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HOMOGENEOUS RIEMANNIAN STRUCTURES ON BERGER 3-SPHERES

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Abstract The homogeneous Riemannian structures on the three-dimensional Berger spheres, their corresponding reductive decompositions and the associated groups of isometries are obtained. The Berger 3-spheres are also considered as homogeneous almost contact metric manifolds.

Keywords: Berger spheres; homogeneous Riemannian structures; almost contact metric structures

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1. Introduction

The Berger spheres $\mathbb{S}^3_\varepsilon$ are homogeneous Riemannian spaces diffeomorphic to the three-dimensional sphere. These spaces, found by Berger [**3**] in his classification of all simply connected normal homogeneous Riemannian manifolds of positive sectional curvature, have non-constant curvature, and their metrics are obtained from the round metric on $\mathbb{S}^3$ by deforming it along the fibres of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ by ε. These spaces are of great interest in Riemannian geometry and provide nice examples; for instance, they served as counterexamples to a conjecture of Klingenberg about closed geodesics (see, for example, [**17**, p. 160]) and to conjectures on the first eigenvalue of the Laplacian on spheres [**2**, **19**].

On the other hand, Cartan's classical characterization of Riemannian symmetric spaces as the spaces of parallel curvature (under certain conditions) is well known. This characterization was extended by Ambrose and Singer [**1**] to the homogeneous Riemannian case; they proved that a connected, simply connected and complete Riemannian manifold (M, g) is homogeneous if and only if there exists a $(1, 2)$-tensor field S on M satisfying certain properties; this tensor field is called a homogeneous Riemannian structure by Tricerri and Vanhecke in [**18**]. Moreover, a homogeneous almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called homogeneous if there exists a connected Lie group G acting transitively on (M, g) as a group of isometries which leave the almost contact structure (φ, ξ, η) invariant; a homogeneous Riemannian structure satisfying an additional condition characterizes such a property.

The purpose of the present paper is to study the homogeneous Riemannian structures on the Berger 3-spheres. After some preliminaries (§ 2), we obtain all the homogeneous Riemannian structures on the Berger spheres (§ 3). These structures define Lie algebras with reductive decompositions, which have associated Lie groups with isometric actions on the spheres (§ 4). Finally, in § 5, we consider natural almost contact metric structures on the Berger spheres $\mathbb{S}^3_\varepsilon$, which are α-Sasakian [**11**], $\alpha = \sqrt{\varepsilon}$, and we obtain that these spheres are homogeneous almost contact metric manifolds.

2. Preliminaries

2.1. Homogeneous Riemannian structures

Let (M, g) be a connected Riemannian manifold. Let ∇ be the Levi-Civita connection of g and let R be its curvature tensor field, for which we adopt the conventions

$$R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z, \qquad R_{XYZW} = g(R_{XY}Z, W),$$

for all vector fields X, Y, Z, W on M. A homogeneous Riemannian structure on (M, g) is [**18**] a tensor field S of type $(1, 2)$ on M such that the connection $\tilde\nabla = \nabla - S$ satisfies

$$\tilde\nabla g = 0, \qquad \tilde\nabla R = 0, \qquad \tilde\nabla S = 0. \tag{2.1}$$

We also denote by S the associated tensor field of type $(0, 3)$ on M defined by $S_{XYZ} = g(S_X Y, Z)$.

Suppose that (M, g) is a homogeneous Riemannian manifold, that is $M = G/H$, where G is a connected Lie group acting transitively and effectively on M as a group of isometries and H is the isotropy group at a point $o \in M$. Then the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. If H is connected, the invariance condition $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The vector space $\mathfrak{m}$ is identified with $T_o(M)$ through the isomorphism

$$\left.\begin{aligned} \mu : \mathfrak{m} &\to T_o(M), \\ X &\mapsto X^*_o, \end{aligned}\right\} \tag{2.2}$$

where X^* is the Killing vector field generated on M by the one-parameter subgroup $\{\exp sX\}$ of G acting on M. Then, the canonical connection $\tilde\nabla$ of $M = G/H$ (with regard to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$) is determined by

$$(\tilde\nabla_{X^*} Y^*)_o = -([X, Y]_{\mathfrak{m}})^*_o, \quad X, Y \in \mathfrak{m}, \tag{2.3}$$

and $S = \nabla - \tilde\nabla$ is the homogeneous Riemannian structure associated with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Conversely, consider a homogeneous Riemannian structure S on a connected, simply connected and complete Riemannian manifold (M, g), fix a point $o \in M$ and put

$\tilde{\mathfrak{m}} = T_o(M)$. If $\tilde{R}$ is the curvature tensor of the connection $\tilde{\nabla} = \nabla - S$, the holonomy algebra $\tilde{\mathfrak{h}}$ of $\tilde{\nabla}$ is the Lie subalgebra (of the Lie algebra of antisymmetric endomorphisms of $(\tilde{\mathfrak{m}}, g_o)$) generated by the operators $\tilde{R}_{XY}$, where $X, Y \in \tilde{\mathfrak{m}}$. Then, a Lie bracket is defined on the vector space direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{m}}$ by

$$\left.\begin{aligned} &[U,V] = UV - VU, && U, V \in \tilde{\mathfrak{h}}, \\ &[U,X] = U(X), && U \in \tilde{\mathfrak{h}},\ X \in \tilde{\mathfrak{m}}, \\ &[X,Y] = \tilde{R}_{XY} + S_X Y - S_Y X, && X, Y \in \tilde{\mathfrak{m}}, \end{aligned}\right\} \tag{2.4}$$

and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is the reductive pair associated with the homogeneous Riemannian structure S. Let $\tilde{G}$ be the connected simply connected Lie group whose Lie algebra is $\tilde{\mathfrak{g}}$ and let $\tilde{H}$ be the connected Lie subgroup of $\tilde{G}$ whose Lie algebra is $\tilde{\mathfrak{h}}$. Then $\tilde{G}$ acts transitively by isometries on M and M is diffeomorphic to $\tilde{G}/\tilde{H}$. If $\varGamma$ is the set of the elements of $\tilde{G}$ which act trivially on M, then $\varGamma$ is a discrete normal subgroup of $\tilde{G}$, and the Lie group $G = \tilde{G}/\varGamma$ acts transitively and effectively on M as a group of isometries, with isotropy group $H = \tilde{H}/\varGamma$. Then M is diffeomorphic to the homogeneous Riemannian manifold G/H.

