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FREE SUBGROUPS OF UNITS IN GROUP RINGS

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ABSTRACT. In this paper we give necessary and sufficient conditions under which the group of units of a group ring of a finite group G over a field K does not contain a free subgroup of rank 2.
Some extensions of this results to infinite nilpotent and FC groups are also considered.

1. **Introduction.** Let RG be the group ring of a group G over a commutative unital ring R; we denote by $U(RG)$ the group of units of this ring. When $R = Z$, the ring of rational integers, and G is finite, Hartley and Pickel [2] gave necessary and sufficient conditions for $U(ZG)$ to contain no free subgroup of rank two.

In this note we give a similar result when R is a field K of characteristic $p \geq 0$ and G is a finite group.

We shall also consider certain classes of infinite groups. When G is infinite and $p = 0$ we consider nilpotent and FC groups and when $p > 0$ we shall give some results for nilpotent groups.

It is interesting to observe that as a consequence, it will follow that if G is finite, and either $p = 0$, or $p > 0$ and K is not algebraic over the prime field $GF(p)$ then $U(KG)$ has no free subgroup of rank two if and only if $U(KG)$ is solvable.

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2. **The main results**

LEMMA 2.0. *Let D be a division ring finite dimensional over its center Z. Then $D^* = D - \{0\}$ contains a free subgroup of rank two.*

Proof. We will consider two cases:

(i) Char $D = 0$.

Suppose not. Since D is a linear group over Z, by [8], Theorem 1, there

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exists a normal solvable subgroup H of D^* such that the index $[D^*:H]$ is finite. Therefore by [5], Theorem 2, H is contained in Z^*, the center of D^*. By [3], Corollary of Theorem VII 12.3 D is commutative, a contradiction.

(ii) Char $D = p > 0$.

Suppose not. Then by [8], Theorem 2, there exists a normal solvable subgroup H of D^* such that D^*/H is locally finite. Arguing as above, we get a contradiction.

THEOREM 2.1. *Let G be a finite group and K a field of characteristic zero. Then $U(KG)$ does not contain a free group of rank two if and only if G is abelian.*

Proof. Since K has characteristic zero we can assume that K contains Q, the field of rational numbers.

Now, $QG \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i)$, where $M_{n_i}(D_i)$ denotes a full $n_i \times n_i$ matrix ring over a division ring D_i.

If, for some i, $n_i > 1$, set $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Then, by [1], A and B freely generate a free group of rank two, which can be obviously embedded in $M_{n_i}(D_i)$. Therefore we must have $QG \cong \bigoplus_{i=1}^{r} D_i$.

Since each D_i is finite dimensional over its center, by Lemma 2.0 each D_i is a field.

COROLLARY 2.2. *Let G be a nilpotent or FC group. If $U(KG)$ contains no free subgroup of rank two then:*

(i) *T, the torsion subgroup of G, is an abelian subgroup.*

(ii) *every subgroup of T is normal in G.*

As a partial converse we have that if (i) and (ii) hold, then $U(QG)$ has no free subgroup of rank two.

Proof. Since T is locally finite, Theorem 2.1 implies that T is abelian. Now, let $x \in G \setminus T$, $a \in T$ and consider $H = \langle a, x \rangle$. Since $U(KH)$ has no free subgroup of rank two, by (7), Lemma 3.12, every idempotent of $K\langle a \rangle$ is central in KH. Therefore x normalizes $\langle a \rangle$.

Conversely, if (i) and (ii) hold, then by [7], Proposition 1.16, every idempotent of QT is central in QG and as in [7], Theorem VI 4.12 it follows that $U(QG)$ is solvable. hence $U(QG)$ contains no free subgroup of rank two.

THEOREM 2.3. *Let G be a finite group and K a field of characteristic $p > 0$. Then $U(KG)$ does not contain a free subgroup of rank two if and only if one of the following conditions occurs:*

(i) *G is abelian*

(ii) *K is algebraic over its prime field $GF(p)$*

(iii) *$S_p(G)$, the p-Sylow subgroup of G, is normal in G and $G/S_p(G)$ is abelian.*

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Proof. First we observe that if K is algebraic over $GF(p)$, then for any unit $\alpha \in U(KG)$, there exists a finite extension E of $GF(p)$ such that $\alpha \in U(EG)$ is of finite order, hence $U(KG)$ is a torsion group.

