < \mathfrak{d}_\kappa^{h'}(\in^*)$, but also that it is consistent that $\mathfrak{d}_\kappa^{h'}(\in^*) < \mathfrak{d}_\kappa^{h}(\in^*)$.

It is natural to ask if we can do better than this.

QUESTION 4.1. *Is it consistent that there exists a family of functions* $\{h_\xi \mid \xi \in \kappa^{++}\}$ *such that each* $\mathfrak{d}_\kappa^{h_\xi}(\in^*)$ *has a distinct value? Is it consistent that there is a model with* 2^κ *many distinct values for cardinals of the form* $\mathfrak{d}_\kappa^{h}(\in^*)$*?*

Our method of separating cardinals uses a forcing $\mathbb{S}_\kappa^h$ that requires $2^\kappa = \kappa^+$ in the ground model to have the $<\kappa^{++}$-c.c., hence if we start with a family of functions of size κ^{++}, our forcing may collapse κ^{++}. This makes it hard to answer the above question using the method presented in this paper.

Another limitation of our method, is that we restrict our attention to forcings that have the F-Sacks properties where $F(\alpha) = F(\alpha)^{|\alpha|}$. Essentially, we know how

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to separate cardinals with a parameter h from cardinals with a parameter 2^h, and thus we make a jump on the order of a power set operation. It is unclear whether a finer structure can be discovered between these cardinals, motivating the following question:

QUESTION 4.2. *Is it consistent that there exist* $h_0, h_1, h_2 \in {}^\kappa\kappa$ *such that* $|h_0(\alpha)| < |h_1(\alpha)| < 2^{|h_0(\alpha)|} = h_2(\alpha)$ *and* $\mathfrak{d}_\kappa^{h_2}(\in^*) < \mathfrak{d}_\kappa^{h_1}(\in^*) < \mathfrak{d}_\kappa^{h_0}(\in^*)$?

The localisation cardinals $\mathfrak{d}_\kappa^h(\in^*)$ have a natural dual form $\mathfrak{b}_\kappa^h(\in^*)$ defined in the introduction, where duality is taken with respect to relational systems, as described in [2]. In general, for many cardinal characteristics $\mathfrak{x}, \mathfrak{y}$ with duals $\mathfrak{x}', \mathfrak{y}'$ it is the case that if $\mathfrak{x} < \mathfrak{y}$ is consistent, then $\mathfrak{y}' < \mathfrak{x}'$ is consistent as well. This motivates the following question, which has also been asked as Question 71 from [3]:

QUESTION 4.3. *Do there exist functions* h, h' *such that* $\mathfrak{b}_\kappa^h(\in^*) < \mathfrak{b}_\kappa^{h'}(\in^*)$ *is consistent*?

Indeed, despite being able to separate κ^+ many cardinals of the form $\mathfrak{d}_\kappa^h(\in^*)$, it is currently unknown how to separate even two cardinals of the form $\mathfrak{b}_\kappa^h(\in^*)$. The main obstacle to this is the lack of preservation theorems for forcings related to the generalised Baire space. Without such theorems, the preservation of, for example, $\mathfrak{b}_\kappa^{\mathsf{id}}(\in^*) = \kappa^+$ under a forcing that increases $\mathfrak{b}_\kappa^{\mathsf{pow}}(\in^*)$ needs an analogue of Lemma 2.5. However, it is unclear what this forcing should be.

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