2.2. The Berger spheres

As usual, we identify the sphere $\mathbb{S}^3$ and the Lie group $SU(2)$ by the map that sends $(z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2$ to $\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right) \in SU(2)$. We consider the basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ given by

$$X_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{2.5}$$

Then

$$[X_1, X_2] = 2X_3, \qquad [X_2, X_3] = 2X_1, \qquad [X_3, X_1] = 2X_2. \tag{2.6}$$

The one-parameter family $\{g_\varepsilon : \varepsilon > 0\}$ of left-invariant Riemannian metrics on $\mathbb{S}^3 = SU(2)$ given at the identity element $I \in SU(2)$, with respect to the basis of left-invariant vector fields X_1, X_2, X_3, by

$$g_\varepsilon = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

are called the Berger metrics on $\mathbb{S}^3$; if $\varepsilon = 1$ we have the canonical (bi-invariant) metric. The Berger spheres are the simply connected complete Riemannian manifolds $\mathbb{S}^3_\varepsilon = (\mathbb{S}^3, g_\varepsilon)$, $\varepsilon > 0$.

The Levi-Civita connection of g_ε is given by

$$2g_\varepsilon(\nabla_X Y, Z) = g_\varepsilon([X,Y], Z) - g_\varepsilon([Y,Z], X) + g_\varepsilon([Z,X], Y),$$

for all $X, Y, Z \in \mathfrak{su}(2)$. So, the covariant derivatives between generators are given by $\nabla_{X_i} X_i = 0$ and

$$\left.\begin{array}{lll} & \nabla_{X_1} X_2 = (2-\varepsilon) X_3, & \nabla_{X_1} X_3 = (\varepsilon - 2) X_2, \\ \nabla_{X_2} X_1 = -\varepsilon X_3, & & \nabla_{X_2} X_3 = X_1, \\ \nabla_{X_3} X_1 = \varepsilon X_2, & \nabla_{X_3} X_2 = -X_1. & \end{array}\right\} \tag{2.7}$$

The components of the curvature tensor field are given by

$$\begin{array}{lll} R_{X_1 X_2} X_1 = \varepsilon^2 X_2, & R_{X_1 X_2} X_2 = -\varepsilon X_1, & R_{X_1 X_2} X_3 = 0, \\ R_{X_1 X_3} X_1 = \varepsilon^2 X_3, & R_{X_1 X_3} X_2 = 0, & R_{X_1 X_3} X_3 = -\varepsilon X_1, \\ R_{X_2 X_3} X_1 = 0, & R_{X_2 X_3} X_2 = (4 - 3\varepsilon) X_3, & R_{X_2 X_3} X_3 = (3\varepsilon - 4) X_2. \end{array}$$

3. Homogeneous Riemannian structures on $\mathbb{S}^3_\varepsilon$

If S is a homogeneous Riemannian structure on $\mathbb{S}^3_\varepsilon$ and $\tilde{\nabla} = \nabla - S$, the condition $\tilde{\nabla} g = 0$ in (2.1) is equivalent to $S_{XYZ} + S_{XZY} = 0$ for all $X, Y, Z \in \mathfrak{su}(2)$. Then, if $\{\alpha^1, \alpha^2, \alpha^3\}$ is the basis of left-invariant forms dual to X_1, X_2, X_3, the tensor field S of type $(0, 3)$ can be written as

$$S = \rho \otimes (\alpha^1 \wedge \alpha^2) + \sigma \otimes (\alpha^1 \wedge \alpha^3) + \tau \otimes (\alpha^2 \wedge \alpha^3), \tag{3.1}$$

where

$$\rho(Z) = S_{Z X_1 X_2}, \qquad \sigma(Z) = S_{Z X_1 X_3}, \qquad \tau(Z) = S_{Z X_2 X_3}, \tag{3.2}$$

for each vector field Z on $\mathbb{S}^3$.

Moreover, the condition $\tilde{\nabla} R = 0$ is equivalent to

$$(\nabla_Z R)_{XYVW} = -R_{S_Z XYVW} - R_{X S_Z YVW} - R_{XY S_Z VW} - R_{XYV S_Z W}, \tag{3.3}$$

for all $Z, X, Y, V, W \in \mathfrak{su}(2)$. Replacing (X, Y, V, W) in (3.3) by (X_1, X_3, X_2, X_3) and (X_1, X_2, X_2, X_3), respectively, we obtain that

$$(\varepsilon - 1)\rho = \varepsilon(\varepsilon - 1)\alpha^3, \qquad (\varepsilon - 1)\sigma = -\varepsilon(\varepsilon - 1)\alpha^2. \tag{3.4}$$

It is easy to see that the condition $\tilde{\nabla} R = 0$ holds if and only if Equations (3.4) are satisfied. In the case of the canonical metric ($\varepsilon = 1$) this condition holds automatically.