Now suppose that K is not algebraic over $GF(p)$. Then there exists an element $\lambda \in K$ which is transcendental over $GF(p)$. We can construct a locally compact field E with a valuation $| \ |$ such that $|\lambda| \neq 1$.

Let $O_p(G)$ be the maximum normal p-subgroup of G and $J(EG)$ be the Jacobson radical of EG. Denote by $\psi: EG \to EG/J(EG)$ the natural projection; since $J(EG)$ is a nilpotent ideal we know that ψ induces an epimorphism

$$\psi: U(EG) \to U(EG/J(EG)) \cong \bigoplus_{i=1}^{r} U(M_{n_i}(D_i)).$$

But, if $n_i > 1$ for some index i, we can produce a free subgroup of rank two in $M_{n_i}(E)$. Set

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \quad \text{and} \quad B = P\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}P^{-1}, \quad \text{where} \quad P = \begin{bmatrix} 1+\lambda & \lambda \\ -\lambda & 1-\lambda \end{bmatrix}$$

Then by [8], Proposition 3.12, there exists an integer $m > 0$ such that A^m and B^m freely generate a free group, which can be obviously embedded in $M_{n_i}(E)$. Therefore

$$U(EG/J(EG)) \cong \bigoplus_{i=1}^{r} D_i^*.$$

We claim that $G/O_p(G)$ has no p-elements.

In fact, if $x \in S_p(G)/O_p(G)$ then there exists $n \in N^*$ such that $x^{p^n} \in O_p(G)$. Since $O_p(G) \subseteq 1 + J(EG)$ we have that

$$\psi(x^{p^n}) = (1, 1, \ldots, 1) \quad \text{and}$$

$(\psi(x) - (1, 1, \ldots, 1))^{p^n} = (0, 0, \ldots, 0)$. Hence $\psi(x) = (1, 1, \ldots, 1)$ and $x \in O_p(G)$.

Now, let $\phi: EG - E(G/S_p(G))$ be the natural epimorphism. Using the same arguments as before we conclude that:

$$\phi: EG \to E(G/S_p(G)) = \bigoplus_{i=1}^{r'} D_i'.$$

Since each D_i' is finite dimensional over its center, by Lemma 2.0 each D_i' is a field.

Therefore $G/S_p(G)$ is abelian.

The converse is trivial.

PROPOSITION 2.4. *Let G be a nontorsion nilpotent group, T be the torsion subgroup of G and $S_p(T)$ the p-Sylow subgroup of T. If $U(KG)$ has no free subgroup of rank two then:*

(i) *$T/S_p(T)$ is abelian, with every subgroup normal in $G/S_p(T)$.*

Also, if (i) holds and, moreover, $T/S_p(T)$ is in the center of $G/S_p(T)$, then $U(KG)$ has no free subgroup of rank two.

Proof. We remark that we can assume that G is finitely generated, since only local properties are involved. But, in this case, T is of finite order, $S_p(T)$ is normal in G and we can assume, considering $G/S_p(T)$, that G has no p-elements.

Now, by [4], Theorem 2.24, G contains a central element x of infinite order. Then we consider:

$$GF(p)\langle x, T\rangle = GF(p)(\langle x\rangle \times T) \cong GF(p)\langle x\rangle \underset{GF(p)}{\otimes} GF(p)T$$

$$\cong GF(p)\langle x\rangle \underset{GF(p)}{\otimes} \left[\bigoplus_{i=1}^{r} M_{n_i}(E_i)\right] = \bigoplus_{i=1}^{r} M_{n_i}\left(GF(p)\langle x\rangle \underset{GF(p)}{\otimes} E_i\right).$$

If, for some i, $n_i > 1$ then, in $U(M_{n_i}(GF(p)\langle x\rangle \otimes_{GF(p)} E_i))$ we have the matrices A and B, as in the proof of Theorem 2.3, a contradiction. Therefore, for every i, $n_i = 1$ and T is abelian.

Let now h be an element of T and assume that $y \in G$ does not normalize $\langle h\rangle$. Then, as in [2], Lemma 4, there is a monomorphism $\phi: U(M_2(GF(p) \times x\rangle)) \to U(KG)$, a contradiction.

Conversely, if (i) holds and T is central in G, then by [6], Proposition 4.5, $U(KG)$ is solvable. Therefore $U(KG)$ has no free subgroup of rank two.

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