In order to determine the conditions for the 1-forms ρ, σ, τ such that $\tilde{\nabla} S = 0$, we first compute the connections forms $\tilde{\omega}_{ij}$ of $\tilde{\nabla}$, defined by $\tilde{\nabla}_Z X_j = \sum_{i=1}^3 \tilde{\omega}_{ij}(Z) X_i$. They are given by $\tilde{\omega}_{ii} = 0$ and

$$\left.\begin{array}{lll} & \tilde{\omega}_{21} = \varepsilon\alpha^3 - \rho, & \tilde{\omega}_{31} = -\varepsilon\alpha^2 - \sigma, \\ \tilde{\omega}_{12} = -\alpha^3 + \dfrac{1}{\varepsilon}\rho, & & \tilde{\omega}_{32} = (2-\varepsilon)\alpha^1 - \tau, \\ \tilde{\omega}_{13} = \alpha^2 + \dfrac{1}{\varepsilon}\sigma, & \tilde{\omega}_{23} = (\varepsilon - 2)\alpha^1 + \tau. & \end{array}\right\} \tag{3.5}$$

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Since $S_{ZXX} = 0$, by using these last equations, from (3.2) we obtain that

$$(\tilde{\nabla}_Z S)_{VX_1X_2} = (\tilde{\nabla}_Z \rho)(V) - (\varepsilon\alpha^2 + \sigma)(Z)\tau(V) + ((\varepsilon - 2)\alpha^1 + \tau)(Z)\sigma(V), \tag{3.6}$$

$$(\tilde{\nabla}_Z S)_{VX_1X_3} = (\tilde{\nabla}_Z \sigma)(V) - (\varepsilon\alpha^3 - \rho)(Z)\tau(V) + ((2 - \varepsilon)\alpha^1 - \tau)(Z)\rho(V), \tag{3.7}$$

$$(\tilde{\nabla}_Z S)_{VX_2X_3} = (\tilde{\nabla}_Z \tau)(V) + \left(\alpha^3 - \frac{1}{\varepsilon}\rho\right)(Z)\sigma(V) + \left(\alpha^2 + \frac{1}{\varepsilon}\sigma\right)(Z)\rho(V). \tag{3.8}$$

In particular, if $\varepsilon = 1$ we have the following theorem.

Theorem 3.1. *The homogeneous Riemannian structures on the sphere $\mathbb{S}^3$ with the canonical metric are given by (3.1), where ρ, σ and τ are differential 1-forms on $\mathbb{S}^3$ satisfying*

$$\tilde{\nabla}\rho = (\alpha^1 - \tau) \otimes \sigma + (\alpha^2 + \sigma) \otimes \tau,$$
$$\tilde{\nabla}\sigma = (\alpha^3 - \rho) \otimes \tau - (\alpha^1 - \tau) \otimes \rho,$$
$$\tilde{\nabla}\tau = -(\alpha^2 + \sigma) \otimes \rho - (\alpha^3 - \rho) \otimes \sigma.$$

Suppose now that $\varepsilon \neq 1$. By (3.4), the condition $\tilde{\nabla}R = 0$ is equivalent to $\rho = \varepsilon\alpha^3$ and $\sigma = -\varepsilon\alpha^2$. Then, Equations (3.5) reduce to

$$\tilde{\omega}_{23} = (\varepsilon - 2)\alpha^1 + \tau = -\tilde{\omega}_{32}, \qquad \tilde{\omega}_{ij} = 0 \text{ in all other cases.} \tag{3.9}$$

Since $(\tilde{\nabla}_Z \alpha^i)(X_j) = -\tilde{\omega}_{ij}(Z)$, by (3.9) we have

$$\tilde{\nabla}_Z \alpha^1 = 0, \qquad \tilde{\nabla}_Z \alpha^2 = ((2-\varepsilon)\alpha^1 - \tau)(Z)\alpha^3, \qquad \tilde{\nabla}_Z \alpha^3 = ((\varepsilon-2)\alpha^1 + \tau)(Z)\alpha^2, \tag{3.10}$$

and by (3.6), (3.7) and (3.10), we have $(\tilde{\nabla}_Z S)_{VX_1X_2} = (\tilde{\nabla}_Z S)_{VX_1X_3} = 0$, and hence, by (3.8), $\tilde{\nabla}S = 0$ if and only if $\tilde{\nabla}\tau = 0$. Now, if we put $\tau = f_1\alpha^1 + f_2\alpha^2 + f_3\alpha^3$, by using (3.10), the equation $\tilde{\nabla}\tau = 0$ is equivalent to the equations

$$Z(f_1) = 0, \qquad Z(f_2) + f_3((\varepsilon - 2)\alpha^1 + \tau)(Z) = 0, \qquad Z(f_3) + f_2((2 - \varepsilon)\alpha^1 - \tau)(Z) = 0,$$

for every vector field Z on $\mathbb{S}^3$. Then f_1 is a constant. Replacing Z by X_2 and X_3 in each one of the last two equations above and using the structure equations (2.6), we obtain that $f_2 = f_3 = 0$. So, τ is a scalar multiple of α^1, and we conclude with the following theorem.

Theorem 3.2. *For $\varepsilon \neq 1$, the homogeneous Riemannian structures on the Berger sphere $\mathbb{S}^3_\varepsilon$ are given by*

$$S_{\varepsilon,t} = \varepsilon\alpha^3 \otimes (\alpha^1 \wedge \alpha^2) - \varepsilon\alpha^2 \otimes (\alpha^1 \wedge \alpha^3) + t\alpha^1 \otimes (\alpha^2 \wedge \alpha^3), \quad t \in \mathbb{R}. \tag{3.11}$$

As a consequence, the components of the $(1,2)$-tensor field corresponding to $S = S_{\varepsilon,t}$ in (3.11) are given by

$$\left.\begin{array}{lll} S_{X_1}X_1 = 0, & S_{X_1}X_2 = tX_3, & S_{X_1}X_3 = -tX_2, \\ S_{X_2}X_1 = -\varepsilon X_3, & S_{X_2}X_2 = 0, & S_{X_2}X_3 = X_1, \\ S_{X_3}X_1 = \varepsilon X_2, & S_{X_3}X_2 = -X_1, & S_{X_3}X_3 = 0. \end{array}\right\} \tag{3.12}$$

Remark 3.3. In [**18**], Tricerri and Vanhecke gave a classification of homogeneous Riemannian structures into eight different classes, which are provided by the decomposition of the space of the tensors S of type $(0,3)$ satisfying $S_{XYZ}+S_{XZY}=0$ into three irreducible components $\mathcal{T}_i$, invariant under the action of the orthogonal group. Now, for each $p\in\mathbb{S}^3$, let $c_{12}(S)_p$ be the map defined on the tangent space $T_p(\mathbb{S}^3)$ by

$$c_{12}(S)_p(Z)=\sum_{i=1}^{3}S_{e_ie_iZ},$$

where $\{e_i\}$ is an orthonormal basis of $T_p(\mathbb{S}^3)$. If we choose $e_1=\sqrt{\varepsilon}^{-1}X_{1|p}$, $e_2=X_{2|p}$, $e_3=X_{3|p}$, we see that $c_{12}(S)_p$ vanishes for every $S=S_{\varepsilon,t}$. According to Tricerri–Vanhecke's classification, this implies that each $S=S_{\varepsilon,t}$ is of type $\mathcal{T}_2\oplus\mathcal{T}_3$. Moreover, if $t=\varepsilon$, we have $S_XY+S_YX=0$, then $S_{\varepsilon,\varepsilon}$ is of type $\mathcal{T}_3$, which means that $\mathbb{S}^3_\varepsilon$ is a naturally reductive Riemannian space. If $t=-2\varepsilon$, we have that each cyclic sum $\mathfrak{S}_{XYZ}S_{XYZ}$ vanishes, and hence $S_{\varepsilon,-2\varepsilon}$ is of type $\mathcal{T}_2$, which may be also expressed by saying that $\mathbb{S}^3_\varepsilon$ is a cotorsionless manifold (see [**9**]).

Remark 3.4. If $\varepsilon=1$, particular solutions (ρ,σ,τ) of equations in Theorem 3.1 are $(\alpha^3,-\alpha^2,t\alpha^1)$, $(\alpha^3,-t\alpha^2,\alpha^1)$, $(t\alpha^3,-\alpha^2,\alpha^1)$ and $(t\alpha^3,-t\alpha^2,t\alpha^1)$, $t\in\mathbb{R}$. They give rise to four one-parameter families of homogeneous Riemannian structures on the standard sphere $\mathbb{S}^3$. The first of them is given by (3.11). If $t=0$ in the fourth family, we get the solution $S=0$; this is equivalent to saying that $\nabla R=0$ and it expresses the well-known fact that the standard sphere is a Riemannian symmetric space.

4. Reductive decompositions and isometric actions on Berger spheres

We shall determine the reductive decompositions associated with the homogeneous Riemannian structures on the Berger spheres $\mathbb{S}^3_\varepsilon$ given by (3.11). We fix the point $o=(1,0)\in\mathbb{S}^3\subset\mathbb{C}^2$, which corresponds to the identity matrix $I\in SU(2)$, and set $\tilde{\mathfrak{m}}=T_o(\mathbb{S}^3)=T_I(SU(2))\equiv\mathfrak{su}(2)$.

If $S=S_{\varepsilon,t}$ is the homogeneous Riemannian structure defined by (3.11), then the connection $\tilde\nabla=\tilde\nabla_{\varepsilon,t}=\nabla-S$ is given, with respect to the basis $\{X_1,X_2,X_3\}$ of $\mathfrak{su}(2)$, by

$$\tilde\nabla_{X_1}X_2=(2-\varepsilon-t)X_3,\qquad \tilde\nabla_{X_1}X_3=(\varepsilon-2+t)X_2,\tag{4.1}$$

with the rest vanishing, and the components of the curvature tensor are

$$\tilde R_{X_1X_2}=\tilde R_{X_1X_3}=0\quad\text{and}\quad \tilde R_{X_2X_3}=2(2-\varepsilon-t)(\alpha^2\otimes X_3-\alpha^3\otimes X_2).$$

The holonomy algebra $\tilde{\mathfrak{h}}=\tilde{\mathfrak{h}}_{\varepsilon,t}$ of $\tilde\nabla$ is the Lie algebra of antisymmetric endomorphisms of $\tilde{\mathfrak{m}}\equiv\mathfrak{su}(2)$ generated by the curvature operators $\tilde R_{X_iX_j}$, and the reductive pair associated with the homogeneous Riemannian structure $S=S_{\varepsilon,t}$ is $(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}})$, where $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{\varepsilon,t}=\tilde{\mathfrak{h}}\oplus\tilde{\mathfrak{m}}$ is a Lie algebra with structure equations defined by (2.4).

If $t=2-\varepsilon$, the holonomy algebra $\tilde{\mathfrak{h}}$ of $\tilde\nabla$ is trivial and the reductive decomposition associated with the homogeneous Riemannian structure $S=S_{\varepsilon,2-\varepsilon}$ is $\tilde{\mathfrak{g}}\equiv\{0\}\oplus\mathfrak{su}(2)$

with structure equations (2.6). Then $\tilde{\mathfrak{g}}_{\varepsilon,2-\varepsilon} \equiv \mathfrak{su}(2)$ and the isometry group associated with $S = S_{\varepsilon,2-\varepsilon}$ is $SU(2)$, acting on itself by left translations.

Suppose now that $t \neq 2-\varepsilon$. We put $a_t = (2-\varepsilon-t)/2$, $b_t = (\varepsilon+t)/2 = 1-a_t$. Then $U = (1/(2a_t))\tilde{R}_{X_2X_3} = 2(\alpha^2 \otimes X_3 - \alpha^3 \otimes X_2)$ generates the holonomy algebra $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_{\varepsilon,t}$ of $\tilde{\nabla} = \tilde{\nabla}_{\varepsilon,t}$ and the reductive decomposition associated with the homogeneous Riemannian structure $S = S_{\varepsilon,t}$ is $\tilde{\mathfrak{g}}_{\varepsilon,t} \equiv \tilde{\mathfrak{h}}_{\varepsilon,t} \oplus \mathfrak{su}(2) = \langle\{U, X_1, X_2, X_3\}\rangle$, with structure equations, by (2.4), given by

$$\begin{aligned} &[X_1, X_2] = 2b_tX_3, \qquad [X_2, X_3] = 2a_tU + 2X_1, \qquad [X_3, X_1] = 2b_tX_2,\\ &[U, X_1] = 0, \qquad [U, X_2] = 2X_3, \qquad [U, X_3] = -2X_2. \end{aligned}$$

If we set $\hat{U} = b_tU - X_1$, $\hat{X}_1 = a_tU + X_1$, $\hat{X}_2 = X_2$, $\hat{X}_3 = X_3$, then

$$[\hat{X}_1, \hat{X}_2] = 2\hat{X}_3, \qquad [\hat{X}_2, \hat{X}_3] = 2\hat{X}_1, \qquad [\hat{X}_3, \hat{X}_1] = 2\hat{X}_2, \qquad [\hat{U}, \hat{X}_j] = 0 \quad (1 \leqslant j \leqslant 3),$$

so $\tilde{\mathfrak{g}}_{\varepsilon,t}$ is the direct product of the Lie algebra $\mathfrak{r} = \langle\{\hat{U}\}\rangle$ of $\mathbb{R}$ and $\mathfrak{su}(2) = \langle\{\hat{X}_1, \hat{X}_2, \hat{X}_3\}\rangle$. The corresponding connected simply connected Lie group is $\tilde{G} = \mathbb{R} \times SU(2)$.

Now, $\tilde{G} = \mathbb{R} \times SU(2)$ acts transitively and almost effectively on each Berger sphere by

$$((s, g), p) \mapsto gp \begin{pmatrix} \mathrm{e}^{-\mathrm{i}s} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}s} \end{pmatrix}. \tag{4.2}$$

The isotropy group at the point $o = I \in \mathbb{S}^3 = SU(2)$ is

$$\tilde{H} = \left\{ \left(s, \begin{pmatrix} \mathrm{e}^{\mathrm{i}s} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}s} \end{pmatrix} \right) : s \in \mathbb{R} \right\}.$$

Let $\psi : \tilde{G} \to U(2)$ be the covering homomorphism defined by $\psi(s, g) = \mathrm{e}^{-\mathrm{i}s}g$. It induces a twofold covering homomorphism $\mathbb{S}^1 \times SU(2) \to U(2)$. The normal subgroup $N = \{(2k\pi, I) : k \in \mathbb{Z}\}$ of $\tilde{G}$ acts trivially on $SU(2)$ and the action (4.2) induces a transitive and almost effective isometric action of $\tilde{G}/N \cong \mathbb{S}^1 \times SU(2)$ on $\mathbb{S}^3_\varepsilon$. The set Γ of all the elements of $\tilde{G}$ which act trivially on $\mathbb{S}^3$ is the discrete normal subgroup of $\tilde{G}$ given by

$$\Gamma = \{(k\pi, (-1)^kI) : k \in \mathbb{Z}\} = \ker \psi \subset \tilde{H}$$

and $G = \tilde{G}/\Gamma \cong U(2)$ acts transitively and effectively on $\mathbb{S}^3_\varepsilon$ as a group of isometries. Moreover, $H = \tilde{H}/\Gamma \cong \psi(H) = \{1\} \times U(1) \subset U(2)$ is the isotropy group at $o = (1, 0) \equiv I$ of the action of $U(2)$ induced by (4.2).

We will show that these actions define reductive decompositions whose canonical connection, given by (2.3), defines the homogeneous Riemannian structures in (3.11). The Lie algebras of the groups $\tilde{G}$, $\tilde{G}/N$ and $\tilde{G}/\Gamma$ are isomorphic to the direct product Lie algebra $\mathfrak{g} = \mathbb{R} \times \mathfrak{su}(2)$. We consider the basis $\{B_0, B_1, B_2, B_3\}$ of $\mathfrak{g}$ given by $B_0 = (D, 0)$, $B_1 = (0, X_1)$, $B_2 = (0, X_2)$, $B_3 = (0, X_3)$, where D is the canonical base vector of $\mathbb{R} = T_0(\mathbb{R})$ and $X_1, X_2, X_3 \in \mathfrak{su}(2)$ are again given by (2.5). Each one-parameter group $\{\exp sB_j\}$ generates a Killing vector field B_j^* $(1 \leqslant j \leqslant 4)$ on the sphere: $B_0^* = -X_1$

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is left invariant and, for $1 \leqslant j \leqslant 3$, B_j^* is the right-invariant vector field on $SU(2)$ defined by $X_{j|I}$. On the other hand, if $U_0 = B_0 + B_1$ and we consider the exponential map $\exp : \mathfrak{g} \to \tilde{G} = \mathbb{R} \times SU(2)$, then $\{\exp sU_0 : s \in \mathbb{R}\} = \tilde{H}$. We set $\mathfrak{h} = \langle\{U_0\}\rangle$, $E_t = -a_t B_0 + b_t B_1$, and $\mathfrak{m} = \langle\{E_t, B_2, B_3\}\rangle$; we have $[U_0, E_t] = 0$, $[U_0, B_2] = 2B_3$, $[U_0, B_3] = -2B_2$, and so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition. The isomorphism $\mu : \mathfrak{m} \to T_o(\mathbb{S}^3) = T_I(SU(2)) \equiv \mathfrak{su}(2)$ in (2.2) satisfies

$$\mu(E_t) = (a_t + b_t)X_{1|I} \equiv X_1, \quad \mu(B_2) = X_{2|I} \equiv X_2 \quad \text{and} \quad \mu(B_3) = X_{3|I} \equiv X_3.$$

Moreover, $[E_t, B_2] = 2b_t B_3$, $[E_t, B_3] = 2a_t U_0 + 2E_t$, $[E_t, B_3] = -2b_t B_2$, and, by (2.3), we get

$$\left.\begin{array}{lll} (\tilde{\nabla}_{E_t^*} E_t^*)_I = 0, & (\tilde{\nabla}_{E_t^*} B_2^*)_I = -2b_t X_{3|I}, & (\tilde{\nabla}_{E_t^*} B_3^*)_I = 2b_t X_{2|I}, \\ (\tilde{\nabla}_{B_2^*} E_t^*)_I = 2b_t X_{3|I}, & (\tilde{\nabla}_{B_2^*} B_2^*)_I = 0, & (\tilde{\nabla}_{B_2^*} B_3^*)_I = -2X_{1|I}, \\ (\tilde{\nabla}_{B_3^*} E_t^*)_I = -2b_t X_{2|I}, & (\tilde{\nabla}_{B_3^*} B_2^*)_I = 2X_{1|I}, & (\tilde{\nabla}_{B_3^*} B_3^*)_I = 0. \end{array}\right\} \tag{4.3}$$

For each

$$p = (z, w) \equiv \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in \mathbb{S}^3 = SU(2),$$

we can write the left-invariant vector fields X_1, X_2, X_3 in terms of the fundamental vector fields B_1^*, B_2^*, B_3^* as

$$\left.\begin{aligned} X_{1|p} &= (|z|^2 - |w|^2)B_{1|p}^* + 2\operatorname{Im}(zw)B_{2|p}^* - 2\operatorname{Re}(zw)B_{3|p}^*, \\ X_{2|p} &= 2\operatorname{Im}(z\bar{w})B_{1|p}^* + \operatorname{Re}(z^2 + w^2)B_{2|p}^* + 2\operatorname{Im}(z^2 + w^2)B_{3|p}^*, \\ X_{3|p} &= 2\operatorname{Re}(z\bar{w})B_{1|p}^* + \operatorname{Im}(w^2 - z^2)B_{2|p}^* + 2\operatorname{Re}(z^2 - w^2)B_{3|p}^*. \end{aligned}\right\} \tag{4.4}$$

Applying now (4.4) and (4.3), some computations show that

$$\begin{aligned} (\tilde{\nabla}_{X_1} X_2)_I &= (\tilde{\nabla}_{E_t^*} B_2^*)_I + 2B_{3|I}^* = (2 - 2b_t)X_{3|I} = 2a_t X_{3|I}, \\ (\tilde{\nabla}_{X_1} X_3)_I &= -2B_{2|I}^* + (\tilde{\nabla}_{E_t^*} B_3^*)_I = (-2 + 2b_t)X_{2|I} = -2a_t X_{2|I}, \end{aligned}$$

and the remaining $(\tilde{\nabla}_{X_j} X_k)_I$ vanish. As a consequence, using (2.7), we see that the homogeneous Riemannian structure $S = \nabla - \tilde{\nabla}$ is given by Equations (3.12). Now, if $\psi_* : \mathfrak{g} = \mathbb{R} \times \mathfrak{su}(2) \to \mathfrak{u}(2)$ is the Lie algebra isomorphism induced by the covering homomorphism ψ, then

$$\begin{aligned} \psi_*(U_0) &= \begin{pmatrix} 0 & 0 \\ 0 & -2\mathrm{i} \end{pmatrix}, \qquad \psi_*(E_t) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & (1 - \varepsilon - t)\mathrm{i} \end{pmatrix}, \\ \psi_*(B_2) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \psi_*(B_3) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \end{aligned}$$

and $\mathfrak{u}(2) = \psi_*(\mathfrak{h}) \oplus \psi_*(\mathfrak{m})$ is the reductive decomposition of $\mathbb{S}_\varepsilon^3 = U(2)/U(1)$ associated with $S_{\varepsilon,t}$. We conclude with the following theorem.

Theorem 4.1. *Let $S = S_{\varepsilon,t}$ be the homogeneous Riemannian structure on the Berger sphere $\mathbb{S}^3_\varepsilon$ defined by (3.11). If $t = 2-\varepsilon$, the associated reductive decomposition is trivial and the corresponding group of isometries is $SU(2)$ acting on itself by left translations.*

The group $\mathbb{S}^1 \times SU(2)$ acts transitively and almost effectively on $\mathbb{S}^3 = SU(2)$ by

$$((\mathrm{e}^{\mathrm{i}s}, g), p) \mapsto gp \begin{pmatrix} \mathrm{e}^{-\mathrm{i}s} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}s} \end{pmatrix},$$

and induces a transitive and effective action of $U(2)$ on $\mathbb{S}^3_\varepsilon$ as a group of isometries, whose reductive decomposition

$$\mathfrak{u}(2) = \left\langle \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \right\} \right\rangle \bigoplus \left\langle \left\{ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & (1-\varepsilon-t)\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\} \right\rangle$$

is the associated one with the homogeneous Riemannian structure S if $t \neq 2-\varepsilon$; if $t = \varepsilon$ it is a naturally reductive decomposition.

Remark 4.2. In the case of the standard sphere ($\varepsilon = 1$), $S = 0$ is a homogeneous Riemannian structure, the associated canonical connection is $\tilde{\nabla} = \nabla$, the holonomy algebra is $\tilde{\mathfrak{h}} \equiv \mathfrak{so}(3)$, the corresponding reductive decomposition is $\tilde{\mathfrak{g}} \equiv \mathfrak{so}(4) \equiv \mathfrak{so}(3) \oplus \tilde{\mathfrak{m}}$, and it defines the representation of $\mathbb{S}^3$ as the symmetric space $\mathbb{S}^3 = SO(4)/SO(3)$. As we noticed in Remark 3.4, different families of homogeneous Riemannian structures on the standard sphere $\mathbb{S}^3$ are also given by putting $(\rho, \sigma, \tau) = (\alpha^3, -\alpha^2, t\alpha^1)$, $(\alpha^3, -t\alpha^2, \alpha^1)$ or $(t\alpha^3, -\alpha^2, \alpha^1)$ in Equation (3.1); if $t \neq 1$, all of them correspond to the isometric action of $U(2)$ on $\mathbb{S}^3$, with different reductive decompositions of $\mathbb{S}^3 = U(2)/U(1)$; if $t = 1$, all of them define the same homogeneous Riemannian structure $S = \alpha^3 \otimes (\alpha^1 \wedge \alpha^2) - \alpha^2 \otimes (\alpha^1 \wedge \alpha^3) + \alpha^1 \otimes (\alpha^2 \wedge \alpha^3)$, which corresponds to the isometric action of $SU(2)$ on itself by left translations.

Remark 4.3. Tricerri and Vanhecke pointed out the following facts in [**18**]. Given a homogeneous Riemannian manifold (M, g) with a group G of isometries, the Ambrose–Singer method determines a tensor field S. This determines conversely a group G' of isometries which is in general not isomorphic to G. A simple but interesting example is the Euclidean plane $\mathbb{R}^2$; from the connected group of isometries $G = SO(2) \cdot \mathbb{R}^2$ one obtains $S = 0$; now, the construction of G' starting from $S = 0$ gives only the group of translations of $\mathbb{R}^2$, since $\mathbb{R}^2$ is flat. So an interesting problem is to understand for which spaces one has $G = G'$ (see [**14**, **18**]). We note that the discussion in the present section shows that Berger spheres provide examples of such spaces.

5. The Berger spheres as homogeneous almost contact metric manifolds

An almost contact structure on a $(2n+1)$-dimensional manifold M is a triple (φ, ξ, η), where φ is a tensor field of type $(1,1)$, ξ a vector field (called the characteristic vector field), and η a differential 1-form on M such that

$$\varphi^2 = -\,\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

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Then $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and φ has rank $2n$. If g is a Riemannian metric on M such that $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ for all vector fields X and Y on M, then (φ, ξ, η, g) is said to be an almost contact metric structure on M and g is called a compatible metric; in this case, $g(X, \xi) = \eta(X)$ and ξ has length 1. If

$$\mathrm{d}\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y])$$

is equal to $2g(X, \varphi Y)$ for all vector fields X and Y on M, then (η, g) is called a contact metric (or contact Riemannian) structure; in particular, $\eta \wedge (\mathrm{d}\eta)^n \neq 0$, that is, η is a contact form on M. If

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X)$$

for a function α on M, then (φ, ξ, η, g) is called α-Sasakian, and the manifold M with such a structure is an α-Sasakian manifold. If $\alpha = 1$, then it is Sasakian. Sasakian manifolds can also be characterized as normal contact metric manifolds and they are in some sense odd-dimensional analogues of Kähler manifolds (see Blair [**4**, **5**]).

If (φ, ξ, η, g) is an almost contact metric structure on M and $(M = G/H, g)$ is a homogeneous Riemannian manifold such that φ is invariant under the action of the connected Lie group G (and hence so are ξ and η), then $(M, \varphi, \xi, \eta, g)$ is called a homogeneous almost contact metric manifold (see [**6**, **10**] and also [**13**]). From the results of Kiričenko in [**12**] it follows that if (M, g) is a connected, simply connected and complete Riemannian manifold and (φ, ξ, η) is an almost contact structure on M such that g is a compatible metric, then $(M, \varphi, \xi, \eta, g)$ is a homogeneous almost contact metric manifold if and only if there exists a tensor field S of type $(1, 2)$ on M satisfying (2.1) and the additional condition $\tilde{\nabla}\varphi = 0$ (and hence $\tilde{\nabla}\xi = 0$ and $\tilde{\nabla}\eta = 0$), where $\tilde{\nabla} = \nabla - S$. Such a homogeneous Riemannian structure S is called a homogeneous almost contact metric structure on $(M, \varphi, \xi, \eta, g)$.

For each $\varepsilon > 0$, we will define an almost contact structure (φ, ξ, η) on the sphere $\mathbb{S}^3$ such that the Berger metric g_ε may be compatible. In order to get that $(\varphi, \xi, \eta, g_\varepsilon)$ can be homogeneous, the characteristic vector field ξ must be invariant by the isometries defined by the action of $SU(2)$ on itself, then ξ must be left invariant; and by (4.1), in order to have $\tilde{\nabla}\xi = 0$ for each homogeneous Riemannian structure $S_{\varepsilon,t}$, $t \neq 2 - \varepsilon$, the vector field ξ must be a scalar multiple of X_1; furthermore, it must have length 1, then $\xi = \sqrt{\varepsilon}^{-1} X_1$; moreover, in order to obtain $g(X, \xi) = \eta(X)$ for each vector field X, the distribution defined by $\eta = 0$ must be generated by the left-invariant vector fields X_2, X_3. We set

$$\varphi = \alpha^2 \otimes X_3 - \alpha^3 \otimes X_2, \qquad \xi_\varepsilon = \frac{1}{\sqrt{\varepsilon}} X_1, \qquad \eta_\varepsilon = \sqrt{\varepsilon}\alpha^1.$$

Remark 5.1. Notice that $\eta_\varepsilon \wedge \mathrm{d}\eta_\varepsilon = -2\varepsilon\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \neq 0$, so η_ε is a contact form on the 3-sphere, and, moreover, $\mathrm{d}\eta_\varepsilon(X, Y) = 2\sqrt{\varepsilon} g_\varepsilon(X, \varphi Y)$; in particular, (η_1, g_1) is a contact Riemannian structure. The simply connected 3-manifolds which admit homogeneous Riemannian contact structures (that is, those for which there exists a connected Lie group acting transitively as a group of isometries which leave the contact form invariant) have been classified by Perrone in [**16**].

The Levi-Civita connection ∇ of g_ε satisfies

$$(\nabla_{X_2}\varphi)X_1 = -\varepsilon X_2, \qquad (\nabla_{X_3}\varphi)X_1 = -\varepsilon X_3, \qquad (\nabla_{X_2}\varphi)X_2 = (\nabla_{X_3}\varphi)X_3 = X_1,$$

and $(\nabla_{X_i}\varphi)X_j = 0$ in the other cases. So, we have

$$(\nabla_X\varphi)Y = \sqrt{\varepsilon}(g_\varepsilon(X,Y)\xi - \eta_\varepsilon(Y)X)$$

for all vector fields X and Y on $\mathbb{S}^3$, hence $(\varphi, \xi_\varepsilon, \eta_\varepsilon, g_\varepsilon)$ is a $\sqrt{\varepsilon}$-Sasakian structure on $\mathbb{S}^3$.

If $\varepsilon \neq 1$, each homogeneous Riemannian structure $S = S_{\varepsilon,t}$ on $\mathbb{S}^3_\varepsilon$ is given by (3.11) and the canonical connection $\tilde{\nabla} = \nabla - S$ satisfies (4.1). This implies $\tilde{\nabla}\varphi = 0$, and we have the following theorem.

Theorem 5.2. *If $\varepsilon \neq 1$, all the homogeneous Riemannian structures on $\mathbb{S}^3_\varepsilon$ are homogeneous almost contact metric structures on the $\sqrt{\varepsilon}$-Sasakian manifold $\mathbb{S}^3(\varphi, \xi_\varepsilon, \eta_\varepsilon, g_\varepsilon)$.*

Suppose now that $\varepsilon = 1$. Each homogeneous Riemannian structure S on the standard sphere $\mathbb{S}^3$ is given by (3.1), where ρ, σ and τ are 1-forms on $\mathbb{S}^3$ satisfying the equations in Theorem 3.1. If $\tilde{\nabla} = \nabla - S$, then

$$\begin{aligned}
\tilde{\nabla}_Z X_1 &= \nabla_Z X_1 - \rho(Z)X_2 - \sigma(Z)X_3,\\
\tilde{\nabla}_Z X_2 &= \nabla_Z X_2 + \rho(Z)X_1 - \tau(Z)X_3,\\
\tilde{\nabla}_Z X_3 &= \nabla_Z X_3 + \sigma(Z)X_1 + \tau(Z)X_2,
\end{aligned}$$

and as a consequence we have that $\tilde{\nabla}\varphi = 0$ if and only if $\rho = \alpha^3$ and $\sigma = -\alpha^2$; if this is the case, the first and second equations in Theorem 3.1 are automatically satisfied and the third equation remains $\tilde{\nabla}\tau = 0$. We find that this is equivalent to requiring that τ be a scalar multiple of α^1 (such as in the case that $\varepsilon \neq 1$ in § 3). Then we have the following theorem.

Theorem 5.3. *The homogeneous almost contact metric structures on the Sasakian manifold $\mathbb{S}^3(\varphi, \xi_1, \eta_1, g_1)$ are given by*

$$S_t = \alpha^3 \otimes (\alpha^1 \wedge \alpha^2) - \alpha^2 \otimes (\alpha^1 \wedge \alpha^3) + t\alpha^1 \otimes (\alpha^2 \wedge \alpha^3), \quad t \in \mathbb{R}.$$

If $t \neq 1$, the corresponding group of isometries leaving invariant the almost contact structure $(\varphi, \xi_1 = X_1, \eta_1 = \alpha^1)$ is $U(2)$ and the associated reductive decomposition is

$$\mathfrak{u}(2) = \left\langle \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \right\} \right\rangle \bigoplus \left\langle \left\{ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -t\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\} \right\rangle.$$

If $t = 1$, the associated reductive decomposition is trivial and the corresponding group of isometries is $SU(2)$.

Remark 5.4. In [**15**], Padrón, Chinea and González have given a classification of homogeneous almost contact metric structures. They give a decomposition of the space of the tensors S of type $(0,3)$ on a $(2n+1)$-dimensional vector space satisfying

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$S_{XYZ} + S_{XZY} = 0$ into 18 irreducible components $\mathcal{H}_i$, invariant under the action of the group $U(n) \times 1$. Moreover, in [**8**], Fino has given two classifications of homogeneous almost contact metric structures: one refines the Tricerri–Vanhecke classification in [**18**]; the other is concerned with the classification of almost contact metric structures found by Chinea and González in [**7**], and it coincides with the one obtained in [**15**]. Now, one can see that any homogeneous almost contact metric structure S on $\mathbb{S}^3(\varphi, \xi_\varepsilon, \eta_\varepsilon, g_\varepsilon)$ satisfies

$$S_{XYZ} = \eta_\varepsilon(X) S_{\xi_\varepsilon YZ} + \eta_\varepsilon(Y) S_{X\xi_\varepsilon Z} + \eta_\varepsilon(Z) S_{XY\xi_\varepsilon}.$$

This implies that S belongs to the class $\mathcal{H}_5 \oplus \mathcal{H}_6 \oplus \mathcal{H}_9$ in [**15**] (which is isomorphic to the class $2\mathbb{R} \oplus [\![\sigma^{2,0}]\!]$ given in [**8**]).

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